NOTES ON A SYMMETRIC BILINEAR FORM ASSOCIATED WITH REGULAR DIRICHLET FORM

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ABSTRACT. We will show how bilinear form $\mathcal{E}_\mu$ related with some smooth measures can be extended to the $L^2(\mathbb{R}^n, \mathbb{C})$ setting.

1. Introduction

We consider a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ where $X$ is locally compact separable metric space and $m$ is a positive Radon measure on $X$ with $\text{Supp}[m] = X$. $\mathcal{S}$ is the family of all smooth measures on $X$. Let $M = (\Omega, X_t, \zeta, P_x)$ be a Hunt process on $X$ which is $m$-symmetric and associated with $(\mathcal{E}, \mathcal{F})$. For a given smooth measure $\mu$, we denote by $A_\mu$ the unique positive continuous additive functional such that $\mu$ is the Revuz measure of $A_\mu$. Let $\mu = \mu_+ - \mu_-$ be a signed Borel measure on $X$. If $\mu_+$ and $\mu_-$ are smooth measures, then we write $\mu \in \mathcal{S} - \mathcal{S}$. For a Borel measure $\nu$ on $X$, $L^2(X, \nu)$ is sometimes written $L^2(\nu)$ when the underlying context is clear.

For $\mu \in \mathcal{S} - \mathcal{S}$, we put

$$\mathcal{E}_\mu(f, g) = \mathcal{E}(f, g) + \int_X f(x)g(x)\mu(dx)$$

for all $f, g \in \mathcal{F} \cap L^2(|\mu| + m)$. We consider the case where $X$ is the Euclidean space $\mathbb{R}^n$ and $m$ is a Lebesgue measure on $\mathbb{R}^n$. It is essential to quantum mechanics.
that functions are from the space $L^2(\mathbb{R}^n, \mathbb{C})$ of square-integrable (with respect to Lebesgue measure), complex-valued functions.

In this paper we extend $\mathcal{E}_\mu$ to $L^2(\mathbb{R}^n, \mathbb{C})$ setting and find self-adjoint operator which represent the extension of $\mathcal{E}_\mu$.

2. Extension of $\mathcal{E}_\mu$ to $L^2(\mathbb{R}^n, \mathbb{C})$

Let us use the short notation $L^2(\mu)$ for $L^2(X, \mu)$, for $\mu \in S - S$, we put

$$\mathcal{E}_\mu(u, v) = \mathcal{E}(u, v) + \int_X u(x)v(x)\mu(dx)$$

for all $u, v \in \mathcal{F} \cap L^2(|\mu| + m)$

**Theorem 1.** If $\mathcal{E}_\mu$ is bounded below, densely defined and closed, then there exist a unique, densely defined self-adjoint operator $H^\mu$ which is bounded below and satisfies $(H^\mu u, v) = \mathcal{E}_\mu(u, v)$ for all $u \in D(H^\mu)$ and $v \in D(\mathcal{E}_\mu)$

**Proof.** See [4] Theorem 2.6

For $\alpha \geq 0$, $\mu$ and $\nu$ in $S - S$, $f \in B(X)$, we set

$$U_\nu^{\alpha + \mu}f(x) = E_x[\int_0^\infty \exp\{-\alpha t - A_t^\nu\}f(X_t)dA_t^\nu]$$

provided the right hand side makes sense. When $\nu = m$, we simply write $U^{\alpha + \mu}f$ for $U_\nu^{\alpha + \mu}f$.

In the following Theorem 2, we consider the case where $X$ is the Euclidean space $\mathbb{R}^n$ and $m$ is a Lebesgue measure on $\mathbb{R}^n$. If $\psi$ is a function in $L^2(\mathbb{R}^n, \mathbb{C})$ (space of square integrable, complex valued functions), we denote by $\psi_1$ its real part and by $\psi_2$ its imaginary part; i.e., $\psi = \psi_1 + i\psi_2$
Theorem 2. Let $\mu \in S - S$ be such that

$$U^{\alpha + \mu}(L^2(m)) \subset L^2(m)$$

for some $\alpha > 0$. Suppose that $\mathcal{E}_\mu$ is closed. If we define $\mathcal{E}_\mu^C$ by

$$\mathcal{E}_\mu^C(\psi, \varphi) = \mathcal{E}_\mu(\psi_1, \varphi_1) + \mathcal{E}_\mu(\psi_2, \varphi_2) + i[\mathcal{E}_\mu(\psi_2, \varphi_1) - \mathcal{E}_\mu(\psi_1, \varphi_2)]$$

for all $\psi, \varphi \in D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu) \subset L^2(\mathbb{R}^n, \mathbb{C})$, then $\mathcal{E}_\mu^C$ is densely defined, bounded below and closed.

Proof. Since $D(\mathcal{E}_\mu)$ is dense in $L^2(\mathbb{R}^n)$, $D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu)$ is dense in $L^2(\mathbb{R}^n, \mathbb{C})$. Since $U^{\alpha + \mu}(L^2(m)) \subset L^2(m)$, $\mathcal{E}_\mu$ is bounded below [1. Theorem 4.1].

Let $A$ be some real number satisfying $\mathcal{E}_\mu(u, u) \geq A\|u\|^2$ for all $u \in D(\mathcal{E}_\mu)$ and let $\psi = \psi_1 + i\psi_2 \in D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu)$. Then we have $\mathcal{E}_\mu^C(\psi, \psi) \geq A[\|\psi_1\|^2 + \|\psi_2\|^2] = A\|\psi\|^2$ by the symmetry of $\mathcal{E}_\mu$.

To verify $\mathcal{E}_\mu^C$ is closed, it suffices to show that $D(\mathcal{E}_\mu^C)$ is complete under the norm

$$\|\psi\|^2 = \mathcal{E}_\mu^C(\psi, \psi) + (-A + 1)\|\psi\|^2$$

Let $(\psi_n)$ be a sequence in $D(\mathcal{E}_\mu^C)$ such that $\|\psi_n - \psi_m\| \to 0$ as $n, m \to \infty$. Then $\psi_n = \psi_{n,1} + i\psi_{n,2}$ for each $n \in N$, where $\psi_{n,1}, \psi_{n,2}$ are in $D(\mathcal{E}_\mu)$. By the symmetry of $\mathcal{E}_\mu$,

$$\|\psi_n - \psi_m\|^2 = \mathcal{E}_\mu^C(\psi_n - \psi_m, \psi_n - \psi_m) + (-A + 1)\|\psi_n - \psi_m\|^2$$

$$= \mathcal{E}_\mu(\psi_{n,1} - \psi_{m,1}, \psi_{n,1} - \psi_{m,1}) + \mathcal{E}_\mu(\psi_{n,2} - \psi_{m,2}, \psi_{n,2} - \psi_{m,2})$$

$$+ (-A + 1)\|\psi_{n,1} - \psi_{m,1}\|^2 + (-A + 1)\|\psi_{n,2} - \psi_{m,2}\|^2$$
\[ = \mathcal{E}_\mu(\psi_{n,1} - \psi_{m,1}, \psi_{n,1} - \psi_{m,1}) + (-A + 1)\|\psi_{n,1} - \psi_{m,1}\|^2 \]
\[ + \mathcal{E}_\mu(\psi_{n,2} - \psi_{m,2}, \psi_{n,2} - \psi_{m,2}) + (-A + 1)\|\psi_{n,2} - \psi_{m,2}\|^2 \]
\[ = \|\psi_{n,1} - \psi_{m,1}\|^2 + \|\psi_{n,2} - \psi_{m,2}\|^2. \]

Since \( \|\psi_n - \psi_m\| \to 0 \), \( \|\psi_{n,1} - \psi_{m,1}\| \to 0 \) and \( \|\psi_{n,2} - \psi_{m,2}\| \to 0 \). Since \( \mathcal{E}_\mu \) is closed, there exist \( \psi_1, \psi_2 \) in \( D(\mathcal{E}_\mu) \) such that \( \|\psi_{n,1} - \psi_1\| \to 0 \) and \( \|\psi_{n,2} - \psi_2\| \to 0 \) as \( n \to \infty \).

This means that \( \|\psi_n - \psi\| \to 0 \) as \( n \to \infty \). And since \( \psi = \psi_1 + i\psi_2 \in D(\mathcal{E}_\mu^C) \), we conclude that \( \mathcal{E}_\mu^C \) is closed.

Let \( H^\mu \) be a self-adjoint operator as in Theorem 1. If we define \( H^\mu_C \) on \( D(H^\mu) + iD(H^\mu) \) by \( H^\mu_C(\psi_1 + i\psi_2) = H^\mu\psi_1 + iH^\mu\psi_2 \), then \( H^\mu_C \) is a self-adjoint operator on \( D(H^\mu) + iD(H^\mu) \subset D(\mathcal{E}_\mu^C) \).

**Theorem 3.** **Under the conditions of Theorem 2,**

\[ (H^\mu_C\psi, \varphi) = \mathcal{E}_\mu^C(\psi, \varphi) \]

**for all** \( \psi \in D(H^\mu_C) \) **and** \( \varphi \in (\mathcal{E}_\mu^C) \).

**Proof.** By Theorem 1, there exist a unique densely defined self-adjoint operator \( H^* \) which is bounded below and satisfies \( (H^*\psi, \varphi) = \mathcal{E}_\mu^C(\psi, \varphi) \) for all \( \psi \in D(H^*) \) and for all \( \varphi \in (\mathcal{E}_\mu^C) \). From the linearity of \( H^\mu_C \), \( (H^\mu_C\psi, \varphi) = \mathcal{E}_\mu^C(\psi, \varphi) \) for \( \psi = \psi_1 + i\psi_2 \in D(H^\mu_C) \) and \( \varphi = \varphi_1 + i\varphi_2 \in D(\mathcal{E}_\mu^C) \).

Using consequences [[4], Corollary 2.4 and Theorem 2.6, p.323] of the first representation Theorem and the simple fact (see e.q. [[5], p.279]) that two self-adjoint operators, one of which extends the other, are actually equal, one has \( H^* = H^\mu_C \).
REFERENCES


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