Numerical measures of Indicating Placement of Posets on Scale from Chains to Antichains

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〈Abstract〉

In this paper we obtain several function defined on finite partially ordered sets(posets) which may indicate constraints of comparability on sets of teams(tasks, etc.) for which evaluation is computationally simple, a relatively rare condition in graph-based algorithms. Using these functions a set of numerical coefficients and associated distributions obtained from a computer simulation of certain families of random graphs is determined. From this information estimates may be made as to the actual linearity of complicated posets. Applications of these ideas is to all areas where obtaining rankings from partial information in rational ways is relevant as in, e.g., team__, scaling__, and scheduling theory as well as in theoretical computer science.

Theoretical consideration of special and desirable properties of various functions is provided permitting judgment concerning sensitivity of these functions to changes in parameters describing (finite) posets.

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I. Introduction

Often the effectiveness of scheduling, clustering, and allocation methodologies employed in the areas of parallel and distributed computing can best be determined by utilizing statistical information obtained when these methods are applied directly to a sizable population of randomly-generated task systems that take the from of directed graphs. Such statistical analysis is especially important when determining the effectiveness of heuristic algorithms used to solve such NP complete problems since the ad-hoc nature of these methodologies do not readily lend themselves to analytical analysis.

There are many algorithmic recipes that are possible to generate directed graphs in a random manner. With most of these recipes the generation of any type of graph is always possible, but only graphs that fall into a relatively small subset of the general population of directed graphs is ever probable. Therefore it is always important to choose a recipe that is biased in favor of graphs that are typical in structure to those that represent the real-world systems that are expected to be applied to the methodology.

For example, consider the case where it is desirable to generate, in a random manner, graphs that represent systems whose tasks are to be allocated to a set of parallel processing elements by some newly-created heuristic. An obvious consideration in this case is the amount of inherent parallelism present within typical graph. If the task allocation methodology is evaluated using an algorithmic recipe that is biased toward producing graphs that are highly sequential, then the statistical evaluation of the data collected would not reflect how well the allocation methodology performs when applied to the more interesting types of task structures i.e. those possessing reasonable amounts of parallelism.

To address this concern, the population of graphs produced by each recipe must be carefully analyzed before the resulting graphs are used as part of any analysis. A complicating factor is that the recipes themselves are often too complex to be analyzed analytically. Fortunately, Mount Carlo techniques can be used to gain knowledge about the particular algorithmic recipe by studying the population of directed graphs that it produces.

I. Structure of Directed Graphs

There are many ways to characterize the structure of acyclic directed graphs. A major concern in the area of parallel processing is the degree of parallelism present in a graph. Some parameters which characterize this include the critical path length and

maximum number usable processors. A more complete measure may be obtained by relating these acyclic structures to the mathematical entity called a finite partially ordered set [1], or poset.

A finite poset, P is a set composed of elements x_i, which are subject to a binary relationship, ($\leq$), such that the following conditions hold true for all elements in the poset ;

(1) $x_i \leq x_i$;

(2) if $x_i \leq x_j$ and $x_j \leq x_i$ then $x_i = x_j$

(3) if $x_i \leq x_j$ and $x_j \leq x_k$ then $x_i \leq x_j$

The first condition describes the reflexive property, the second anti-symmetry and the third transitivity.

Whenever the condition $x_i \leq x_j$ and $x_i \neq x_j$, then the binary relationship ($\leq$) is often replaced with the more strict relationship, ($<$). This may be the more appropriate representation of posets obtained from task graphs since this implies a strict ordering. (The ($\leq$) relationship might be viewed as allowing the possibility that more than one task can be processed on the same processing element at the same time).

Posets are usually described by a set of covering relationships that are defined to include all ordering relationships between the elements of a poset $x_i \leq x_k$ such that there exists no other relationship in the form

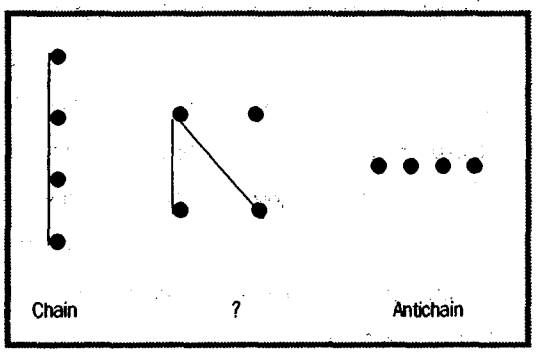
$$x_i \leq x_j \leq x_k$$

Posets are often represented visually in the form of Hasse diagrams, where each element of the poset is portrayed as a vertex, and directed arcs are placed between vertices to show the covering relationships that exit between the elements. The arcs are assigned directions such that a vertex x_j points to another vertex x_i whenever there is a covering relationship of the form $x_i \leq x_j$. In a Hasse diagram the arc directions are not shown explicitly ; these directions are always assumed to be from top to bottom of the diagram (thus Vertex x_i is below Vertex x_j whenever $x_i \leq x_j$).

Two types of structures which can be classified quite readily in terms of their parallelization potential are the chain and antichain. A chain is a poset which for each pair of elements(x_i, x_j), one of the following relationships, $x_i \leq x_j$ or $x_j \leq x_k$, holds true. This implies a strict linear order is present between all of the elements. At the other extreme is the antichain, which is a poset that is not bound by any constraints (no covering relationships) ; all its elements are incomparable with one another. These two structures have definite meaning when they represents task systems targeted for parallel computation ; a chain represents the case of total sequentially and an antichain represents the case of perfect (inherent) parallelism.

Clearly structures which exhibit antichain-like characteristics, are more desirable candidates for parallel processing than those which more closely resemble chains. Figure 1 shows the Hasses Diagram of a chain and an antichain along with a poset that has an intermediate structure.

(Figure 1) Possible Range of Poset Structures



Ⅱ. Alignment Functions

A useful method of classifying the intermediate poset structures is to describe, by some measure, their degree of 'chaininess' (or 'antichaininess'). Such a technique is presented by [1], who extended the concept of families introduced by [3] in a manner that measures the linearity of posets through the calculation of special purpose alignment functions. The method is based upon the idea of assigning ratings to each element of a poset.

Ratings are numbers that assign to individual elements of a poset a value that allows the elements to be compared or ranked in some way with one another. Such ratings are usually based upon incomplete information which is analogous to the partial order associated with the elements of posets. The alignment functions described by Bae incorporate the rating functions, r, of height, above and below, each of which can be termed as being natural in that they all can be obtained in a precise manner directly from the Hasse diagram.

The height rating, h, of a vertex is intuitively defined as the maximum direct 'distance' from the bottom of the Hasse diagram to the current element. Such direct distances are measured by counting the minimum number of arcs which must be transversed in the reverse direction from elements on the bottom of the diagram to the one in question. Expressing it in a more formal manner, the height of element x_i in poset P is defined to be $h_p(x_i) = r$, when there exists a chain described by the covering relationships $x_1 \le x_2 \le x_3 \cdots x_r \le x_i$ and there is no other chain of greater cardinality. The height of the elements varies from the minimum value of zero for the element(s) on the bottom of the Hasse diagram to some maximum which is often termed the height of the poset, $[h_p(p) = \max\{h_p(x_i) : x_i \in p\}]$.

The ratings above, a_p, and below, b_p, describe the number of successor and descendent vertices, respectively, associated with an element (plus the element itself).

Above is defined for an element x_i in poset P as $a_p(x_i) = |\{x_r : x_r \geq x_i\}|$, and in a dual manner the rating below is defined as. Both ratings are integer-based and have a minimum value of 1 because the ($\geq$) and ($\leq$) operators always include the element x_i. The maximum value for both functions is bounded by the number of elements in the poset |P|. In the case where $a_p(x_i) = b_p(x_i) = 1$ then element x_i is a singleton (i.e. an unconnected vertex on the Hasse diagram) which of course implies that $h_p(x_i) = 0$. It is interesting to note that a_p and b_p are duals of one another in the sense that if the relationship ($\leq$) were replaced by the relationship ($\geq$) in describing the poset then $a_p \equiv b_p$ and $b_p \equiv a_p$, where the asterisk indicates the new value of above and below respectively.

Ratings often are produced in a manner that correlate in some way with the ordering relationships describing a poset. Rating functions r_p, which produce values that are consistent with the order (i.e. when $x_i \leq x_j$, then $r_p(x_i) \leq r_p(x_j)$) are termed order-covariant. Those that produce values which are inversely correlated with the ordering relationship are considered to be order-contravariant. Considering the ratings functions just described, height and below are order-covariant and above is order-contravariant. Alignment functions can be constructed by employing mathematical entities called indicator functions, I_p, that produce a numeric value for each ordered pair, (x_i, x_j), of elements in a poset. Such indicator functions form a basis upon which any two elements of a poset can be compared with one another, often including those which are incompletely specified by the ordering relationships that describe the poset (i.e. the so-called free-pair elements, x_i and x_j). Indicator functions often employ one or more rating functions which can be very diverse in nature; the general form of the indicator function being employed by Bae is shown in Equation(1) ;

$$I_p[\delta_1 \cdots \delta_\alpha : \theta_1 \cdots \theta_\beta](x_i, x_j) = \prod_{k=1}^{\alpha}[\delta_k(x_i) - \delta_k(x_j)] \cdot \prod_{k=1}^{\beta}[\theta_k(x_i) - \theta_k(x_j)] \quad (1)$$

where (x_i, x_j) are any two elements of the poset, $\delta_1 \cdots \delta_\alpha$ are any number of order-covariant rating functions, and $\theta_1 \cdots \theta_\beta$ are order-contravariant ones.

The alignment function, λ_p, presented by [1], examines all possible ordered pairs of elements, applies the aforementioned indicator function to each ordered pair, totals the number of instances that the indicator function produces non-negative results, and reports this number in the form of the proportion of the total population of ordered pairs. Its general form is shown in Equation

(2):

$$\lambda_p[\delta_1 \cdots \delta_\alpha : \theta_1 \cdots \theta_\beta]$$

$$= \frac{|\{x_k x_i \in x_k x_j [\delta_1 \cdots \delta_\alpha : \theta_1 \cdots \theta_\beta](x_i x_j \geq P)\}|}{|P|^2} \quad (2)$$

where (x_i, x_j) are any two elements of poset, P, $\delta_1 \cdots \delta_\alpha$ are any number of order-covariant rating functions, $\theta_1 \cdots \theta_\beta$ and are order-contravariant ones.

The possible indicator functions that utilize the ratings of height, above, and satisfy the requirements imposed in Equation(2) are the functions $I_p[h_p, a_p]$, $I_p[b_p, a_p]$, and $I_p[h_p, b_p, a_p]$. (In this case consideration is given only to indicator functions which employ at least one co-variant and one contra-variant rating function). It has been proven that these alignment functions have a lower bound value that corresponds to posets forming a chain and an upper bound value for posets which are antichains[2] (see Equation (3) - (5)).

$$\lambda_p[h_p, a_p]_{CHAIN} = \frac{1}{|P|} \lambda_p[h_p, a_p]_{ANTICHAIN} = 1 \quad (3)$$

$$\lambda_p[b_p, a_p]_{CHAIN} = \frac{1}{|P|} \lambda_p[b_p, a_p]_{ANTICHAIN} = 1 \quad (4)$$

$$\lambda_p[h_p, b_p, a_p]_{CHAIN} = \frac{(|P| + 1)}{2 \cdot |P|} \lambda_p[h_p, b_p, a_p]_{ANTICHAIN} = 1 \quad (5)$$

There are several considerations when relating these alignment functions to acyclic task systems. First, it must be remembered that these alignment functions do not take into account node or are weightings. For this reason, one can consider them as measuring the connection structure, not necessarily the execution efficiency associated with a task graph. These alignment functions do correlate well, however, to cases where there is a fairly even distribution of task execution times, and arc weights are fairly uniform : Another consideration is that alignment functions are not exact in nature. This is highlighted by the fact that there are slight differences between the relative values of chaininess assigned to certain posets structures by the three alignment functions which have been described. It must be emphasized that these differences are slight, and that alignment functions appear to be a very efficient method for such measurements to be made across an arbitrary class of posets (the execution of the algorithm is order n^2 time where $n = |P|$).

Ⅳ. Random Directed graph Generation

The usefulness of alignment functions in the area of evaluating allocation methodologies stems from their ability to quantity in a numerical manner the structure

of posets. This permits randomly generated posets to be placed into groups based upon their structure (or their expected structure), allowing competing allocation methodologies to be ranked by analyzing, in a statistical manner, how well each performs on a certain range of groups.

There are many methods possible to randomly generate such posets, with some methods being based upon the development of random but possibly redundant ordering relationships, and others which are concerned only with the creation of random covering relationships. The p-methods is the former type because it creates random posets by considering all possible connections between the elements (this closely matches the data flow model of a task system, not necessarily its precedence structure). It requires that data be encoded internally in the form of an adjacency matrix where,

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,j} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,j} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,j} & \cdots & a_{k,n} \end{pmatrix}$$

where,

a_{ij} = 1 when $a_i \rightarrow a_j$ ($a_i \geq a_j$ for posets);

a_{ij} = 0 otherwise.

The p-method, assigns zeros to the diagonal elements and to the elements that make up the lower triangular portion of the matrix. This assures that the resulting graph will be acyclic in nature, possessing no self-loops or loops between other elements. The method then considers, in a row-by-row sequence, each of the remaining upper triangular elements and assigns a value of one or zero based upon the value of a random number generator. A more precise way of describing the method is shown below :

$$\begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ 0 & 0 & a_{2,3} & \cdots & a_{2,j} & \cdots & a_{2,n} \\ 0 & 0 & 0 & \cdots & a_{3,j} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{i,j} & \cdots & a_{k,n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

for each (i, j),

a_{ij} = 1 when $i<j \wedge \text{rand}( ) \leq P$;

a_{ij} = 0 otherwise.

Here the rand() function returns an evenly distributed random value between the value of 0 and 1. The value of p is a probability constant which is set at the start of the process, to represent the probability at each of creating an are between two

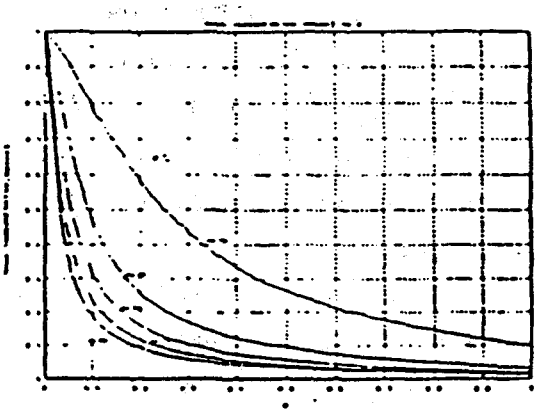
vertices. A probability constant of one creates a chain and a zero assures an antichain; any value in between results in an indeterminate type of structure.

V. Empirical Analysis

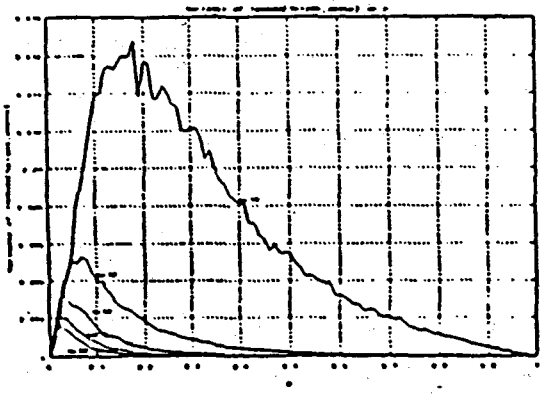
Employing this technique, random graphs of varying sizes have been constructed using values of p that vary over the range of 0 to 1 in steps of 0.01. For each value of p, the three alignment functions have been computed for each poset in a sample consisting of 1000 random structures and the sample mean and variance have been recorded. These are shown in the figures that follow(Figure 2-7) for posets of sizes n=10, n=30, n=50, n=70, and n=90, where n=|P|. It is interesting to note that the figures of all three alignment functions are very similar to one another. This reflects their general utility and accuracy in approximating the chaininess of poset structures.

From these figures, several general observations can be made. First as the size of poset, n=|P|, increases the figures that show the mean value of the alignment functions each have a progressively smaller and sharper transition region between values that reflect chain-like poset behavior and those dominated by antichain-like properties. (It should be noted that the threshold

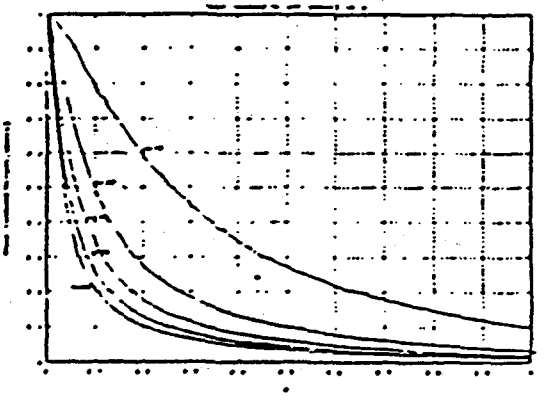
〈Figure 2〉 Mean for Random Posets



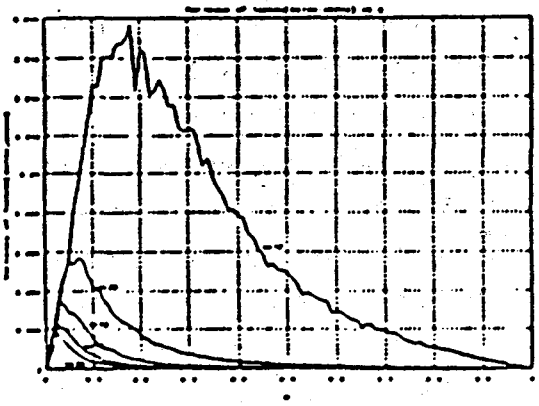
〈Figure 3〉 Variance for random Posets



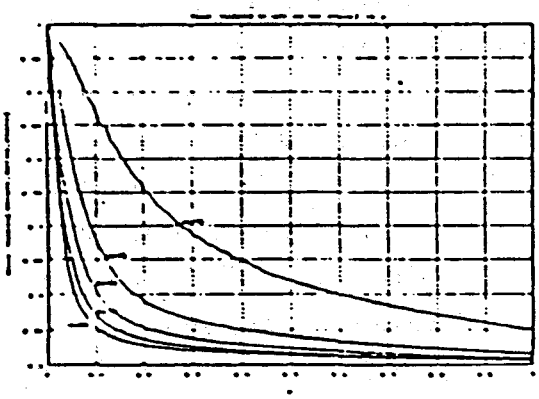
〈Figure 4〉 Mean for Random Posets



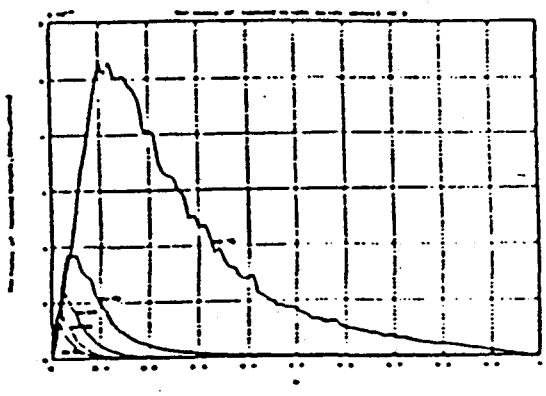
〈Figure 5〉 Variance for random Posets



〈Figure 6〉 Mean $\lambda_p[h_p, b_p, a_p]$ for Random Posets



〈Figure 7〉 Variance $\lambda_p[h_p, b_p, a_p]$ for Random Posets



behavior described by these figures is not an uncommon phenomenon, it is similar to that displayed by other random processes in the area of graph theory and partially-ordered sets). This means that generating posets in the intermediate range(somewhere between chains and antichains) requires an increasingly careful selection of the value of p. The tendency of many researchers that have used this method has been to chose values of p that are too large, resulting in a sample of posets being produced that are dominated by chain-like posets. Such domination is very likely to adversely affect the results of any statistical analysis.

It is also apparent that the value of p in the figures is closely correlated with the size of the poset in that as n increases, the p value with the greatest variance becomes progressively smaller. For each value of n, the value of p with greatest variance occurs within the antichain/chain transition region. This seems reasonable given the fact that the point of maximum variance is where the widest range of poset structures are possible. It can be proven that for any given value of p, $0 < p \leq 1$, when n approaches infinity the value of all three alignment functions each approach one [4]. The product $n \cdot p$, however, appears to be bounded by a constant. From the figures, the relationship $n \cdot p = 2$ appears to always approximate the point of greatest variance. Assuming this

conjecture is correct, the value of p can be determined directly from the size of the poset. This could be of major significance in effectively employing this method to evaluate allocation methodologies and applying it to other problem domains.

Another observation is that as n increases, the magnitude of the variance decreases. It can also be seem from the figures that for the curves that represent posets larger than n=30 elements, the graphs of the mean and variances tend to be smooth, indicating that they are not suffering from the effects of small sampling size. For this reason, it is wise to only generate random posets of sizes greater than 40 elements when testing allocation methodologies. This is not a major problem since the large size systems provide more opportunity for parallel processing.

Ⅵ. Conclusions and Future Research

The major problem with the p-method of generating acyclic directed graphs is that for most values of p the directed graphs that are produced are either very chain-like or very antichain-like. With the p-method the more interesting structures that are produced lie within a very narrow antichain/chain transition region. This region becomes even sharper for the larger acyclic directed graphs. Unfortunately it is a common mistake of many researchers to select values of p that are too large causing the population of directed graphs to become biased in favor of chains. The value of p that defines the center of the antichain/chain transition region appears to be very closely correlated with the number of vertices in the directed graphs. If the conjecture that $n \cdot p_c$ = constant is true then the utility of this method for use in the evaluation of allocation methodologies (and many other problem domains) is greatly increased. Otherwise, other methods of generating directed graphs may be superior to the p-method for this application.

Two additional methods of generating random acyclic structures merit future research. They include methods based upon the intersection of randomly-generated linear orderings[5-6] and the so-called box methods[7]. Initial empirical results slow that slight alterations of the original methods can lead to much larger antichain/chain transition regions.

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