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# The Factor Space in Financial Markets*

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## Abstract

We show assets can be classified into diversifiable risks and non-diversifiable risks based on aggregate endowment and spanning so that in equilibrium agents eliminate diversifiable risks which must have zero values. Consequently, the benchmark portfolio that represents a pricing operator should have only a non-diversifiable risk, aggregate endowment should earn a positive risk premium over a riskless asset, and, even in incomplete markets, there should be a pricing operator represented by a function of aggregate endowment if any asset mean-independent of aggregate endowment is diversifiable. These results apply to both the CAPM and a representative agent model.

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Most equilibrium-based asset pricing theories utilize agents' optimization and market clearing conditions in order to determine the additional properties of a linear pricing rule implied by the no arbitrage condition. Since the no arbitrage by itself implies a linear pricing operator which has an inner product representation with some benchmark portfolio, equilibrium theories seek to find additional restrictions on the benchmark portfolio. For example, the CAPM developed by Sharpe(1964), Lintner(1965), Mossin(1966) and Black(1972) concludes the market portfolio is mean-variance efficient. This conclusion is equivalent to saying that the benchmark portfolio should lie in a two-dimensional space spanned by a riskless asset and the market portfolio. Since the no arbitrage condition does not specify any particular subspace for the benchmark portfolio, equilibrium restriction reduces the cardinality from the span of assets to a two-dimensional subspace. Another example is a representative agent model constructed by Lucas(1978) and Constantinides(1982), where the benchmark portfolio is a marginal rate of substitution for a representative agent. Since the marginal rate of substitution is some function of aggregate endowment the model says that the benchmark portfolio should lie in a space spanned by functions of aggregate endowment.

The additional restrictions in these two models, however, are not obtained without cost. The cost is the controversial assumptions the models need to make. For example, in a representative agent model we assume complete markets, which are not realistic due to the well-known market failure problem. Even though the CAPM does not include a complete market assumption, it alternatively assumes mean-variance efficient behavior which is also highly restrictive. It is only through either quadratic preferences or a spanning restriction, such as elliptically distributed assets, that we can justify this behavior in terms of expected utility maximization. Furthermore, these models do not provide a clue to how their conclusions will change without their controversial assumptions. If the conclusions of these models were rejected by empirical tests, and if any other equilibrium restriction could not be obtained without these assumptions, then, as in the Arbitrage Pricing Theory by Ross(1976), we could not but ignore equilibrium conditions and rely solely on the no arbitrage condition to explain asset prices. Hence, characterizing an equilibrium restriction without these specialized assumptions is a fundamentally important but unresolved task.

In this article we analyze an equilibrium restriction in incomplete markets without assuming mean-variance efficient behavior. The key idea in our approach is to generalize the notion of diversification that was used in the CAPM to describe idiosyncratic risks that must have a zero price. Instead of assuming an exogenously given factor structure we rely on both uncertainty in aggregate endowment and a span of assets in order to decompose any asset into two different risks: a non-diversifiable risk which we call a factor and a diversifiable risk which we call an idiosyncratic risk. We make this decomposition in such a way that equilibrium conditions force each agent to diversify away any idiosyncratic risk so that all idiosyncratic risks must have zero value. For this purpose we define a factor space (a space spanned by non-diversifiable risks) as the minimal space of the subspaces which include aggregate endowment and also imply that an asset which is uncorrelated to any element of the subspace becomes also mean-independent. The space spanned by idiosyncratic risks, which we call an idiosyncratic space, is defined as a collection of assets which are mean-independent of any factor.

The idiosyncratic space according to our definition is basically a set of assets that any risk-averse agent wants to diversify away, because through diversification risk-averse agents can shun mean-preserving spreads. A well-known principle of decision-making under uncertainty formalized by Rothschild and Stiglitz(1970) will justify this behavior. Another implication of our decomposition is that all agents can also eliminate any idiosyncratic risk collectively because aggregate endowment is required to be in the factor space. The beauty of our decompositon, however, lies in the fact equilibrium conditions require agents to diversify away all idiosyncratic risks. To see this, let's suppose some agents would hold idiosyncratic risks in equilibrium. Then the idiosyncratic risk held by some agent must have (strictly) negative value because, otherwise, agents could always improve his utility by selling it. Consequently, the sum of the idiosyncratic risks held by each agent must also have negative value dut to the linear pricing rule. This, however, cannot happen in equilibrium because the sum is a zero-vector according to both our definition and market clearing conditions, and the zero-vector must have zero value under the linear pricing rule.

If agents do not take any idiosyncratic risk, then all idiosyncratic risks must have a zero price in equilibrium. This is so because, if any idiosyncratic risk had a positive (negative) price, selling (buying) it would be a fair bet and, according to Samuelson(1967), agents will take a small enough piece of a favorable gamble. Since all idiosyncratic risk must have zero value, the benchmark portfolio should be a factor under the linear pricing rule. Hence we identify the factor space as a subspace in the span of assets where the benchmark portfolio can be found *a priori* in equilibrium.

Our analysis also provides an equilibrium justification of why aggregate endowment must earn a positive risk premium over a riskless asset. Even though agents diversify away all idiosyncratic risks, agents must still hold some additional risks which must have negative value in equilibrium. For example, when there is a riskless asset in the span of asset, a mean-zero component of a future consumption choice is an additional risk because it is mean-independent of a riskless asset, and this component must have a negative price because, otherwise, agents improve their utilities by selling it. Then market clearing and the linear pricing rule imply that a mean-zero component of aggregate endowment must also have negative value. As a result, aggregate endowment must earn a positive risk premium over a riskless asset.

Our concept of the factor space and its related results also provide a unifying means of understanding equilibrium restriction. For example, according to our definition, in complete markets the factor space is spanned by all functions of aggregate endowment. Another example can be found in the CAPM with distributional assumptions, where the factor space according to our definition is spanned by the market portfolio directly follows from our results. These two examples illustrate that when we drop their specialized assumptions from the models, our conclusions can be substituted for the conclusions of these two models.

The equilibrium restriction in both the CAPM and the representative agent model, however, has another property: the benchmark portfolio is a function of aggregate endowment (or a contingent claim written on aggregate endowment). This property is of importance to empirical studies because they can focus on estimating the functional form of the benchmark portfolio without resorting to other macroeconomic variables. This property, however, does not always follow in incomplete markets, even though the benchmark portfolio lies in the factor space, because all factors cannot always be associated with some contingent claims on aggregate endowment. Hence, one might wish to determine what condition must exist in order to have a linear pricing functional that has an inner product representation with some contingent claims on aggregate endowment as a benchmark. In this article, we provide a sufficient condition for this. The condition states that any asset mean-independent of aggregate endowment is an idiosyncratic risk. Under this sufficient condition, we prove that any factor can be associated with some contingent claims written on aggregate endowment by an orthogonal projection map. Then our previous results show that the benchmark portfolios can be associated in a similar manner. Indeed, in both the CAPM and a representative agent model we can show that our sufficient condition is met. Thus our sufficient condition is implicit in these two models.

Our article is closely related to Connor(1984), Milne(1988) and Chamberlain(1988). Connor(1984) assumes a factor structure assumption and finds that the benchmark portfolio should be a factor in equilibrium. In this article, we improve his results in at least three important ways. First, we define the factor space instead of assuming it. Because of this, the factor space and its related results can serve as a unifying means of understanding equilibrium restrictions. Second, we prove our results directly from equilibrium conditions without resorting to the constrained Pareto optimality. Third, we analyze how factors are determined by both uncertainty in aggregate endowment and a spanning restriction, and provide a sufficient condition for the benchmark portfolio being associated with aggregate endowment. Milne (1988) extends Connor's results to more general preferences and also points out that the CAPM with a normal distribution assumption satisfies the factor structure assumption. This idea is more thoroughly explored in our analysis of the CAPM, where elliptical distribution assumptions and non-traded endowments are additionally considered. Chamberlain(1988) shows how Connor's results can be generalized into a multi-period model. His results suggest that our two-period model can extend to a multi-period model.

The organization of the remainder of the article is as follows. Section 1 derives our main results in the simplest possible model, where the familiar two-period world with a finite number of states is assumed. In this setting we discuss the factor space, its related results, and our sufficient condition. Sections 2 shows how our results can be applied to other asset pricing theories, such as the CAPM with distributional assumptions and a representative agent model. Section 3 explains how our simple moedel can be extended to allow for non-traded endowments. Section 4 concludes with some future research directions.

## 1. The Basic Model

We begin with the simplest possible model, where the standard asset pricing theories, such as the CAPM or a representative agent model, do not provide a meaningful equilibrium restriction on the benchmark portfolio. Our basic model, however, will extend to more general cases in Sections 2 and 3. The framework we will use is a familiar two-period (indexed by today and future, or, t=0, 1) world with J securities and S states of nature. At each time and in each state we assume a single consumption commodity which acts as *numeraire*. The following additional assumptions characterize the special economy $((u^h, e^h), A)$ we consider in this section.

(A1 Incomplete Markets) $M \equiv \text{span}[A] \subseteq X \equiv \mathbb{R}^S$

(A2 von-Neuman Morgenstern Utility Functions) $U^h(x)=u^h(x_0)+\delta^h \sum_{s=1}^{S} \pi_s^h u^h(x_{1s})$ where $u^h$ are strictly concave, weakly monotonic, differentiable functions from $\mathbb{R}_+$ to $\mathbb{R}$ for h=1,...,H.

(A3 Common Priors) $\pi_s^h=\pi_s$ for all s=1,...,S and h=1,...,H.

(A4 Endowment Spanning) $e_1^h \in M$ for all h=1,...,H.

In (A1) the matrix of gross payoffs per unit of investment is given by A (S×J), the columns of which represent the payoffs of the given securities. Since we are particularly interested in incomplete markets, span[A], denoted by M, is usually a J-dimensional subspace of $\mathbb{R}^S$. Unless otherwise mentioned, a riskless asset, or a vector of 1s $\in \mathbb{R}^S$ denoted by $\tilde{1}$, need not be spanned by assets. Without loss of generality we assume that each asset is in zero supply. In (A2) each agent $h \in H=\{1,...,H\}$ is risk-averse and has a time-additive state-independent preference.[1] Other than specified in (A2),

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1) Milne(1988) shows that we can allow more general preferences than von-Neuman Morgenstern Utility functions. Since our preference assumption has a main impact only on the results in Lemmas 1 and 2 in Section 1.1, we can allow more general preferences as long as these preferences are consistent with Lemmas 1 and 2.

we do not assume any particular functional form for the von-Neumann Morgenstern Utility function. Each agent $h \in H$ also has an endowment in both periods, denoted by $e^h = (e_0^h, e_1^h) \in \mathbb{R}^{S+1}$. In (A4) we assume that $e_1^h$ is spanned by assets for all $h \in H$.

In this basic model, only by trading assets can agent h choose a consumption plan differently from initial endowment. The budget constraint agent h faces can be written

$$B^h(q) = \{(x^h, \theta^h) \mid x_0^h - e_0^h + \sum_{j=1}^{J} q_j \theta_j^h \leq 0, \\ x_1^h - e_1^h - \sum_{j=1}^{J} \theta_j^h a_j \leq 0\}$$

where $\theta^h \in \mathbb{R}^J$ denotes a portfolio choice and $q \in \mathbb{R}^J$ denotes asset prices at time 0. Agent h takes asset prices q as given and solves for optimal consumption and portfolio choices

$$(x^h, \theta^h) \in \text{argmax } U^h(x^h) \text{ s.t. } (x^h, \theta^h) \in B^h(q). \tag{1}$$

Agent h's first-order optimization condition can be written

$$q_j = E\left(\frac{\delta^h Du^h(x_1^h)}{Du(x_0^h)} a_j\right) = E(MRS^h a_j), \tag{2}$$

for all $j \in J$, where D denotes a differentiation operator. Note that a marginal rate of substitution ($MRS^h$) in Equation (2) need not be in M, nor be the same among agents due to incomplete markets. Equation (2), however, implies that an agent h who evaluates asset prices using $MRS^h$ cannot disagree on the prices of assets.

In this article, we are interested in how security prices are determined at a perfectly competitive equilibrium. A collection $(q, (x^h, \theta^h)_{h \in H}) \in \mathbb{R}^{J+H(S+1+J)}$ is an equilibrium in this economy $((u^h, e^h)_{h \in H}, A)$ if, for all $h \in H$, $(x^h, \theta^h)$ solves (1), and if two market clearing conditions,

$$\sum_{h=1}^{H} x^h = \sum_{h=1}^{H} e^h \tag{3}$$

and

$$\sum_{h=1}^{H} \theta^h = 0, \tag{4}$$

are met. In our economy Ross(1978a) points out that asset prices should not allow for any arbitrage opportunity as a precondition of equilibrium; if there were any arbitrage opportunity, the maximization problem (1) would not be well-defined and markets could not clear. This no arbitrage condition alone, as he went on to say, implies a linear pricing rule, and asset prices are determind by a continuous liner functional that has an inner product representation with some unique benchmark portfolio;[2] in other words, for any asset j in M, its price at time zero is determined by

$$q_j = E(ca_j), \tag{5}$$

where c denotes a benchmark portfolio in M. Although this linear pricing rule is quite useful for pricing derivative assets in M relative to the given prices q of primitive assets, its usefulness is limited when we are interested in how the primitive assets themselves are priced in equilibrium. This limitation is reflected in Equation (5), where the benchmark portfolio can be found in M instead of any other subspace in M. Therefore, we want to utilize equilibrium conditions (Equations (2), (3), (4), and (5)) in order to specify a subspace in M where the benchmark portfolio can be found. Furthermore, we want to determine when the benchmark portfolio can extend to a function of aggregate endowment.

### 1.1 Some Preliminry Results on Diversification and Asset Pricing

Instead of taking into account all equilibrium conditions at once, we will first consider only an agent's optimization condition (Equation (2)) in order to discuss how risks are priced in equilibrium. We will relate Equation (2) to two important principles of decision-making under uncertainty to show that risks diversified away in equilibrium must have a zero price, whereas additional risks held by agents in equilibrium must have a (strictly) negative price.

We begin with a few definitions. Given two random variables x, $\varepsilon : S \rightarrow \mathbb{R}$,

2) Refer to Chamberlain and Rothschild(1983) for this representation. This representation directly follows from the Riesz representation theorem. For further explanation about the Riesz representation theorem, see Reed and Simon(1980) p.43.

and a probability measure $\pi$ on S, we say that $\varepsilon$ is mean-independent of x with respect to $\pi$, iif for any $c \in \mathbb{R}$, $E(\varepsilon|x=c)=0$, i.e., $\sum_{\{s \in S : x(s)=c\}} \varepsilon(s)\pi(s)=0$. Of course one immediate consequence of this definition is that $E(\varepsilon)=0$. Following the lead of Rothschild and Stiglitz(1970), we say that the random variable $y : S \to \mathbb{R}$ is a mean-preserving spread of x with respect to $\pi$ iff $y=x+\varepsilon$, where $\varepsilon$ is mean-independent of x. The concept of mean-preserving spread has an important consequence for decision-making under uncertainty that we now state formally in Lemma 1. This lemma first appeared in Rothschild and Stiglitz(1970), which we adapt to our model.

*Lemma 1(Risk averse agents shun mean-preserving spreads). Let $u: \mathbb{R}_+ \to \mathbb{R}$ be strictly concave, and let $y: S \to \mathbb{R}_+$ be a mean-preserving spread of $x: S \to \mathbb{R}_+$ with respect to the probability measure $\pi$ on S. Then $Eu(x)=\sum_{s \in S} \pi(S)u(x_s) \geq \sum_{s \in S} \pi(S)u(y_s) \equiv Eu(y)$. Moreover, if $\pi$ is strictly positive and if $y \neq x$, the inequality is strict.*

*Proof*. Let $C \subset \mathbb{R}_+$ be the support of the random variable x, that is, all c such that there is $s \in S$ with $\pi_s > 0$ and $x_s=c$. For each $c \in C$, Let $S_c$ denote the set of s for which $x_s=c$. Then S is the disjoint union of sets of the form $S_c$, and a set $S_0$ with $\pi(S_0)=0$. Consider now that for each $c \in C$, $E(y|S_c)=c$. Hence from the definition of concavity(sometimes called Jensen's inequality) $E_\pi(u(y)|S_c) \leq u(c)$, where the inequality is strict if y is not constant on $S_c$. Hence $\sum_{s \in S} u(y_s)\pi_s=\sum_{c \in C}\sum_{s \in S_c} u(y_s)\pi_s \leq \sum_{c \in C} u(c)\pi(S_c)=\sum_{s \in S} u(x_s)\pi_s$, and the inequality is strict if y is not identically equal to x and if $x >> 0$. □

There is one more property of decision making under uncertainty that will be of importance when interpreting Equation (2). Samuelson(1967) first described this principle in terms of a story. Suppose that the odds of a ship coming in safely are favorable, but that both the risks and the gains are so large that a risk averse agent would not want to take on the gamble that ship would make it by buying the ship. What role can insurance play to induce the investment? A plausible, but wrong answer, is that if there are many ships, with independent risks, then by the law of large numbers, it is almost certain that more ships will come in than not, so it would appear that a risk averse agent would invest in the whole fleet. The trouble with this argument, as Samuelson pointed out, is that in the very unlikely event that all the ships fail to come in, the loss is astronomical. In short, if buying one ship is a bad idea, how could repeating a mistake many times be a good idea? The key idea in insurance, he went on to say, is not the multiplication of risks, but their division into smaller risks. If instead of buying many ships, an agent could buy a small piece of one ship (or even a small piece of all the ships, if their risks were independent) then despite his risk aversion, it would be beneficial for him. We adapt this in Lemma 2 to our model.

***Lemma 2 (Every agent will take a small enough piece of a favorable gamble).*** *Let $u: IR_+ \to IR$ be differentiable and weakly monotonic. Let $x, z: S \to IR_+$ be random variables on S, and let $\pi$ be a probability on S. Let $C \subset IR_+$ be the support of x. Suppose that $x >> 0$. Suppose that for all $c \in C$, $E(z \mid S_c) >> 0$. Then there is a strictly positive $\lambda$ such that $Eu(x+\lambda z) > Eu(x)$.*

*Proof*. Let us evaluate the derivative of Eu(x) in the direction z. Note that $D_Z Eu(x) \equiv \sum_{s \in S} \pi(s) Du(x_s) \bullet z_s = \sum_{c \in C} Du(c) \bullet \sum_{s \in S} z_s \pi(s) = \sum_{c \in C} Du(c) \bullet E(z \mid S_c) > 0$. From the definition of differentiability, the result follows. (Note that since $x >> 0$, the vector $x + \lambda z \geq 0$ for small enough $\lambda$).[3] □

In Lemma 2 we have used the fact that conditional on every c in the support of x, the expectation of z is positive. If we had assumed only that $E(z) >> 0$, the lemma would not generally hold. An important corollary to Lemma 2, which is of relevance to our analysis, is the following lemma on the pricing of diversified risks in equilibrium.

***Lemma 3 (Risks Diversified away in equilibrium must have zero value).*** *Let $(q, (x^h, \theta^h)_{h \in H})$ be an equilibrium in an economy $((u^h, e^h)_{h \in H}, A)$. Suppose that for some agent h, $x^h >> 0$. Suppose that some asset j is mean-independent of $x_1^h$. Then $q_j = 0$.*

*Proof*. The argument is exactly the same as for Lemma 2. If $q_j < 0$, then for some strictly positive $\lambda$ agent h could improve his utility by buying $\lambda$ more of asset j. Up to "first order" he would not change his utility in period 1 at all, since $\sum_{S=1}^{S} Du(x_{1S}) \bullet A_S^j = 0$, and he would strictly gain in period 0. Similarly, if

3) Note that if $u: IR \to IR$ and $x, z: S \to IR$, then we do not need $x >> 0$ for our conclusion. This point matters when we apply our results to the CAPM in Section 2.

$q_j > 0$, then agent h could strictly improve his utility by selling some small fraction $\lambda$ of asset j, contradicting the optimality of $(x^h, \theta^h)$. □

Lemma 3 basically says that diversified risks in equilibrium must have a zero price because, otherwise, taking on small amount of diversified risks would be a fair bet. Agents, however, cannont diversify away all risks and may hold some extra risks in equilibrium. For example, when there is a riskless asset in the span of assets, agent h's future consumption can be written

$$x_1^h = E(x_1^h)\tilde{1} + \varepsilon^h \qquad (6)$$

for all $h \in H$, where $\varepsilon^h$ becomes mean-independent of $E(x_1^h)\tilde{1}$ by construction. In this case agents hold $\varepsilon^h$ in equilibrium unless aggregate endowment is riskless. Another corollary of Lemma 2, which says that $\varepsilon^h$ must have a negative price for all $h \in H$, is the following lemma.

*Lemma 4 (Extra risks held by agents in equilibrium must have negative value). Let $(q,(x^h, \theta^h)_{h \in H})$ be an equilibrium in an economy $((u^h, e^h)_{h \in H}, A)$. Suppose that for some agent h, $x^h >> 0$. Suppose also that there are some assets z and $\varepsilon$ in M so that $z >> 0$, $\varepsilon$ is mean-independent of z, and $x_1^h = z + \varepsilon$. Then $q_\varepsilon < 0$.*

*Proof*. The argument is exactly the same as for Lemma 2. If $q_\varepsilon \geq 0$, then agent h could improve his utility by selling asset $\varepsilon$, contradicting the optimality of $(x^h, \theta^h)$. □

We see from the last four lemmas that although mean-independent additions to consumption are bad, they are bad only in a second order sense. This was to be expected, since if $\varepsilon$ is mean-independent of x, then so is $-\varepsilon$. If $\varepsilon$ were first order bad, then $-\varepsilon$ would necessarily be first order good, contradicting the fact that agents shun mean preserving spreads.

### 1.2 Diversification and Factor Pricing in Incomplete Markets

In this section we will generalize a notion of diversification that was used in the CAPM to describe a set of assets that must have a zero price. We will explain how we can always decompose an asset into two different risks: a non-diversifiable risk and a diversifiable risk. Economic intuition says that diversifiable risk must have zero value because agents diversify it away. To justify this intuition, we will carefully define diversifiable risks so that equilibrium conditions (Equations (2), (3), (4), and (5)) force all diversifiable risks to have zero value. Using a pricing implication of non-diversifiable risks, we will also explain why aggregate endowment must earn a positive risk premium over a riskless asset. Our discussion will indicate that a sensible distinction between two risks can be made only at an economy level.

We begin with a definition of a factor space, a space spanned by non-diversifiable risks. A factor space is defined as a subspace of M which satisfies the following four conditions in Definition 1. The first three conditions are assumed in Connors(1984). Here, we add Condition(4) and define the factor space instead of assuming it. In Definition 1, $e=(e_0, e_1) \in \mathbb{R} \times M$ denotes aggregate endowment, a sum of $e^h$ across agents, and $\oplus$ denotes a direct sum of subspaces.

***Definition 1.*** *F is a factor space in M, if F satisfies*

*(1)* $M = F \oplus E$ *so that* $E(\varepsilon f) = 0$ *for any* $\varepsilon \in E$ *and any* $f \in F$,

*(2)* $e_1 \in F$,

*(3)* $E(\varepsilon | f) = 0$ *for any* $\varepsilon \in E$ *and any* $f \in F$, *and*

*(4) F is a minimal subspace which satisfies (1), (2) and (3).*

The assets contained in F are called factors, and those in E are called idiosyncratic risks. Condition (1) by itself is not restrictive at all because we can always decompose M into two orthogonal subspaces, F and E so that $M=F \oplus E$. Condition (2), which identifies aggregate endowment in the factor space, was first introduced by Connor(1984). Condition (3) imposes mean-independence which is stronger than the orthogonality imposed by Condition (1). Since M trivially satisfies the first three conditions in the definition, existence of the well-defined factor space is not problematic. F is usually smaller than M because of Condition (4): F is a minimal subspace in the sense that there does not exist a smaller subspace which satisfies the othere three conditions. If we cannot find F smaller than M, then we set $F=M$ and $E=\{0\}$.

A consequence of Definition 1 is that any asset can be decomposed into factor and an idiosyncratic risk: for example, each agent's period 1 endowment $e_1^h = f^h + \varepsilon^h$, where $f^h \in F$ and $\varepsilon^h \in E$. Since our decomposition depends on aggregate endowment by Condition (3), this decomposition can be made only at an economy level. The idea of this decomposition is that the income of an agent (or any asset payoff) is determined by two components. One is a global factor that affects the whole of the economy, like the weather, or the political situation, or the state of technology. The contingent contracts that can be written on these events are denoted by F. (This intuitive argument receives more formal treatment in Section 1.3). In addition, an agent's endowment (or any asset) depends on some idiosyncratic risks which one can think of as independent of the global factors.

In Definition 1 we carefully define the idiosyncratic risks as a set of assets agents want to and are able to diversify away. Due to Condition (3), agents want to diversify away idiosyncratic risks through portfolio choice. Due to Condition (2), all agents can eliminate any idiosyncratic risk collectively. The importance of Condition (2) in the definition can easily be seen when there is a riskless asset in M. Note that span$[\tilde{1}]$ satisfies all other conditions except for Condition (2). When aggregate endowment is not riskless, span$[\tilde{1}]$, however, cannot be a factor space because of Conditions (2).

If the idiosyncratic risks according to our definition represent diversifiable risks, the we can expect that agents *indeed* diversify away any idiosyncratic risk in equilibrium. Now, we show that this is indeed the case in equilibrium utilizing equilibrium conditions (Equations (2), (3) and (5)). The idea here is that, if some agents would hold idiosyncratic risks in equilibrium, the idiosyncratic risk held by each agent must have (strictly) negative value due to Lemma 4 (or Equation (2)), and the sum of the idiosyncratic risks across agents must also have negative value due to the linear pricing rule (Equation (5)). This would be impossible because the sum is a zero-vector due to Condition (2) and market clearing (Equation (3)), and the zero-vector must have zero value due to the linear pricing rule (Equation (5) again). Hence, in equilibrium all agents completely eliminate any idiosyncratic risk. In other words, agents' future consumptions become factors, and they are characterized by Ross's (1978b) k-fund separation, the cardinality k being determined by the factor space. The following lemma formally states this.

*Lemma 5 (Agents diversify away any idiosyncratic risk in equilibrium). Let* $(q,(x^h, \theta^h)_{h\in H})$ *be an equilibrium in an economy*$((u^h, e^h)_{h\in H},A)$. *Suppose that for all* $h\in H$, $x^h >> 0$, *Then* $x_1^h \in F$ *for all* $h\in H$.
*Proof*. See above. □

Now, let's consider a pricing implication of Lemma 5. Since agents diversify away any idiosyncratic risk in equilibrium, idiosyncratic risks must have zero value due to Lemma 3. Under a linear pricing rule (Equation (5), this can happen if and only if the benchmark portfolio becomes a factor because of Condition (1) in Definition 1. In other words, we can restrict the benchmark portfolio to the factor space instead of the whole span of assets. Thus although there may be only a few factors, causing the dimension of F to be quite low, the prices of just these factors are enough to determine the value of any portfolio in the span of assets. So this result is *a priori* apparently quite surprising, and quite useful to summarize investment opportunities available in the markets. This result also indicates that we have generalized the notion of diversification used in the CAPM by properly identifying the idiosyncratic risks. The following theorem formally states this factor pricing result.

*Theorem 1 (All idiosyncratic risks must have zero value in equilibrium). Let et* $(q,(x^h, \theta^h)_{h\in H})$ *be an equilibrium in an economy*$((u^h, e^h)_{h\in H},A)$. *Suppose that* $x^h >> 0$ *for all* $h\in H$. *Then* $q_j=E(ca_j)$ *and* $c\in F$ *for all* $a_j\in M$.
*Proof*. See above. □

We can also discuss the meaning of Theorem 1 in terms of the mean-variance efficient set. As Chamberlain and Rothschild(1983) show, the mean-variance efficient set is a two-dimensional subspace spanned by c and m where m is a unique portfolio that represents an expectation operator in M: E(a)=E(ma) for all a∈M.[4] It is immediate to show m∈F, because Condition (3) implies that $E(\varepsilon)=0$ for any $\varepsilon\in E$. Hence, Theorem 1 implies that the mean-variance efficient set is alwasy included in the factor space. Note

4) See Chamberlain and Rothschild(1983) for more explanation on m. Note that m=P1, where P: X →M is a projection.

that the factor space is at least as big as span[$e_1$, m], because aggregate endowment is also included in F. As a result, when F=span[$e_1$, m], Theorem 1 guarantees the mean-variance efficiency of aggregate endowment. When F is greater than span[$e_1$, m], aggregate endowment, however, need not be mean-variance efficient except for some degenerate cases.[5)]

Now, let's analyze a pricing implication of equilibrium conditions on non-diversifiable risks to show that aggregate endowment must earn a positive risk premium over a riskless asset. Even though agents do not take on any idiosyncratic risk in equilibrium, they may still take on some (non-diversifiable) extra risks, as we have already discussed in Equation (6) in Section 1.1. For our discussion, we assume a riskless asset in the span of assets and decompose aggregate endowment in the same way as we did in Equation (6) :

$$e_1=E(e_1)\tilde{1}+\varepsilon_e \qquad (7)$$

where $\varepsilon_e$ is a sum of $\varepsilon^h$ across all agents due to market clearing. If aggregate endowment is risky, $\varepsilon^h$ becomes a factor according to our definition because of Lemma 5. (Also remember that span[$\tilde{1}$] cannot be a factor space). Employing the same argument as for Lemma 5, we now show that all agents indeed hold some non-diversifiable extra risk in equilibrium, or $\varepsilon^h \neq 0$ for all $h \in H$: if some agent would choose a riskless consumption in equilibrium, all mean-zero assets would have zero value due to Lemma 3, and consequently every agent would choose a riskless position. Obviously, this is not consistent with market clearing. The extra risk $\varepsilon^h$ held by each agent, according to Lemma 4, has negative value for all $h \in H$. The linear pricing rule then implies that a risky component of aggregate endowment, or $\varepsilon_e$ must also have negative value. As we can see from the following theorem, this result indicates that aggregate endowment must earn a positive risk premium over a riskless rate.

***Theorem 2 (Aggregate endowment earns a positive risk premium over a riskless rate).*** *Let* $(q,(x^h, \theta^h)_{h\in H})$ *be an equilibrium in an economy* $((u^h, e^h)_{h\in H},A)$. *Suppose*

5) The CAPM with a quadratic preference assumption is an example of these degenerate cases. Due to the strong restriction on preferences, aggregate endowment becomes mean-variance efficient, even though the factor space is larger than span[m, $e_1$]. Refer to Oh(1996) for more details on this conclusion.

*$\tilde{1} \in span[A]$. Suppose also that $x^h >> 0$ for all $h \in H$. Let r denote a riskless rate. Then, for some $\rho > 0$,*

$$q_e = E(ce_1) = \frac{E(e_1)}{1+r+\rho}. \tag{8}$$

*Proof.* From the above discussion we get $E(ce_e) < 0$. Now, we show

$$q_e = E(ce_1) = E(e_1)E(c\tilde{1}) + E(ce_e) < E(e_1)E(c\tilde{1}). \tag{9}$$

Let $E(c\tilde{1}) = 1/(1+r)$. It is immediate to see that we must have $\rho > 0$ to satisfy the equality in Equation (8). □

Theorem 2 provides another equilibrium restriction on the benchmark portfolio. The restriction is that the benchmark portfolio has a zero covariance with aggregate endowment because $Cov(c, e_1) = E(ce_e) < 0$. Even though Theorem 2 relies on the existence of a riskless asset in M, our previous reasoning suggests that we can also get a similar result in the special case when there is no riskless asset in M. If m, defined before as a portfolio that represents an expectation operator in M, is mean-independent of any mean-zero asset in M, then in the previous argument we cna replace a riskless asset with m to get the following corollary of Theorem 2.

***Corollary 1 (Aggregate endowment earns a positive risk premium over a return on m).*** *Let $(q, (x^h, \theta^h)_{h \in H})$ be an equilibrium in an economy$((u^h, e^h)_{h \in H}, A)$. Let $span[A] = span[m] \oplus B$ so that $E(\varepsilon) = 0$ for any $\varepsilon \in B$. Suppose that $E(\varepsilon | m) = 0$ for any $\varepsilon \in B$. Suppose also that $x^h >> 0$ for all $h \in H$. Let r denote a return on m. Then, for some $\rho > 0$,*

$$q_e = E(ce_1) = \frac{E(e_1)}{1+r+\rho}. \tag{10}$$

*Proof.* If we replace $\tilde{1}$ with m, the argument is the exactly same as for Theorem 2. □

### 1.3 Role or Spanning Restriction and Aggregate Endowment in Factor Pricing

In the previous two sections we see that risks can be discussed at an individual level, whereas factors and idiosyncratic risks are determined at an economy level. In this section we will further analyze how both uncertainty in aggregate endowment and spanning restrictions determine the factors and the idiosyncratic risks. This analysis will provide a sufficient condition for the benchmark portfolio to extend to a function of aggregate endowment in our basic model.

We start with a definition of informational span, which can be interpreted as a collection of all contingent claims written on some random variable. Given a single dimensional random variable $x: S \rightarrow \mathbb{R}$, let us define the informational span of x, infospan[x] $\equiv \{y \in \mathbb{R}^S : y_s \neq y_{s'}$, implies that $x_s \neq x_{s'}\}$. This definition basically says that infospan[x] is a collection of all random variable that can be written as a (measurable) function of the random variable x. We interpret a function of x as a contingent claim written on x. Note that this is clearly a linear subspace of $\mathbb{R}^S$ containing x itself. If C is the collection of values in $\mathbb{R}$ that x takes on, then the dimension of infospan[x] is the cardinality of C. Indeed, for each $c \in C$, let $1_c$ be the indicator function of the set $S_c \equiv \{s \in S : X_S = c\}$. Then the vectors $1_c$, $c \in C$, are orthogonal and they span infospan[x]. More generally, if B is any collection of vectors in $\mathbb{R}^S$, then infospan[B] denotes all vectors $y \in \mathbb{R}^S$ such that $y_S \neq y_{S'}$, implies there is some $b \in B$ with $b_S \neq b_{S'}$,. Once again it is clear that infospan[B] is a subspace of $\mathbb{R}^S$ containing span[B].

The concept of informational span is useful to understand the following difference between orthogonality and mean-independence, which markes Condition (3) a stronger requirement than Condition (1) in Definition 1. It is immediate that for single dimensional random variables x, $\varepsilon : S \rightarrow \mathbb{R}$, $\varepsilon$ is mean-independent of x if and only if $E(y\varepsilon)=0$ for all $y \in$ infospan[x]. More generally, for any collection of random variables $B \subset \mathbb{R}^S$, $\varepsilon$ is mean-independent of B if and only if $E(y\varepsilon)=0$ for all $y \in$ infospan[B]. This implies that x being mean-independent of $\varepsilon$ is stronger than the requirement that $E(\varepsilon)=0$, and that $Cov(x, \varepsilon)=0$. To see this, note that a riskless asset is always in the infospan[x]. Consequently, if $E(\varepsilon | x)=0$, then $E(\varepsilon)=0$ and $E(x\varepsilon)=0$, or $Cov(\varepsilon, x)=0$.

Now, we can analyze how factors are related to infospan[$e_1$], a collection of contingent claims written on aggregate endowment, It is immediate to see that all marketed contingent claims on aggregate endowment, denoted by infospan $[e_1]\cap M$, becomes a factor because $E(\varepsilon|e_1)=0$ for any $\varepsilon\in E$. When markets are incomplete, however, the factor space is usually larger than infospan$[e_1]\cap M$. To see this, let's consider a collection of portfolios which are the closest substitutes for some contingent claims on aggregate endowment. We define this collection as a image of infospan$[e_1]$ under projection mapping, or P(infospan$[e_1]$)=$\{z\in M: z=Px$ for all $x\in$infospan$[e_1]$) where $P: \mathbb{R}^S\rightarrow M$ be an orthogonal projection with respect to $\pi$. In this definition the idea of "the closest substitute" is captured by an orthogonal projection. We say that a portfolio z extends to a function of aggregate endowment if $z\in P$ (infospan$[e_1]$), or in other words, a portfolio is a closest substitute for some contingent claim on aggregate endowment. Obviously, infospan$[e_1]\cap M\subseteq P$ (infospan$[e_1]$). P(infospan$[e_1]$), however, can be strctly larger than infospan$[e_1]\cap M$ when there is a portfolio that is not by itself a contingent claim on aggregate endowment but is the closest substitute for some contingent claim. This portfolio is a kind of bundled asset whose payoff has two components: one that is a contingent claim on aggregate endowment and another that is mean-independent of aggregate endowment. When the spanning restriction prevents agents from unbundling these two payoffs in incomplete markets, this bundled portfolio can be the closest substitute for its first component or other contingent claims on aggregate endowment. Now, we prove in the following lemma that this bundled portfolio also becomes a factor.

***Lemma 6 (Any portfolio that extends to a function of aggregate endowment is a factor).*** *Infospan*$[e_1]\cap M\subseteq P$(*infospan*$[e_1]$)$\subseteq F$.
*Proof.* Let $z\in P$(infospan$[e_1]$), or $z=x-y$, where $x\in$infospan$[e_1]$, and y is orthogonal to M. $E(z\varepsilon)=E(x\varepsilon)-E(x\varepsilon)=0$ for any $\varepsilon\in E$ because $E(x\varepsilon)=E(y\varepsilon)=0$. Hence, $z\in F$ and P(infospan$[e_1]$)$\subseteq F$. □

One interesting question raised from Lemma 6 is whether any factor extends to a function of aggregate endowment. This, however, is true only for special cases in incomplete markets. For example, when infospan$[e_1]\cap M=P$(infospan$[e_1]$), or equivalently when there is no bundled portfolio, we can easily check that infospan$[e_1]\cap M$ satisfies all four condition for being a factor space. Hence, we get infospan$[e_1]\cap M=P$(infospan$[e_1]$)$=F$, and any factor is a contingent claim on aggregate endowment. Notabley, in this case any asset mean-independent of aggregate endowment becomes an idiosyncratic risk.

When there is some bundled portfolio and, as a result, $P(\text{infospan}[e_1]) \supset \text{infospan}[e_1] \cap M$, some agent who might choose this bundled factor for future consumptions may want to take on an additional portfolio that is mean-independent of aggregate endowment. To put it differently, some portfolios that are mean-independent of aggregate endowment must be included in the factor space to meet Condition (3) in Definition 1 for those portfolios. Applying the same concept of infospan and projection to this bundled portfolio successively as we did before to aggregate endowment, we can find all portfolios that belong to the factor space. In the process, some portfolios that cannot extend to a function of aggregate endowment are also included in the factor space. To rule out this possibility, we need a condition that any asset mean-independent of aggregate endowment is an idiosyncratic risk. If this condition is met, then the agent whose consumption is a bundled portfolio cannot find any other portfolio that improves his itility, if the other portfolio is mean-independent of aggregate endowment. In this case any factor can extend to a function of aggregate endowment. This is formally proven in Lemma 7.

***Lemma 7 (Any factor extends to a function of aggregate endowment if any asset mean-independent of aggregate endowment is an idiosyncratic risk).*** *Let* $M = P(\text{infospan}[e_1]) \oplus B$, *where* $B$ *is an orthogonal complement of* $P(\text{infospan}[e_1])$ *in* $M$. *If* $B \subseteq E$, *then* $F = P(\text{infospan}[e_1])$.

*Proof.* If $B \subseteq E$, then $P(\text{infospan}[e_1])$ satisfies all conditions in Definition 1 except for Condition (4). Consequently, $P(\text{infospan}[e_1]) \supseteq F$. Due to Lemma 6 $P(\text{infospan}[e_1]) \subseteq F$. Hence $P(\text{infospan}[e_1]) = F$. □

The sufficient condition in Lemma 7 has an important implication on asset pricing in incomplete markets. Note that Theorem 1 alone does not imply that the benchmark portfolio can alwasy extend to a function of aggregate endowment. If this sufficient condition is met, however, the benchmark portfolio does extend to a function of aggregate endowment. As a result, there exists a linear pricing functional which is represented by some contingent claims on aggregate endowment. We state this formally in Theorem 3.

***Theorem 3 (The benchmark portfolio extends to a function of aggregate endowment if any asset mean-independent of aggregate endowment becomes an idiosyncratic risk).*** *Let $(q,(x^h, \theta^h)_{h\in H})$ be an equilibrium in an economy$((u^h, e^h)_{h\in H}, A)$. where $M=span[A]=P(infospan[e_1]) \oplus B$ and $E(\varepsilon|f)=0$ for any $\varepsilon\in B$ and $f\in P(infospan[e_1])$. Suppose that $x^h >> 0$ for all $h\in H$. Then there exists a function, $U$, such that $q_j=E(U(e_1)a_j)$ for any $a_j\in M$.*

*Proof.* From Theorem 1 and Lemma 7 there exists some function U such that $U(e_1)=c+y$, $q_j=E(ca_j)$ and $E(ya_j)=0$ for all $a_j\in M$. Therefore, $E(U(e_1)a_j)=E(ca_j)+E(ya_j)=E(ca_j)=q_j$. □

The conclusion of Theorem 3, however, is weaker than that of a representative agent model in several aspects. First, in incomplete markets, the function in Theorem 3 may not be unique because a projection operator is not a one-to-one mapping. In other words, there can be many contingent claims on aggregate endowment that can explain asset prices equally well. Second, Theorem 3 does not require the benchmark portfolio itself to be a contingent claim on aggregate endowment. Third, Theorem 3 does not require a functional form of U to be either differentiable or monotonic. In contrast, in a representative agent model, the benchmark portfolio is unique function of aggregate endowment that can explain asset prices and the functional form is both differentiable and monotonic. The monotonic functional form implies that aggregate endowment earns a positive risk premium in complete markets. Our Theorem 2, however, has already proven the same result in incomplete markets without requiring monotonicity of U.

## 2. Application of Factor Pricing

In this section we present three examples which illustrate how our factor pricing results can apply to existing asset pricing models such as a representative agent model and the CAPM.

**Example 1 (Complete Markets).**

We first consider a complete market structure, where J=S. In this case, infospan$[e_1] \cap M = P(\text{infospan}[e_1]) = $infospan$[e_1]$. Consequently, infospan$[e_1]$ $=F$, and any asset mean-independent of aggregate endowment becomes an idiosyncratic risk. Theorems 2 and 3 conclude in this case that the benchmark portfolio is a function of aggregate endowment and aggregate endowment has a positive risk premium. However, as we discussed before, these conclusions are weaker than those of a representative agent model.

**Example 2 (Informationally Complete Markets).**

Next we consider an informationally complete market structure, where the equilibrium restrictions are exactly the same as those of the complete markets. We say that the market structure M is informationally complete if infospan$[e_1] \subseteq M$ and $e_1^h \in M$ for all $h \in H$. We can interpret the informationally complete market to mean that, despite a huge number of missing markets, market structures are rich enough to span both each agent's future endowment and every possible contract contingent on aggregate endowment. Note that complete markets are informationally complete, so informationally complete markets are a more general structure than complete markets. In informationally complete markets, infospan$[e_1] \cap M = P(\text{infospan}[e_1]) = $infospan$[e_1]$, and, as a result, infospan$[e_1] = F$. Furthermore, opening up missing markets in informationally complete markets has no real impact because it does not change the factor space. This indicates that Pareto optimality will be achieved in informationally complete markets.

**Example 3 (The CAPM with a Distributional Assumption).**

Here, we will derive the CAPM conclusions based on our results without utilizing the mean-variance efficient behavior.[6] We will show that the distributional assumption in the CAPM imposes two properties on the factor space. One is that the factor space is spanned by two assets, aggregate endowment and a riskless asset(or its equivalent, m, if it does not exist). The other is that any asset mean-independent of aggregate endowment becomes an idiosyncratic risk. This analysis will show that our factor pricing results will indeed replace the CAPM conclusions when we drop the distributional

6) Our approach is an extension of Connor(1984) and Milne(1988), who also analyze the factor space implication of the CAPM distributional assumption.

assumption from the CAPM.

The CAPM, as developed by Sharpe(1964), Lintner(1965), Mossin(1966) and Black(1972), assumes normally distributed assets. Later on, Chamberlain(1983) and Owen and Rabinovitch(1983) extend the CAPM to the class of elliptical distributions,[7] Since a normal distribution belongs to the class of elliptical distributions, we will focus on the CAPM with elliptical distributions. In our analysis we will utilize only two properties of elliptical distributions. One is that any linear transformation of elliptical distributions also has an elliptical distribution. The other is that two random variables with elliptical distributions are mean-independent of each other if they are uncorrelated.

Before we begin, we need to extend our basic model to deal with the infinite number of states the CAPM assumes. The extension, however, is almost trivial, except for some technicalities, if we replace the choice set, $\mathbb{R}^{S+1}$, with $\mathbb{R}\times X$, where X is a separable $L^2$-space, a set of all square integrable random variables in a given probability space with a mean-square inner product $E(x_1y_1)$ for any element $x_1$, $y_1 \in X$.[8] Since any discussion in Section 1 does not depend on the special properties of $\mathbb{R}^S$ that do not hold in the Hilbert space X, all the definitions, theorems and lemmas can be restated with proper modification of language without substance changes. Another assumption change we need to make is that each agent's von-Neuman Morgenstern utility function is well-defined for negative values of its argument. This is necessary because some elliptically distributed random variables take on negative values. This change allows us to drop the assumption we make in Lemmas 2, 3, 4, 5, and Theorems 1, 2, and 3, which was necessary to consider an interior solution, without affecting the conclusion.

We begin with the CAPM with a riskless asset. When all but riskless assets are elliptically distributed and all agents' endowments are spanned by

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7) Refer to Kelker(1970) and Owen and Rabinovitch(1983) for details of elliptical distributions. In this section we will consider only the class of elliptical distributions with second moments. Owen and Rabinovich(1983) discuss how the CAPM can be derived with elliptical distributions without second moments. Readers familiar with the CAPM with the normal distributions can follow the subsequent discussion by replacing elliptical distributions with normal distributions.

8) Refer to Chamberlain and Rothschild(1983) for details of how asset pricing can be discussed in the Hilbert space. Ross's (1978) treatment of no arbitrage condition has been generalized to an infinite dimensional space by Harrison and Kreps(1979), Kreps(1981), and Duffie and Huang(1986).

assets, we can show that infospan$[e_1] \cap M = P(\text{infospan}[e_1]) = \text{span}[e_1, \tilde{1}]$ due to the special properties of elliptical distributions, and consequently infospan$[e_1] \cap M = P(\text{infospan}[e_1]) = F$. Since $P(\text{infospan}[e_1]) = F$, any asset mean-independent of aggregate endowment becomes an idiosyncratic risk. We analyze this formally in Theorem 4.

**Theorem 4 (The CAPM with a riskless asset).** *If any risky portfolio in M has an elliptical distribution and* $\tilde{1} \in M$, *then* $\text{infospan}[e_1] \cap M = P(\text{infospan}[e_1]) = F = \text{span}[\tilde{1}, e_1]$.

*Proof*. Let F be a span$[\tilde{1}, e_1]$, and E be its orthogonal complement in M so that $M = F \oplus E$. Note that $\text{Cov}(f, \varepsilon) = E(f\varepsilon) - E(f)E(\varepsilon) = 0$ for all $f \in F$ and all $\varepsilon \in E$ because $E(f\varepsilon) = 0$ and $E(\varepsilon) = 0$. Since both f and $\varepsilon$ are elliptically distributed, $E(\varepsilon | f) = 0$ (See Kelker(1970), Theorem 6). Hence, $\varepsilon$ is mean-independent of f. It is obvious that F is a minimal subspace. Therefore, F is a factor space. Note that infospan$[e_1] \cap M = \text{span}[\tilde{1}, e_1]$. Hence, infospan$[e_1] \cap M = P(\text{infospan}[e_1]) = F = \text{span}[\tilde{1}, e_1]$. □

When there is no riskless asset in M, as is the case in the Black's(1972) zero-beta CAPM, we can replace $\tilde{1}$ with m. Note that m is not usually an element of infospan$[e_1] \cap M$. Therefore, $P(\text{infospan}[e_1])$ is larger than infospan$[e_1] \cap M$. The elliptical distribution assumption, however, makes m mean-independent of any mean-zero asset and consequently, any asset mean-independent of aggregate endowment becomes an idiosyncratic risk. So we have $P(\text{infospan}[e_1]) = F = \text{span}[m, e_1]$. This is formally analyzed in Theorem 5.

***Theorem 5 (The CAPM Without a Riskless Asset).*** *If any asset in M has an elliptical distribution and endowments are spanned by assets, then* $\text{infospan}[e_1] \cap M \subseteq P(\text{infospan}[e_1]) = F = \text{span}[m, e_1]$.

*Proof*. Note that m is mean-independent of any mean-zero asset because $\text{Cov}(m, a) = E(ma) - E(m)E(a) = 0$ if $E(a) = 0$ and any portfolio in M is elliptically distributed. Now, let F be a span$[m, e_1]$. Following the same argument as for Theorem 4, we can show that F is the factor space. Note that infospan$[e_1] \cap M = \text{span}[e_1]$ and $\text{span}[m, e_1] \subseteq P(\text{infospan}[e_1])$. Therefore, $P(\text{infospan}[e_1]) = F = \text{span}[m, e_1]$. □

Once we have Theorem 4 and 5, it is immediate to derive the CAPM conclusions from our Lemma 5 and Theorem 1, which respectively imply a mutual fund separation and the mean-variance efficiency of aggregate endowment. When there is a riskless asset, Theorems 2 and 3 also respectively imply that the benchmark portfolio become a function of aggregate endowment and that aggregate endowment earns a positive risk premium over a riskless asset. When there is no riskless asset, Corollary 1 and theorem 3 respectively imply that the benchmark portfolio extends to a function of aggregate endowment and that aggregate endowment earns a positive risk premium over a return on m. To see how the benchmark portfolio extends, let $c=km+le_1$ for two constants k and l. Since $m=P\tilde{1}$, we can easily extend c to $k\tilde{1}+le_1$, which is a function of aggregate endowment.

## 3. Diversification with Non-Traded Endowments

In this section we briefly discuss how to relax our endowment spanning assumption (A4) in Section 1. When we allow non-traded endowments into our analysis, we need to extend the concept of the factor space. For this, let's decompose $\mathbb{R}^S$ into two orthogonal subspaces, M and N such that $\mathbb{R}^S=M \oplus N$, where N represents a subspace spanned by non-marketed assets. Using an orthogonal projection, we also decompose agent h's future endowment into two components, a marketed component and a non-marketed component;

$$e_1^h=e_{1m}^h+e_{1n}^h, \text{ where } e_{1m}^h \in M, \text{ and } e_{1n}^h \in N, \tag{11}$$

for $h \varepsilon H=\{1,\ldots,H\}$. Now, we define a non-marketed factor space, $F_2$, as a span of non-marketed components. That is,

$$F_2=\text{span}[e_{1n}^h, \text{ for } h \varepsilon H=\{1,\ldots,H\}]. \tag{12}$$

Using $F_2$ we generalize a definition of the factor space.

**Definiltion 2.** *$F_1$ is a factor space in M if $F_1$ satisfies*

*(1) $M=F_1 \oplus E$ so that $E(\varepsilon f)=0$ for any $\varepsilon \in E$ and any $f \in F_1$,*

*(2) $Pe_1 \in F_1$, where $P: X \rightarrow M$ is an orthogonal projection,*

(3) *$E(\varepsilon|f)=0$ for any $\varepsilon \in E$ and $f \in F_1 \oplus F_2$*
(4) *$F_1$ is a minimal subspace which satisfies (1), (2) and (3).*

Note that the factor space in Definition 2 will be the same as that in Definition 1 without non-marketed components. When agents have non-marketed components, the factor space, however, usually expands because we require the idiosyncratic risks to be mean-independent of not only marketed factors but also all non-marketed components. Note that agents will still diversify away idiosyncratic risks despite the existence of non-marketed components. As a result, Theorem 1 holds for newly defined idiosyncratic risks, but the equilibrium restriction in Theorem 1 becomes weaker because the cardinality of idiosyncratic risks becomes smaller.

The CAPM with distributional assumptions, however, is an exceptional case in the sense that the existence of non-traded endowments in the CAPM does not affect the cardinality of the factor space. When we assume elliptically distributed non-traded endowments in the CAPM, the factor space is still spanned by $Pe_1$ and m due to the special properties of elliptical distributions. Furthermore, any asset mean-independent of aggregate endowment is still an idiosyncratic risk and, as a result, any element in the factor space extends to a function of aggregate endowment. We analyze this formally in Theorem 6.

***Theorem 6 (The CAPM with Non-Traded Endowments).*** *If any risky portfolio in M and each agent's future endowment have elliptical distributions, then $P(infospan[e_1]) = F_1 = span[m, Pe_1]$.*

*Proof.* Let $F_1$ be a span[m, $Pe_1$]. Notice that $Pe_1$ has an elliptical distribution. Let $M=F_1 \oplus E$. Note that $Cov(f_1, \varepsilon)-E(f_1\, \varepsilon)-E(f_1)E(\varepsilon)=0$, for any $f_1 \in F_1$ and $\varepsilon \in E$. Since both $f_1$ and $\varepsilon$ are elliptically distributed, $E(\varepsilon|f_1)=0$ for any $f_1 \in F_1$ and $\varepsilon \in E$. Let $F_2$ be defined as in Equation (12). Any element in $F_2$ is also elliptically distributed, and $F_2$ is orthogonal to E. Applying the same argument as for Theorem 4 results in $E(\varepsilon|f_2)=0$ for any $f_2 \in F_2$ and $\varepsilon \in E$. Obviously, $F_1$ is a minimal space. Therefore, $F_1$ is a factor space. Since $span[m\ Pe_1] \subseteq P(infospan[e_1])$, $P(infospan[e_1])=F_1=span[m, Pe_1]$. □

When Theorem 1 is employed in this case, $Pe_1$ becomes mean-variance efficient. Furthermore, since Theorem 3 still applies to this case, the benchmark portfolio can extend to a function of aggregate endowment: in other words, there exist constants k and 1, such that

$$q_j=E(km+1Pe_j)a_j)=E(k\tilde{1}+1e_j)a_j),$$

for all $a_j \in M$.[9] Theorem 2 also implies that $Pe_1$ earns a positive risk premium over a returen on m.

## 4. Conclusions

In this article we anlyze how and why assets are classified into two different classes, diversifiable risks and non-diversifiable risks, in equilibrium. We define a non-diversifiable risk as an element in a factor space which is the minimal space of the subspaces which include aggregate endowment and also imply that an asset which is uncorrelated to any element of the subspace becomes also mean-independent. Diversifiable risks are, in turn, defined as a collection of assets which are mean-independent of any non-diversifiable risk. We prove that in equilibrium all agents indeed diversify away any diversifiable risk and all diversifable risks must have zero value. Under a linear pricing rule this implies that the benchmark portfolio which represents a pricing operator should lie in the factor space. We also prove that aggregate endowment should earn a positive risk premium over a riskless asset because agents should hold additional risks, and additional risks must have negative value in equilibrium. Finally we prove that, even in incomplete markets, there is a pricing operator that is represented by some contingent claim written on aggregate endowment as long as any asset mean-independent of aggregate endowment is a diversifiable risk. All our results directly apply to both the CAPM and a representative agent model.

Before we conclude, we want to mention some implications of our results on empirical tests. Our results suggest that empirical testing in incomplete markets is a thorny job, even when a pricing operator is represented by a

9) Oh(1992) discusses the same asset pricing result in the CAPM.

contingent claim written on aggregate endowment. In this case, for example, we could try to find some polynomial function of aggregate endowment that explains the asset prices best. Alternatively, we may try to find the best combination of some contingent claims written on aggregate endowment using, for example, S&P 500 index futures and options as proxies for them. However, since our results do not guarantee uniqueness of such a function, other functions can exist which explain asset prices equally well. Therefore, identification can be a serious problem in incomplete markets. Furthermore, our results do not tell us how to find a sensible restriction for the functional form. The CAPM and a representative agent model are two notable exceptions to these problems. In the CAPM, the theory concludes that there is a linear functional form that is spanned by a riskless asset and aggregate endowment. In a representative agent model, the complete market structure provides both a uniqueness and a sensible restriction on the functional form. How to solve this identification problem in incomplete markets for other than these two cases is left as a future research topic.

## References

[1] Black, F., 1972, "Capital Market Equilibrium with Restricted Borrowing," *Journal of Business*, 45, 444-445.

[2] Chamberlain, G., 1983, "A Characterization of the Distributions that Imply Mean-Variance Utility Functions," *Journal of Economic Theory*, 29, 185-201.

[3] Chamberlain, G., 1988, "Asset Pricing in Multiperiod Securities Markets," *Econometrica*, 56, 1283-1300.

[4] Chamberlain, G. and M. Rothschild, 1983, "Arbitrage, Factor Structure, and Mean-Variance Analysis on Large Asset Markets," *Econometrica, 51, 1281-1304.*

[5] *Connor, G., 1984, "A Unified Beta Pricing Theory," Journal of Economic Theory*, 34, 13.-31.

[6] Constantinides, G. 1982, "Intertemporal Asset Pricing with Heterogeneous Consumers and without Demand Aggregation," *Journal of Business*, 55, 253-267.

[7] Duffie, D. and C. Huang, 1986, "Multiperiod Security Markets with Differential Information," *Journal of Mathematical Economics*, 15, 283-303.

[8] Harrison, J. and D. Kreps, 1979, "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory*, 20, 381-408.

[9] Kelker, D., 1970, "Distribution Theory of Spherical Distributions and a Local-Scale Parameter Generalization," *Sankhya*, A32, 419-430.

[10] Kreps, D., 1981, "Arbitrage and Equilibrium in Economics with Infinitely Many Commodities," *Journal of Mathematical Economics*, 8, 15-35.

[11] Lintner, J., 1965, "The Valuation of Risky Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review and Economics and Statistics*, 47, 13-37.

[12] Lucas, R., 1978, "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429-1445.

[13] Milne, F., 1988, "Arbitrage and Diversification in a General Equilibrium Asset Economy," *Econometrica*, 56, 815-840.

[14] Mossin, J., 1966, "Equilibrium in a Capital Asset Market," *Econometrica*, 34, 768-773.

[15] Oh, G., 1996, "Some Results in the CAPM with Non-Traded Endowments," Management Science, 42, 286-293.

[16] Owen, J. and R. Rabinovitch, 1983, "On the Class of Elliptical Distributions and Their Applications to the Theory of Portfolio Choice," *Journal of Finance*, 38, 745-752.

[17] Reed, M. and B. Simon, 1980. *Functional Analysis*, Academic Press, New York.

[18] Ross, S., 1976, "The Arbitrage Pricing Theory of Capital Asset Pricing," *Journal of Economic Theory*, 13, 341-360.

[19] Ross, S., 1978a "A Simple Approach to the Valuation of Risky Streams," *Journal of Business*, 51, 453-475.

[20] Ross, S., 1978b, "Mutual Fund Separation in Financial Theory-The Separating Distributions," *Journal of Economic Theory*, 17, 254-286.

[21] Rothschild, M. and J. Stiglitz, 1970, "Increasing Risk I: A Definition," *Journal of Economic Theory*, 2, 225-243.

[22] Samuelson, P., 1967, "General Proof that Diversification Pays," *Journal of Financial Quantitative Analysis*, 2, 1-13.

[23] Sharpe, W., 1964, "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," *Journal of Finance*, 19, 425-442.