Journal of Statistical
Theory & Methods
1996, Vol. 7, pp 21~35

Conditional Confidence Interval for Parameters in Accelerated Life Testing

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Abstract In this paper, estimation and prediction procedures are discussed for general situation in which the failure time follows the independent density $f_i(\varepsilon_i)$ for the accelerated life testing under Type II censoring. In the context of accelerated life test experiment, procedures are given for estimating the parameters in the Eyring model, and for estimating mean life at a given future stress level. The procedures given are conditional confidence interval procedures, obtained by conditioning on ancillary statistics. A comparison is made of these procedures and procedures based on asymptotic properties of the maximum likelihood estimates.

keywords: Accelerated Life Testing, Conditional Confidence Intervals, Ancillary Statistics, Exponential Regression, Eyring model.

1. Introduction

In studies concerning the length of life of certain types of manufactured items, it is often wished to consider the relationship between length of the life and one or more concomitant (or stress) variables. For a example, in an experiment to study the lifetimes of a certain type of electrical insulation, the relationship between length of life and temperature was studied (see; Escobar and Meeter (1993), and Nelson (1990)).

The estimated relationship between length of life and stress variables allows the prediction of item life under use conditions. This situation commonly arises in accelerated life testing (ALT) where, on the basis of tests run at accelerated test conditions, it is desired to predict item life under use conditions. For parametric approaches to this problem McCool(1980) and Meeker and Hahn (1985) obtained the distribution of pivotal quantities to derive exact confidence intervals for parameters, percentiles, reliabilities, or other quantities of interest. Lawless (1976) also considered the exponential-inverse power rule model and use condition on

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ancillary statistics to yield interval estimates for parameters based on complete and single Type II censoring. Thomas (1964) showed applications of the Eyring model in studying the aging process of electronic parts when the stress is temperature. Most of researchers have investigated mainly for parameter estimation and optimal design problems. The calculation of linear estimates is generally much less laborious than the calculation of maximum likelihood estimates, and confidence intervals based on linear estimates have not been developed for the exponential regression models.

In this paper, we will find more general form pertaining to conditional confidence intervals estimation in the regression model having also been presented for parameters. Using this result, we study the maximum likelihood estimation for parameters of ALT models under Type II censoring, assuming that the log-lifetime is a linear function of the stresses.

In Section 2, we show that the conditional p.d.f. of pivotal quantity given the ancillary statistic is the same of the form the joint p.d.f. pivotal quantity and the ancillary statistic. In Section 3, Using pivotal quantities, we derive conditional confidence intervals for the Eyring model. In Section 4, we consider the performance of the conditional confidence interval and approximate confidence interval for the use condition, $\underline{\beta}_u$, of the Eyring model in comparing p.d.f.'s through the Monte Carlo simulation.

2. Conditional Confidence Interval Procedures

We discuss estimation and prediction procedures for a model including the Eyring model which is commonly used in reliability and life testing work. We consider the Eyring model in the following form: Let the lifetime of an item under environmental condition i have an exponential distribution with mean θ_i. In this model the environmental conditions are specified by means of a single stress x_i, and the relationship

$$\theta_i = x_i^* \exp(\beta_1 + \beta_2 x_i) \tag{1}$$

is assumed, where x_i is temperature, β_1 and β_2 are unknown parameters, $x_i^* = (bK)^{-1}$, $b = 8.617 \times 10^{-5}$ is Boltzmann's constant in electron volts per degree Kelvin, and K is the temperature in degree Kelvin.

For multiplicative error terms ε_i the model can be written as

$$\hat{\theta}_i = \theta_i \varepsilon_i,\ i = 1, 2, \cdots,\ \text{ or }\ \log\hat{\theta}_i = \log x_i^* + \beta_1 + \beta_2 x_i + \log\varepsilon_i,$$

and then the regression model of observations in matrix form is given as

$$\underline{Y} = X\underline{\beta} + \underline{\varepsilon} \tag{2}$$

where $\underline{Y} = (Y_1, Y_2, \cdots, Y_N)'$ is $N \times 1$ observable random vectors, $\underline{\beta} = (\beta_1, \beta_2, \cdots, \beta_N)'$ is $p \times 1$ vector of parameters, and $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_1, \cdots, \varepsilon_N)^-$ is $N \times 1$ unobservable vector of random variables, where the ε_i are independent random variable with p.d.f. $f_i(\varepsilon_i)$.

$$X = \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{p1} \\ x_{12} & x_{22} & \cdots & x_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1(N-p)} & x_{2(N-p)} & \cdots & x_{p(N-p)} \\ x_{1(N-p+1)} & x_{2(N-p+1)} & \cdots & x_{p(N-p+1)} \\ x_{1(N-p+2)} & x_{2(N-p+2)} & \cdots & x_{p(N-p+2)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1N} & x_{2N} & \cdots & x_{pN} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{3}$$

is a $N \times p$ matrix of rank $p(p < N)$ of known fixed numbers(see ; Verhagen(1961)).

We will consider here life test experiments in which n_i items are put on test simultaneously at stress level x_i, $i = 1, 2, \cdots, N$, and the test at x_i is terminated at the failure time of the r_ith item. The model in (2) can be rewritten as

$$Y_i = \underline{x_i'\beta} + \varepsilon_i \tag{4}$$

where ε_i is independent p.d.f. $f_i(\varepsilon_i)$, $i = 1, 2, \cdots, N$. Therefore the p.d.f. of Y_i is represented as

$$h_i(y_i) = f_i(y_i - \underline{x_i'\beta}) \tag{5}$$

The following lemmas are required to prove Theorem 2.1.

Lemma 2.1 Suppose $d^2 \log f_i(\varepsilon_i)/d\varepsilon_i^2$ does not vanish at $A_i = Y_i - x_i'\beta$, and is continuous in some neighborhood $(A_i - h, A_i + h)$ of A_i. Let $g_i(\varepsilon_i) = \log f_i(\varepsilon_i)$, $i = 1,2, \cdots, N$. Then the $u_j(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N) = \sum_{i=1}^{N} x_{ji} g_i'(\varepsilon_i)$, $j = 1,2, \cdots, p$ and their partial derivatives are continuous on the open N-dimensional ball, $B_N = (A_1, A_2, \cdots, A_N; h)$ of radius h and center $(A_1, A_2, \cdots, A_N)$.

Proof. Let $\varepsilon > 0$ be given and $\underline{\varepsilon}_0 = (\varepsilon_1^0, \varepsilon_2^0, \cdots, \varepsilon_N^0)$ be an arbitrary point in

$B_N(A_1, A_2, \cdots, A_N; h)$. By the assumption, there exists $\delta > 0$ such that $|x_{ji}||g_i'(\varepsilon_i) - g_i'(\varepsilon_i^0)| < \varepsilon/N$ whenever $|\varepsilon_i - \varepsilon_i^0| < \delta$. Note that $\|\underline{\varepsilon} - \underline{\varepsilon}^0\| < \delta$ implies $|\varepsilon_i - \varepsilon_i^0| < \delta$ for $i = 1,2, \cdots N$. So for $\|\underline{\varepsilon} - \underline{\varepsilon}^0\|$ we have

$$\begin{aligned}|u_j(\underline{\varepsilon}) - u_j(\underline{\varepsilon}^0)| &= \left|-(\sum_{i=1}^{N} x_{ji} g_i'(\varepsilon_i) + \sum_{i=1}^{N} x_{ji} g_i'(\varepsilon_i^0)\right| \\ &= \left|-\sum_{i=1}^{N} x_{ji}\left(g_i'(\varepsilon_i) + g_i'(\varepsilon_i^0)\right)\right| \\ &\le \sum_{i=1}^{N} |x_{ji}||g_i'(\varepsilon_i) - g_i'(\varepsilon_i^0)| \le \varepsilon.\end{aligned}$$

This establishes that the u_j are continuous on $B_N(A_1, A_2, \cdots, A_N; h)$. The partial derivative of $u_j(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N)$ with respect to ε_N is

$$\frac{\partial u_j(\underline{\varepsilon})}{\partial \varepsilon_k} = -\frac{\partial}{\partial \varepsilon_k} \sum_{i=1}^{N} x_{ji} g_i'(\varepsilon_i) = x_{jk} g_i''(\varepsilon_k).$$

By the assumption, there exists $\delta > 0$ such that $|x_{jk}||g_k''(\varepsilon_k) - g_k''(\varepsilon_i^0)| < \varepsilon$ whenever $|\varepsilon_k - \varepsilon_k^0| < \delta$. So for $\|\underline{\varepsilon} - \underline{\varepsilon}^0\| < \delta$ we have

$$\left\|\frac{\partial u_j(\varepsilon_i)}{\partial \varepsilon_k} - \frac{\partial u_j(\underline{\varepsilon}^0)}{\partial \varepsilon_k^0}\right\| = |x_{jk}||g_k''(\varepsilon_k) - g_k''(\varepsilon_i^0)| < \varepsilon$$

This completes the proof.

Lemma 2.2 Suppose $d^2 \log f_i(\varepsilon_i)/d\varepsilon_i^2$ does not vanish at A_i, and $\det(X_2^t) \neq 0$, where X_2 is given by (3). Then the determinant of D is nonzero when it is evaluated at $(A_1, A_2, \cdots, A_N)$, where

$$|D| = \begin{vmatrix} \dfrac{\partial u_1}{\partial \varepsilon_{N-p+1}} & \dfrac{\partial u_1}{\partial \varepsilon_{N-p+2}} & \cdots & \dfrac{\partial u_1}{\partial \varepsilon_N} \\ \dfrac{\partial u_2}{\partial \varepsilon_{N-p+1}} & \dfrac{\partial u_2}{\partial \varepsilon_{N-p+2}} & \cdots & \dfrac{\partial u_2}{\partial \varepsilon_N} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial u_p}{\partial \varepsilon_{N-p+1}} & \dfrac{\partial u_p}{\partial \varepsilon_{N-p+2}} & \cdots & \dfrac{\partial u_p}{\partial \varepsilon_N} \end{vmatrix}.$$

proof. Since the partial derivative of the u_j are

$$\frac{\partial u_j(\varepsilon_1,\varepsilon_2,\cdots\varepsilon_N)}{\partial\varepsilon_k}=-x_{jk}\frac{d^2\log f_k(\varepsilon_k)}{d\varepsilon_k^2}=-x_{jk}g_k^{''}(\varepsilon_k)$$

for k=1,2,···,N and j=1,2,···,p, it can be shown from Lemma 2.1 that the $u_j(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N)$, $j=1,2,\cdots,p$ and their first derivatives are continuous on the N-dimensional open ball $B_N=(A_1,A_2,\cdots,A_N;h)$. And then, using partial derivatives of the u_j and the determinant of D, we obtain

$$D=\left(\prod_{i=N-p+1}^{N}\frac{d^2\log f_i(\varepsilon_i)}{d\varepsilon_i^2}\right)=\det\begin{pmatrix}-x_{1(N-p+1)} & -x_{1(N-p+2)} & \cdots & -x_{1N}\\ -x_{2(N-p+1)} & -x_{2(N-p+2)} & \cdots & -x_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ -x_{p(N-p+1)} & -x_{p(N-p+2)} & \cdots & -x_{pN}\end{pmatrix}$$

$$=(-1)^p\left(\prod_{i=N-p+1}^{N}\frac{d^2\log f_i(\varepsilon_i)}{d\varepsilon_i^2}\right)\det\left(X_2^t\right)$$

Hence, under assumptions of lemmas, this completes the proof.

The following theorems relevant to conditional confidence interval estimation for parameters in regression model are provided.

Theorem 2.1 Let $\underline{\hat{\beta}}$ be the M. L. estimator of $\underline{\beta}$ in (2) and $\underline{\beta}_u=(\beta_{u1},\beta_{u2},\cdots,\beta_{up})^t$ denote the true parameter of $\underline{\beta}$. Then one can get the followings:

1. $\underline{Z}=\underline{\hat{\beta}}-\underline{\beta}_u$ is a pivotal quantity.
2. $A_i=Y_i-\underline{x}_i^t\,\underline{\hat{\beta}}$, $i=1,2,\cdots,N$ are ancillary statistics.
3. If $d^2\log f_i(\varepsilon_i)/d\varepsilon_i^2$ does not vanish at A_i, exists and is continuous in some neighborhood of A_i, and $\det(X_2^t)\neq 0$, then $A_1,A_2,\cdots,A_{N-p}$ of the ancillaries are independent.

Proof. 1 and 2 proofs easily obtained by Lawless(1982).

3. Since

$$A_i=Y_i-\underline{x}_i^t\underline{\hat{\beta}}(Y_1,Y_2,\cdots,Y_N),\ i=1,2,\cdots,N \tag{6}$$

one can get as

$$u_j(A_1,A_2,\cdots,A_N)=0,\ j=1,2,\cdots,p.$$

By using Lemma 2.1 and Lemma 2.2, the implicit function theorem guarantees that numbers l_1 and l_2 exist such that for each $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{N-p}$, $|\varepsilon_i-A_i|<l_1$ for

$i=1,2,\cdots,N-p$. Therefore, there is unique solution $(\varepsilon_1,\varepsilon_2,\cdots\varepsilon_N)$ of

$$u_j(A_1,A_2,\cdots,A_N)=0,\ j=1,2,\cdots,p \tag{7}$$

for which $|\varepsilon_i - A_i| < l_2$, $i=N-p+1,\cdots,N$. Since $\varepsilon_{N-p+1},\varepsilon_{N-p+2},\cdots,\varepsilon_N$ are functions of $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{N-p}$, say,

$$\begin{aligned}\varepsilon_{N-p+1} &= v_{N-p+1}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{N-p})\\ \varepsilon_{N-p+2} &= v_{N-p+2}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{N-p})\\ &\vdots\\ \varepsilon_N &= v_N(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{N-p})\end{aligned}$$

it follows $(A_1,A_2,\cdots,A_N)$ that

$$\begin{aligned}A_{N-p+1} &= v_{N-p+1}(A_1,A_2,\cdots,A_{N-p})\\ A_{N-p+2} &= v_{N-p+2}(A_1,A_2,\cdots,A_{N-p})\\ &\vdots\\ A_N &= v_N(A_1,A_2,\cdots,A_{N-p})\end{aligned}$$

Hence, only $N-p$ of the A_i are independent, and this completes the proof.

The following theorem concerning the distribution of the pivotals and ancillaries also provide an alternate proof of Theorem 2.1.

Theorem 2.2 Under the assumptions of Theorem 2.1, the joint p.d.f. $\underline{Z}$ and $\underline{A}=(A_1,A_2,\cdots,A_{N-p})$ is of the form

$$k(\underline{a},X)\prod_{i=1}^{N} f_i(a_i+\underline{x}_i^t\underline{Z})\cdot$$

Moreover, the conditional p.d.f. of $\underline{Z}$ given $\underline{A}=(a_1,a_2,\cdots,a_{N-p})$ is the same form.

Proof. The joint p.d.f. of $Y_1,Y_2,\cdots,Y_N$ is

$$\prod_{i=1}^{N} h_i(y_i)=\prod_{i=1}^{N} f_i(y_i-\underline{x}_i^t\underline{\beta}_u)$$

From $A_i=Y_i-\underline{x}_i^t\underline{\beta}_u$, $i=1,2,\cdots,N$, the inverse transformation can be written as

$$Y_1 = x_{11}\hat{b}_1 + x_{21}\hat{b}_2 + \cdots + x_{p1}\hat{b}_p + A_1$$
$$Y_2 = x_{12}\hat{b}_1 + x_{22}\hat{b}_2 + \cdots + x_{p2}\hat{b}_p + A_2$$
$$\vdots$$
$$Y_{N-p} = x_{1(N-p)}\hat{b}_1 + x_{2(N-p)}\hat{b}_2 + \cdots + x_{p(N-p)}\hat{b}_p + A_{N-p}$$
$$Y_{N-p+1} = x_{1(N-p+1)}\hat{b}_1 + x_{2(N-p+1)}\hat{b}_2 + \cdots + x_{p(N-p+1)}\hat{b}_p + V_{N-p+1}(A_1, A_2, \cdots, A_{N-p})$$
$$\vdots$$
$$Y_N = x_{1N}\hat{b}_1 + x_{2N}\hat{b}_2 + \cdots + x_{pN}\hat{b}_p + V_N(A_1, A_2, \cdots, A_{N-p})$$

Since the Jacobian matrix $J = \partial(Y_1, Y_2, \cdots, Y_N)/\partial(A_1, A_2, \cdots, A_{N-p}, \underline{\hat{\beta}})$ depends only on $\underline{A}$ and the matrix X given in (3) it can be denoted as

$$|J| = k(\underline{a}, X).$$

Hence the joint p.d.f. of $\underline{A}$ and $\underline{\hat{\beta}}$ is represented

$$g_i(\underline{A}, \underline{\hat{\beta}}) = k(\underline{a}, X)\prod_{i=1}^{N} f_i(a_i + \underline{x}_i^t(\underline{\hat{\beta}} - \underline{\hat{\beta}}_u)).$$

This competes the proof.

3. Confidence Interval Procedure for the Eyring Model

In this section we deals with a conditional confidence interval estimations for the Eyring model using a generalized form of Section 2.

In life tests, suppose that n_i items are put on test at stress level x_i and the test is terminated at the failure time of the $r_i^{th}(< n_i)$, $i = 1,2, \cdots, N$, item failure. The density function of a lifetime t at stress level x_i is

$$f_i(t, \underline{\beta}) = \frac{1}{\theta_i}\exp(-\frac{t}{\theta_i}), \quad 0 < t < \infty \tag{8}$$

where

$\theta_i = x_i^* \exp(\underline{x}_i^t \underline{\beta})$, $\underline{x}_i^t = (1, x_i)$, $\underline{\beta} = (\beta_1, \beta_2)$, $x_i = x_i^* - \bar{x}, \bar{x} = \sum_{i=1}^{N} r_i x_i^* / \sum_{i=1}^{N} r_i$, and $x_i^* = 1/b K_i$ at which the i th Type II censored test was conducted. Here the relationship of a parameter θ_i and a stress x_i, $\theta_i = x_i^* \exp(\underline{x}_i^t \underline{\beta})$ is called the Eyring model. If the observed lifetimes at x_i are $Y_{i1} \le Y_{i2} \le \cdots \le Y_{ir_i}$

$(i=1,2,\cdots,N)$. Then the likelihood function the data assuming the Eyring model is

$$L(\underline{\beta}) = \left[\prod_{i=1}^{N}\frac{n_i!}{(n_i-r_i)!}\right]\left[\left(\prod_{i=1}^{N}x_i^*\right)^{-1}\exp\left(-\sum_{i=1}^{N}r_i(\beta_1+\beta_2 x_i)\right)\right]$$
$$\times\left[\exp-\left(\sum_{i=1}^{N}S_i\Big/x_i^*\exp(\beta_1+\beta_2 x_i)\right)\right]$$

where $S_i=\sum_{j=1}^{r_i}y_{ij}+(n_i-r_i)y_{ir_i}$ is called the total times on i-th test. So it follows that the statistics $S_1,S_2,\cdots,S_N$ are jointly sufficient for (β_1,β_2). Therfore the joint density of $S_1,S_2,\cdots,S_N$ is given as

$$\prod_{i=1}^{N}\frac{1}{\Gamma(r_i)\theta_i^{r_i}}s_i^{r_i-1}\exp\left(-\frac{s_i}{\theta_i}\right)\quad \text{for } 0<s_i<\infty,\ i=1,2,\cdots,N. \tag{9}$$

The procedures to be developed will be disscussed more naturally if we consider log failure times. The logarithms of the individual failure times follows a extreme value distribution, although for our purposes here it is sufficient to consider the logarithms of the total times $S_1,S_2,\cdots,S_N$. For the Eyring model, $\theta_i=x_i^*\exp(\beta_1+\beta_2 x_i)$ with the transformation, $Y_i=\log S_i+\log bK_i$, $i=1,2,\cdots,N$, the joint density of $Y_1,Y_2,\cdots,Y_N$ is given as

$$\begin{aligned}L_n(\underline{\beta}) &= \left[\prod_{i=1}^{N}\frac{e^{Y_i(r_i-1)}}{\Gamma(r_i)\theta_i^{r_i}(bK_i)^{r_i-1}}\exp\left(-\frac{e^{Y_i}}{(bK_i)\theta_i}\right)\right]\left[\prod_{i=1}^{N}(bK_i)^{-1}\exp(Y_i)\right]\\ &= \prod_{i=1}^{N}\frac{1}{\Gamma(r_i)}\exp\left[\left(r_i(Y_i-\underline{x}_i'\underline{\beta})\right)-\exp(Y_i-\underline{x}_i'\underline{\beta})\right],\end{aligned} \tag{10}$$

where $-\infty<y_i<\infty$.

Let β_1 and β_2 be maximum likelihood estimators of β_1 and β_2 based on the sample $Y_1,Y_2,\cdots,Y_N$, and let β_{u1} and β_{u2} denote the true values of β_1 and β_2. Then from Theorem 2.1 it is easily verify that $\hat{\beta}_1-\beta_{u1}$ and $\hat{\beta}_2-\beta_{u2}$ are pivotal quantity, and $A_i=\log s_i+\log bK_i-\beta_1-\beta_2 x_i$ are ancillary statistics and functional independent . Since $A_1,A_2,\cdots,A_{N-2},\beta_1,\beta_2$ are jointly sufficient for (β_1,β_2), $A_1,A_2,\cdots,A_{N-2}$, $Z_1=\beta_1-\beta_{u1}$, $Z_2=\beta_2-\beta_{u2}$ are jointly sufficient for (β_1,β_2). Hence by Theorem 2.2, we have that the joint density of Z_1 and Z_2 given $A_1=a_1,A_2=a_2,\cdots,A_{N-2}=a_{N-2}$ is given as

$$f(z_1,z_2|\underline{a})=k(\underline{a},X)\prod_{i=1}^{N}\frac{1}{\Gamma(r_i)}\exp\left(r_i[a_i+z_1+z_2x_i]-\exp[a_i+z_1+z_2x_i]\right) \tag{11}$$

where $-\infty < z_1 < \infty$ and $-\infty < z_2 < \infty$. The expressions below are simplified when the x_i's are centralized so that $\sum_{i=1}^{N} r_i x_i = 0$ and (11) reduce to

$$f(z_1, z_2 | \underline{a}) = k(\underline{a}, X)\exp\left(rz_1 - \exp z_1\left[a_i + z_2 x_i\right]\right), \tag{12}$$

where $r = \sum_{i=1}^{N} r_i$ and $k(\underline{a}, X) = \left(\prod_{i=1}^{N} 1/\Gamma(r_i)\right)\exp\left(\sum_{i=1}^{N} r_i a_i\right)$. In this model, $a_i = y_i - \underline{x}_i^t \hat{\underline{\beta}} = \log s_i + \log bK_i - \hat{\beta}_1 - \hat{\beta}_2 x_i$, where $x_i = x_i^* - \overline{x}$, and $x_i^* = 1/b\,K_i$.

Now we will derive the marginal distribution functions $Z_1 = \hat{\beta}_1 - \beta_{u1}$ and $Z_2 = \hat{\beta}_2 - \beta_{u2}$, given $a_1, a_2, \cdots, a_{N-2}$ to make the confidence intervals for β_{u1} and β_{u2}, respectively. By using the transformation $u = e^{z_1} \sum_{i=1}^{N} \exp(a_i + z_2 x_i)$ it follows from (12) that the p.d.f. of Z_2, given $\underline{a}$ is

$$h(z_2 | \underline{a}) = k_1(\underline{a}, X) \Big/ \left(\sum_{i=1}^{N} \exp(a_i + z_2 x_i)\right)^r \tag{13}$$

where $r = \sum_{i=1}^{N} r_i$ and

$$k_1^{-1}(\underline{a}, X) = \int_{-\infty}^{\infty} \left[\left(\sum_{i=1}^{N} \exp(a_i + z_2 x_i)\right)^r\right]^{-1} dz_2 \,.$$

Confidence interval statement about β_{u2} follows directly from probability statements about z_2.

To determine the marginal distribution for Z_1 given $\underline{a}$, we consider

$$P(Z_1 \le t | \underline{a}) = \int_{-\infty}^{\infty} \int_{-\infty}^{t} k_1(\underline{a}, X)\exp\left(rz_1 - \exp \sum_{i=1}^{N} \exp(a_i + z_2 x_i)\right) dz_1 dz_2 .$$

Setting $u = e^{z_1}$, the above equation is given as

$$\begin{aligned} P(Z_1 \le t | \underline{a}) &= \int_{-\infty}^{\infty} k_1(\underline{a}, X) \int_0^{\exp(t)} u^{N-1} \exp\left(-u \exp \sum_{i=1}^{N} \exp(a_i + z_2 x_i)\right) du\, dz_2 \\ &= k_1(\underline{a}, X) \int_{-\infty}^{\infty} \frac{G_r\left(-\sum_{i=1}^{N} \exp(a_i + z_2 x_i + t)\right)}{\left(-\sum_{i=1}^{N} \exp(a_i + z_2 x_i)\right)^r} dz_2, \end{aligned} \tag{14}$$

where $r = \sum_{i=1}^{N} r_i$, $G_r(s) = (1/\Gamma(r))\int_0^s u^{r-1} e^{-u} du$ is the incomplete gamma function.

4. Numerical Comparisons

In this section we consider the performance of the conditional confidence interval and approximate confidence interval for $\underline{\beta}_u$ of the Eyring model in comparing p.d.f.'s through the Monte Calro simulation. Since the denominator of (11) tends to be rather large if N is large, it is difficult for us to obtain the confidence interval for β_{u2}.

We consider the confidence interval for β_{u2}. For ease of computation of confidence interval for β_{u2}, z_1 and z_2 of the densities (12), (13) and (14) are replaced by z and using the method of Lawless(1982), $h(z|\underline{a})/h(0|\underline{a})$, we define as the ratio $f(z|\underline{a})$ as

$$f(z|\underline{a})=\frac{h(z|\underline{a})}{h(0|\underline{a})}=\left(\frac{\sum_{i=1}^{N}\exp(a_i)}{\sum_{i=1}^{N}\exp(a_i+zx_i)}\right)^r \tag{15}$$

where $f(z|\underline{a})$ is the p.d.f. of z, given $\underline{a}$. Therefore the distribution function of z, given $\underline{a}$ is represented as

$$P(Z_2\le t|\underline{a})=\int_{-\infty}^{t}h(z|\underline{a})\,dz=f(z|\underline{a})\int_{-\infty}^{t}f(z|\underline{a})\,dz=f(z|\underline{a})F(t|\underline{a}), \tag{16}$$

where $F(t|\underline{a})=\int_{-\infty}^{t}f(z|\underline{a})dz$. Since $F(\infty|\underline{a})=1/h(0|\underline{a})$, it follows that

$$P(Z_2\le t|\underline{a})=\frac{F(t|\underline{a})}{F(\infty|\underline{a})}=\int_{-\infty}^{\infty}f(z|\underline{a})dz$$

Hence, solving the equations

$$P(Z_2\le t_1|\underline{a})-\frac{\alpha}{2}=0 \text{ and } P(Z_2\le t_2|\underline{a})-\frac{\alpha}{2}=0,$$

one can obtains the 100(1-α)% confidence interval for β_{u2} as

$$P(\hat{\beta}_2-t_2\le\beta_{u2}\le\hat{\beta}_2-t_1)=1-\alpha$$

On the other hand, to find conditional confidence interval for β_{u1} we require to compute the equations

$$\begin{aligned}P(Z_1\le t_1|\underline{a})&=\int_{-\infty}^{\infty}h(z|\underline{a})G_r\left\{\sum_{i=1}^{N}\exp(a_i+z\,x_i+t)\right\}dZ\\&=h(0|\underline{a})\int_{-\infty}^{\infty}\frac{h(z|\underline{a})}{h(0|\underline{a})}G_r\left\{\sum_{i=1}^{N}\exp(a_i+z\,x_i+t)\right\}dz\\&=F(\infty|\underline{a})\int_{-\infty}^{\infty}f(z|\underline{a})G_r\left\{\sum_{i=1}^{N}\exp(a_i+z\,x_i+t)\right\}dz.\end{aligned} \tag{17}$$

Hence solving the equations

$$P(Z_1 \le t_1 | \underline{a}) - \frac{\alpha}{2} = 0 \text{ and } P(Z_1 \le t_2 | \underline{a}) - \frac{\alpha}{2} = 0$$

one can obtains the $100(1-\alpha)\%$ confidence interval for β_{u1} as

$$P(\hat{\beta}_1 - t_2 \le \beta_{u1} \le \hat{\beta}_1 - t_1) = 1 - \alpha$$

Yoon (1995) studied the accuracy of approximate procedures based on asymptotic theory for M. L. estimates. Some evidence below indicates that these procedures are suitable if n is even moderately large, and there are not too many different levels for stress variables.

An approximate $100(1-\alpha)\%$ confidence intervals for β_{u1} and β_{u2} are given as

$$\hat{\beta}_1 \pm z_{(1-2/\alpha)} \sqrt{\left(n \sum_{i=1}^{N} \pi_i \, p_i\right)^{-1}} \text{ and } \hat{\beta}_2 \pm z_{(1-2/\alpha)} \sqrt{\left(n \sum_{i=1}^{N} \pi_i \, p_i \, x_i^2\right)^{-1}},$$

where $z_{(1-(2/\alpha))}$ is the $100(1-(2/\alpha))\%$th percentile of the standard normal distribution and π_i is represented by the proportion of $n(=\sum_{i=1}^{N} n_i)$ units allocated to the i th stress level. Also, p_i is represented by the proportion of allocated components failing at the stress level i.

Hence, for the Eyring model at the use condition, the relationship of parameter θ and stress x is

$$\log \hat{\theta}_u = \log x_u^* + \hat{\beta}_1 + x_u \hat{\beta}_2$$

Recall that for the Eyring models when the use condition $x_u = x_u^* - \bar{x}$, where

$$\bar{x} = \sum_{i=1}^{N} \pi_i \, p_i \, x_i^* \Big/ \sum_{i=1}^{N} \pi_i \, p_i \, .$$

Example 1 As a first example of the use of the procedure of Section 2, we consider some data discussed by Tobias and Trindade(1986). For the Eyring model with $K_1 = 358^\circ K$, $K_2 = 378^\circ K$, $K_3 = 398^\circ K$, where $K_u = 300^\circ K$, and $\beta_1 = 0.42$, $\beta_2 = 1$, and $\hat{\theta}_u = x_u^* \exp(\hat{\beta}_1 + \hat{\beta}_2 x_u)$, the accelerated life test data are generated for $N = 3$ levels of the stress, $\pi_i = 1/3$, $i = 1,2,3$, $p_1 = 3/7$, $p_2 = 5/7$, $p_3 = 1$ and $n = 21$. The conditional confidence interval(C.C.I.) and approximate confidence intervals(A.C.I.) are given in Table 1, and the graphs of the conditional p.d.f. of $Z_2 | A$ and the approximate p.d.f. Z_2 are given in Figure 1.

Example 2 For the Eyring model with $\beta_1 = 0.42$, $\beta_2 = 1$, $K_1 = 318^\circ K$,

$K_2 = 338°K$, $K_3 = 358°K$, $K_4 = 378°K$, $K_5 = 398°K$, where $K_u = 300°K$, the accelerated life test data are generated for $N = 5$ levels of the stress, $\pi_i = 1/5$, $i = 1,2, \cdots, 5$. $p_1 = 2/5$, $p_2 = 3/5$, $p_3 = 3/10$, $p_4 = 9/10$, $p_5 = 1$, and $n = 50$. The conditional confidence interval(C.C.I.) and approximate confidence intervals(A.C.I.) are given in Table 2, and the graphs of the conditional p.d.f. of $Z_2|A$ and the approximate p.d.f. Z_2 are given in Figure 2.

Example 3 For $p_2 = 1/2$, $p_2 = 1/2$, $p_3 = 2/3$, $p_4 = 5/6$, $p_5 = 5/6$, and $n = 150$. The conditional confidence interval(C.C.I.) and approximate confidence intervals (A.C.I.) are given in Table 3, and the graphs of the conditional p.d.f. of $Z_2|A$ and the approximate p.d.f. Z_2 are given in Figure 3.

From figures and tables, one can observe the following facts.(1) In all of the cases, the widths of confidence intervals decrease as the number of sample size increases. (2) The widths of confidence intervals decrease as the number of stress levels increases. (3) For each stress levels, the total testing times are diminish as the degree of temperature increases. (4) The conditional and approximate p.d.f.'s agree fairly closely for the total sample sizes of at least 21 under the some conditions.

Table 1. Total Test Times and 90% Confidence Intervals for the Eyring Model : N=3, n=21

Stress	n_i	r_i	x_i	s_i	y_i	a_i
1	7	3	2.0921	786.4786	3.1889	0.6768
2	7	5	0.3770	331.3585	2.3789	1.5820
3	7	7	-1.1658	81.3107	1.0255	1.7714

Parameter	L.C.I	U.C.I	Parameter	True Value	M.L.E
$\beta_1(C.C.I)$	-0.0977	0.7783	β_1	0.4200	0.2526
$\beta_1(A.C.I)$	0.2798	0.7850	β_2	1.0000	0.9485
$\beta_1(A.C.I)$	0.2798	0.7850	θ_u	213467.6000	118381.1000
$\beta_2(C.C.I)$	0.6064	1.3400			

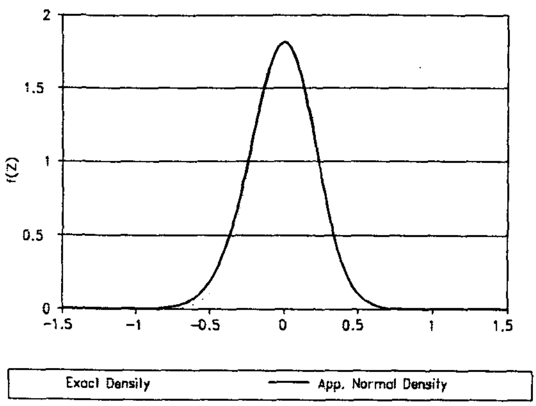

Figure 1. Marginal Density for $Z=\hat{\beta}_2-\beta_{u2}$ with the Eyring Model : $N=3,\ n=21,\ n_i=7,\ i=1,2,3,\ r_1=3,\ r_2=5,\ r_3=7$

Table 2. Total Test Times and 90% Confidence Intervals for the Eyring Model : $N=5,\ n=50$

Stress	n_i	r_i	x_i	s_i	y_i	a_i
1	10	4	4.6385	16416.8300	6.1089	1.0504
2	10	6	2.4791	3828.6300	4.7141	1.8150
3	10	7	0.5610	518.1335	2.7716	1.7906
4	10	9	-1.1541	115.1977	1.3224	2.0565
5	10	10	-2.6969	28.4827	-0.0234	2.2535

Parameter	L.C.I	U.C.I	Parameter	True Value	M.L.E
$\beta_1(C.C.I)$	0.0702	0.6256	β_1	0.4200	0.3118
$\beta_1(A.C.I)$	0.0432	0.5805	β_2	1.0000	0.9844
$\beta_2(C.C.I)$	0.8728	1.1049	θ_u	213467.6000	168527.8000
$\beta_2(A.C.I)$	0.8770	1.0917			

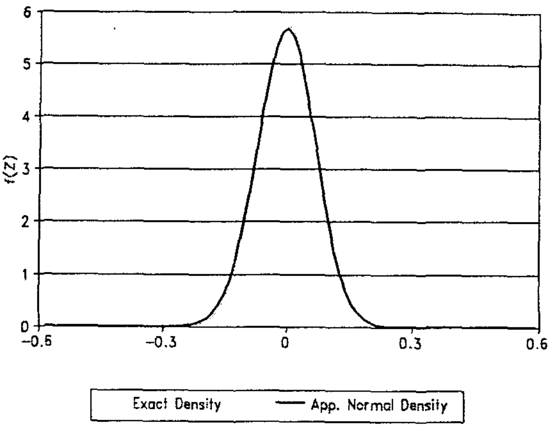

Figure 2. Marginal Density for $Z=\hat{\beta}_2-\beta_{u2}$ with the Eyring Model : $N=5,\ n=50,\ n_i=10,\ i=1,2,\ \cdots\ ,5,\ r_1=4,\ r_2=6,\ r_3=7,\ r_4=9,\ r_5=10$

Table 3 Total Test Times and 90% Confidence Intervals for the Eyring Model : $N=5,\ n=150$

Stress	n_i	r_i	x_i	s_i	y_i	a_i
1	30	15	4.4214	66753.8690	7.5116	2.6702
2	30	15	2.2620	7979.5534	5.4485	0.7665
3	30	20	0.3439	1336.8570	3.7194	2.9555
4	30	25	-1.3712	284.6358	2.2269	3.1781
5	30	25	-2.9140	61.0434	0.7389	3.2328

Parameter	L.C.I	U.C.I	Parameter	True Value	M.L.E
$\beta_1(C.C.I)$	0.2567	0.5872	β_1	0.4200	0.4090
$\beta_1(A.C.I)$	0.2540	0.5641	β_2	1.0000	0.9995
$\beta_2(C.C.I)$	0.9353	1.0666	θ_u	213467.5600	210182.1000
$\beta_2(A.C.I)$	0.9373	1.0616			

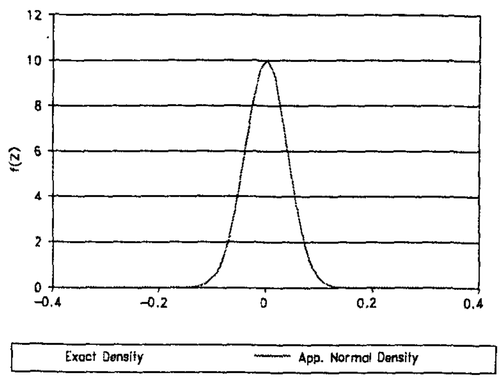

Figure 3. Marginal Density for $Z = \hat{\beta}_2 - \beta_{u2}$ with the Eyring Model :
$N = 5,\ n = 150,\ n_i = 30,\ i = 1,2,\ \cdots\ ,5,\ r_1 = 15,\ r_2 = 15,\ r_3 = 20,\ r_4 = 25,\ r_5 = 25$

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