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A Distribution for Regulated μ-Brownian Motion Process with Control Barrier at x_0

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Abstract Consider a natural model for stochastic flow systems is Brownian motion, which is Brownian motion on the positive real line with constant drift and constant diffusion coefficient, modified by an impenetrable reflecting barrier at x_0. In this paper, we investigate the joint distribution functions and study on the distribution of the first-passage time. Also we find out the distribution of $\mu - RBMPx_0$.

Keywords : Brownian motion process, first-passage time, $\mu - BMP$, reflection principle, Radon-Nikodym derivative, $\mu - RBMPx_0$.

1. Introduction

Brownian motion has been studied with reference to scientific questions, in connection with financial problems and in the aspect of engineering. It is important to be able to evaluate the probability that, within a given time, a certain 'barrier' is reached, as this corresponds in the case of stock-market investments to finding the chance that the value of the investments falls below a certain level. And it is also important to find out the distribution of output process in stochastic flow sysem.

The definitions of Brownian motion process are follows;

(1) Brownian motion process (*BMP*) is a stochastic process $\{X_t, t \geq 0\}$ with following properties:

(a) Every increment $X_{t+s} - X_s$ is normally distributed with mean 0 and variance σ_t^2 ; σ^2 is a fixed parameter.

(b) For every pair of disjoint time intervals $[t_1, t_2]$, $[t_3, t_4]$, say $t_1 < t_2 \leq t_3 < t_4$, the increments $X_{t_4} - X_{t_3}$ and $X_{t_2} - X_{t_1}$ are independent random variables with distributions given in (a), and similarly for n disjoint time intervals where n is an

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arbitrary positive integer.

(c) $X_0 = 0$ and X_t is continuous at t=0.

Under the condition that $X_0 = 0$, the variance of X_t is σ_t^2. Hence σ^2 is sometimes called the diffusion parameter of the process. The process $Z_t = X_t / \sigma\sqrt{t}$ is a Brownian motion process having a variance parameter of one, called standard Brownian motion process (Ross (1983)).

(2) Brownian motion process with drift $\mu(\mu - BMP$ or $(\mu,\sigma)BMP)$ is a stochastic process $\{X_t, t \geq 0\}$ with following properties (Wendel(1980)) :

(a) Every increment $X_{t+s} - X_s$ is normally distributed with mean μ_t and variance σ_t^2; μ and σ^2 are fixed parameters.

(b) For every pair of disjoint time intervals $[t_1,t_2],[t_3,t_4]$, say $t_1 < t_2 \leq t_3 < t_4$, the increments $X_{t_4} - X_{t_3}$ and $X_{t_2} - X_{t_1}$ are independent random variables with distributions given in (a), and similarly for n disjoint time intervals where n is an arbitrary positive integer.

(c) $X_0 = 0$ and X_t is continuous at t=0.

2. A joint distribution

Let $N_t = \sup\{X_s,\ 0 \leq s \leq t\}$, $N_t^* = \sup\{X_s + x_0,\ 0 \leq s \leq t\}$ and then define the joint distribution function

$$G_t(x,y) = P\{X_t + x_0 \leq x,\ N_t^* \leq y\}$$

Because $X_0 = 0$ by hypothesis, one need only calculate $G_t(x,y)$ for $y \geq 0$ and $x \leq y$.

We shall compute G for standard Brownian motion in this section and then extend the calculation to general μ and σ in section3. Fitting μ=0 and σ=1 throughout this section, note first that

$$\begin{aligned}G_t(x,y) &\equiv P\{X_t + x_0 \leq x,\ N_t^* \leq y\} \\ &= P\{X_t + x_0 \leq x\} - P\{X_t + x_0 \leq x,\ N_t^* > y\} \\ &= P\{X_t \leq x - x_0\} - P\{X_t \leq x - x_0,\ N_t > y - x_0\} \\ &= \Phi\left((x - x_0)t^{-\frac{1}{2}}\right) - P\{X_t \geq 2y - x - x_0\} \\ &= \Phi\left((x - x_0)t^{-\frac{1}{2}}\right) - \Phi\left((x + x_0 - 2y)t^{-\frac{1}{2}}\right)\end{aligned}$$

Now the term $P\{X_t \leq x,\ N_t > y\}$ can be calculated heuristically using the so-called reflection principle as follows (Harrison (1985)): For every sample path of X that hits level y before time t but finishes below level x at time t, there is another equally probable path that hits y before t and then travels upward at least y-x units to finish above level $y+(y-x)=2y-x$ at time t. Thus $P\{X_t \le x,\ N_t > y\} = P\{X_t \ge 2y-x\}$, and we obtain the following:

Proposition 2.1 If μ=0 and σ=1, then

$$P\{X_t + x_0 \le x,\ N_t^* \le y\} = \Phi\left((x-x_0)t^{-\frac{1}{2}}\right) - \Phi\left((x-2y+x_0)t^{-\frac{1}{2}}\right) \tag{1}$$

Corollary 2.2 If μ=0 and σ=1, then

$$P\{X_t + x_0 \in dx,\ N_t^* \le y\} = g_t(x,y)dx$$

where

$$g_t(x,y) = \left\{\phi\left((x-x_0)t^{-\frac{1}{2}}\right) - \phi\left((x-2y+x_0)t^{-\frac{1}{2}}\right)\right\}t^{-\frac{1}{2}} \tag{2}$$

3. A hitting time distribution

Returning to the analysis begun in section 2, we now use the change of measure theorem to calculate the joint distribution of $X_t + x_0$ and N_t^* in generality.

Proposition 3.1 For general values of μ and σ we have

$$P\{X_t + x_0 \in dx,\ N_t^* \le y\} = h_t(x,y)dx$$

where

$$h_t(x,y) = \frac{1}{\sigma}\exp\left(\frac{\mu(x+x_0)}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right) g_t\left(\frac{x-x_0(1-\sigma)}{\sigma}, \frac{y-x_0(1-\sigma)}{\sigma}\right) \tag{3}$$

and $g_t(x,y)$ is defined by (2).

Proof. Only the case σ=1 will be treated have; the extension to general σ is accomplished by a straightforward rescaling. Suppose initially that X is a standard Brownian motion on (Ω, F, P) so that

$$P\{X_t + x_0 \in dx,\ N_t^* \le y\} = \left\{\phi\left(\frac{x-x_0}{\sqrt{t}}\right) - \phi\left(\frac{x-2y+x_0}{\sqrt{t}}\right)\right\}t^{-\frac{1}{2}}dx \tag{4}$$

by (2). Now fix t >0, let $\mu \in R$ be arbitrary, set

$$\xi \equiv \exp\left(\mu(X_t + x_0) - \frac{1}{2}\mu^2 t\right)$$

and define a new probability measure P^* by taking $P^*(A) = \int_A \xi(\omega)P(d\omega),\ A \in F$. Where ξ is the Radon-Nikodym derivative P^* with respect to P(Billingsley (1986)). The change of measure theorem says that $\{X_s,\ 0 \le s \le t\}$ is a $(\mu, 1)$

Brownian motion under P^*, so the desired result is equivalently stated as

$$P^*\{X_t + x_0 \in dx,\ N_t^* \le y\}$$
$$= \exp\left(\mu(x+x_0) - \frac{\mu^2 t}{2}\right)\left\{\phi\left(\frac{x-x_0}{\sqrt{t}}\right) - \phi\left(\frac{x-2y+x_0}{\sqrt{t}}\right)\right\} t^{-\frac{1}{2}} dx \tag{5}$$

To simplify typography in the proof of (5), let us denote by $1(A)$ the random variable that has value 1 on A and value zero otherwise. Using (4),

$$P^*\{X_t + x_0 \le x,\ N_t^* \le y\}$$
$$= E^*\{1(X_t + x_0 \le x,\ N_t^* \le y)\}$$
$$= E\{\xi \cdot 1(X_t + x_0 \le x,\ N_t^* \le y)\}$$
$$= E\left\{\exp\left(\mu(X_t + x_0) - \frac{\mu^2 t}{2}\right)\cdot 1(X_t + x_0 \le x,\ N_t^* \le y)\right\}$$
$$= \int_{-\infty}^{x} \exp\left(\mu(z+x_0) - \frac{\mu^2 t}{2}\right) P\{X_t + x_0 \in dz,\ N_t^* \le y\}$$
$$= \int_{-\infty}^{x} \exp\left(\mu(z+x_0) - \frac{\mu^2 t}{2}\right)\left\{\phi\left(\frac{z-x_0}{\sqrt{t}}\right) - \phi\left(\frac{z-2y+x_0}{\sqrt{t}}\right)\right\} t^{-\frac{1}{2}} dz$$

Differentiating with respect to x gives (5) as required.

Theorem 3.2 Let $G_t(x,y) \equiv P\{X_t + x_0 \le x,\ N_t^* \le y\}$ as in section 2. For general values of μ and σ we have

$$H_t(x,y) = \exp\left(\frac{2x_0\mu}{\sigma^2}\right)\Phi\left(\frac{x-x_0-\mu t}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2\mu y}{\sigma^2}\right)\Phi\left(\frac{x-2y+x_0-\mu t}{\sigma\sqrt{t}}\right) \tag{6}$$

Proof. By specializing the general formula (3) for h accordingly, we obtain

$$H_t(x,y) = \int_{-\infty}^{x} h_t(z,y)dz$$
$$= \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{t}} \exp\left(\frac{\mu(z+x_0)}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right)\left\{\phi\left(\frac{z-x_0}{\sigma\sqrt{t}}\right) - \phi\left(\frac{z-2y+x_0}{\sigma\sqrt{t}}\right)\right\} dz$$
$$= \exp\left(-\frac{\mu^2 t}{2\sigma^2}\right)\int_{-\infty}^{x} \frac{1}{\sigma\sqrt{t}} \exp\left(\frac{\mu(z+x_0)}{\sigma^2}\right)$$
$$\times\left\{\phi\left(\frac{z-x_0}{\sigma\sqrt{t}}\right) - \phi\left(\frac{z-2y+x_0}{\sigma\sqrt{t}}\right)\right\} dz$$

$$= \exp\left(\frac{\mu(x+x_0)}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right)\int_{-\infty}^{0}\left\{\exp\left(\frac{\mu z}{\sigma^2}\right)\frac{1}{\sigma\sqrt{t}}\phi\left(\frac{z+x-x_0}{\sigma\sqrt{t}}\right)\right.$$

$$\left. -\exp\left(\frac{\mu z}{\sigma^2}\right)\frac{1}{\sigma\sqrt{t}}\phi\left(\frac{z+x-2y+x_0}{\sigma\sqrt{t}}\right)\right\}dz \tag{7}$$

$$= \exp\left(\frac{\mu(x+x_0)}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right)(\Psi(x-x_0) - \Psi(x-2y+x_0))$$

where

$$\Psi(x) = \int_{-\infty}^{0}\exp\left(\frac{\mu z}{\sigma^2}\right)\frac{1}{\sigma\sqrt{t}}\phi\left(\frac{z+x}{\sigma\sqrt{t}}\right)dz$$

How let $k(x,t) = (x-\mu t)/\sigma\sqrt{t}$, we have

$$\Psi(x) = \int_{-\infty}^{0}\exp\left(\frac{\mu z}{\sigma^2}\right)\frac{1}{\sigma\sqrt{t}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(z+x)^2}{2\sigma^2 t}\right)dz$$

$$= \exp\left(\frac{\mu^2 t}{2\sigma^2} - \frac{x\mu}{\sigma^2}\right)\int_{-\infty}^{0}\frac{1}{\sigma\sqrt{t}}\phi\left(\frac{z+x-\mu t}{\sigma\sqrt{t}}\right)dz$$

$$= \exp\left(\frac{\mu^2 t}{2\sigma^2} - \frac{x\mu}{\sigma^2}\right)\int_{-\infty}^{k(x,t)}\phi(u)\,du$$

$$= \exp\left(\frac{\mu^2 t}{2\sigma^2} - \frac{x\mu}{\sigma^2}\right)\Phi(k(x,t))$$

Substituting this into (7) gives the desired formula.

Corollary 3.3 If $x_0 = 0$, then

$$H_t(x,y) = \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2\mu y}{\sigma^2}\right)\Phi\left(\frac{x-2y-\mu t}{\sigma\sqrt{t}}\right) \tag{8}$$

for all $t \geq 0$.

If we define T_y as the first t at which $X_t + x_0 = y$, then obviously $T_y > t$ if and only if $N_t^* < y$.

Theorem 3.4 For general values of μ and σ we have

$$P(T_y > t) = \exp\left(\frac{2x_0\mu}{\sigma^2}\right)\Phi\left(\frac{y - x_0 - \mu t}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2\mu y}{\sigma^2}\right)\Phi\left(\frac{-y + x_0 - \mu t}{\sigma\sqrt{t}}\right) \tag{9}$$

for $y > 0$.

Proof. Letting $x \uparrow y$ in (6) gives

$$\begin{aligned} P\{T_y > t\} &= P\{N_t^* < y\} \\ &= H_t(y, y) \quad \text{for } y > 0 \end{aligned}$$

4. A distribution for $\mu - RBMPx_0$

Definition 4.1 If $\{X_t, t \geq 0\}$ is Brownian motion process with drift μ define an increasing process L and a positive process R by setting

$$L_t = \sup\{-X_s + x_0,\ 0 \leq s \leq t\} \qquad t \geq 0,\ x_0 \geq 0$$

and

$$\begin{aligned} R_t &= X_t + L_t \\ &= \sup\{(X_t - X_s + x_0,\ 0 \leq s \leq t)\} \qquad t \geq 0 \end{aligned} \tag{10}$$

is called regulated μ-Brownian motion process with a lower control barrier at x_0, which we abbreviate to μ.

We focus on $\mu - RBMPx_0$,which is Brownian motion on the positive half line with constant drift μ and constant diffusion coefficient σ^2, modified by an impenetrable reflecting barrier at x_0. A slight modification of the argument used in section 2 and 3 gives the joint distribution of X_t and L_t from which one can obviously calculate the distribution of R_t. But here is an easier way. Fix $t > 0$ and for $0 \leq s \leq t$ let $X_s^* = X_t - X_{t-s}$. Note that $X^* = \{X_s^*,\ 0 \leq s \leq t\}$ has stationary, independent increments with $X_0 = 0$ and $X_s^* \sim N(\mu s,\ \sigma^2 s)$ thus X^* is another $\mu - RBMPx_0$ Brownian motion with starting state zero. Combining this with (10), we get

$$\begin{aligned} R_t &= \sup\{(X_t - X_s + x_0,\ 0 \leq s \leq t)\} \\ &= \sup\{(X_t - X_{t-s} + x_0,\ 0 \leq s \leq t)\} \\ &= \sup\{(X_s^* + x_0,\ 0 \leq s \leq t)\} \\ &\sim \sup\{(X_s + x_0,\ 0 \leq s \leq t)\} \\ &= N_t^* \end{aligned}$$

(Here the symbol $\sim$ denotes equality in distribution.) Thus the distributions of R_t and N_t^* coincide for each fixed t, although the distributions of the complete processes R and N^* are very different.

The marginal distribution of N^* was displayed earlier in (9), and we obtain the following theorem:

Theorem 4.2 If R_t represent the state of $\mu - RBMPx_0$ at time t then

$$P(R_t \le z) = \exp\left(\frac{2x_0\mu}{\sigma^2}\right)\Phi\left(\frac{z - x_0 - \mu t}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2\mu z}{\sigma^2}\right)\Phi\left(\frac{-z + x_0 - \mu t}{\sigma\sqrt{t}}\right) \qquad (11)$$

for all $t \ge 0$.

Thus as $t \to \infty$,

$$P(R_t \le z) \to \begin{cases} \exp\left(\frac{2x_0\mu}{\sigma^2}\right) - \exp\left(\frac{2\mu z}{\sigma^2}\right) & \text{if } \mu < 0 \\ 0 & \text{if } \mu \ge 0 \end{cases} \qquad (12)$$

For $\mu < 0$ and x_0=0, the limit distribution of (12) is exponential with mean $\sigma^2/2|\mu|$.

5. Numerical results

Suppose the level of dam is a μ-Brownian motion process with a lower control barrier at x_0. The three values of formula (3) in Proposition 3.1, x, y and x_0 indicate as following:

x : proper level
y : dangerous level
x_0 : barrier level

The result of formula (3) is shown if Figure 1, given μ=14(1)18, σ=5, x=50, y=70 and x_0=2. Also, if we have two particular cases when μ=20, σ=6, x=25(5)45, y=70 and x_0=2 and μ=10, σ=4(0.5)6, x=50, y=70 and x_0=2, the results are plotted in Figure 2 and 3 respectively.

Figure 4 displays the values of $P(R_t \le z)$ for μ=-1, σ=2, t=1000, and various barrier points x_0. We can confirm that the converges to exponential distribution with mean 2 as x_0 goes to zero through Figure 4.

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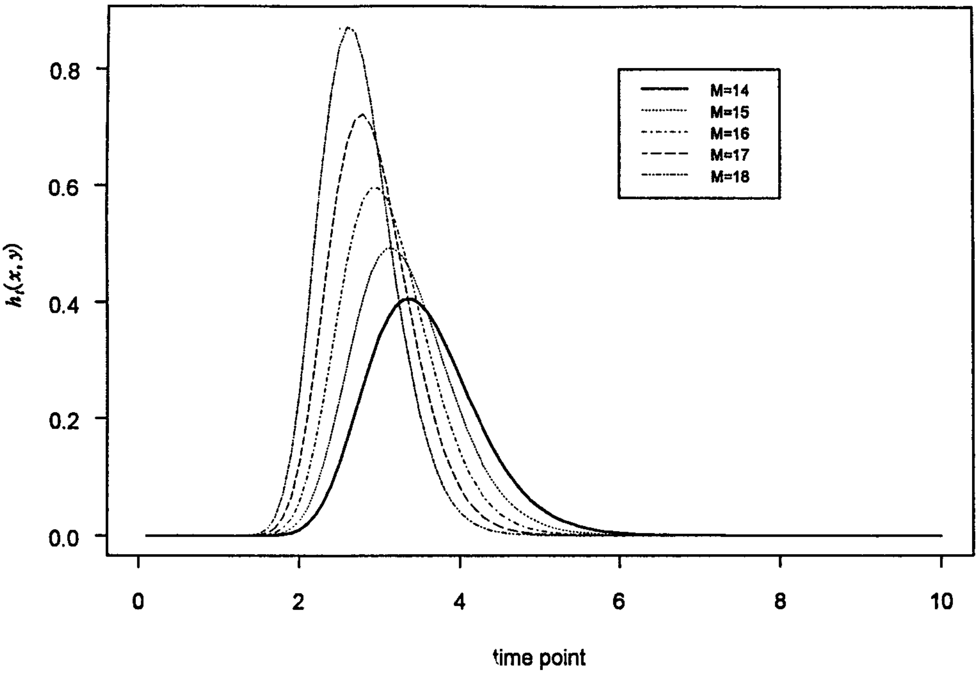

Figure 1 The probability of $h_t(x,y)$ $(\mu \equiv M)$

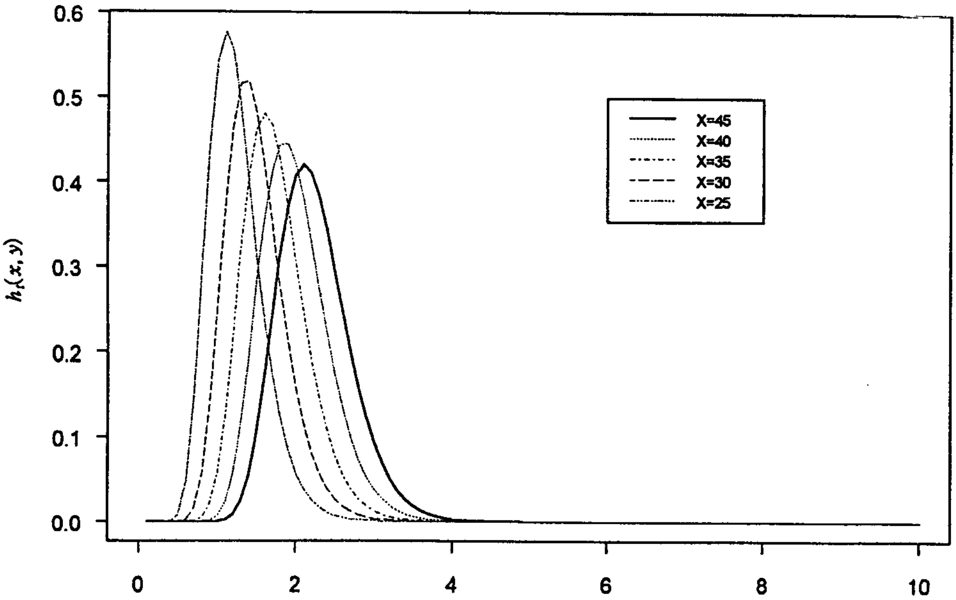

Figure 2 The probability of $h_t(x,y)$

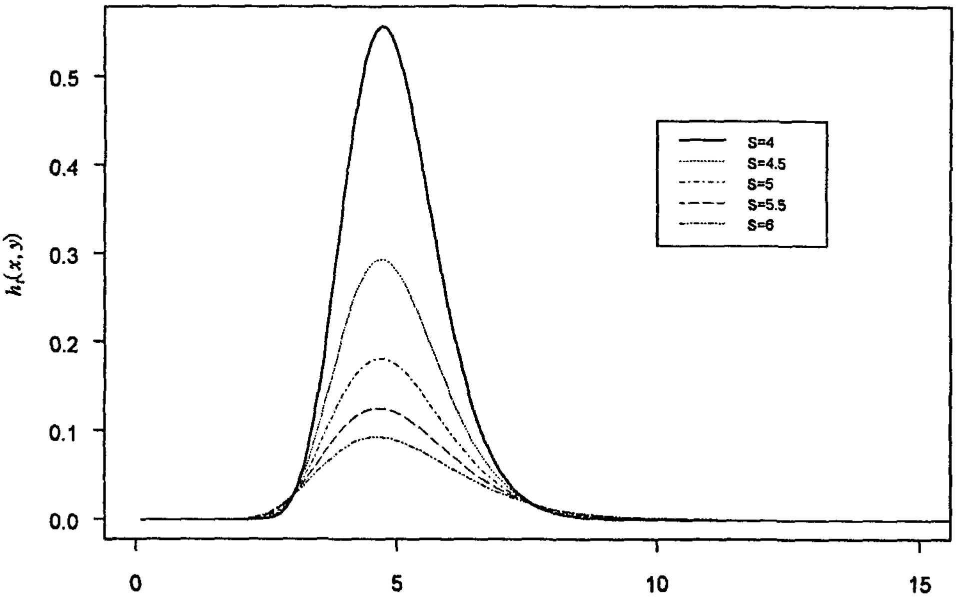

Figure 3 The probability of $h_t(x,y)$ $(\sigma \equiv s)$

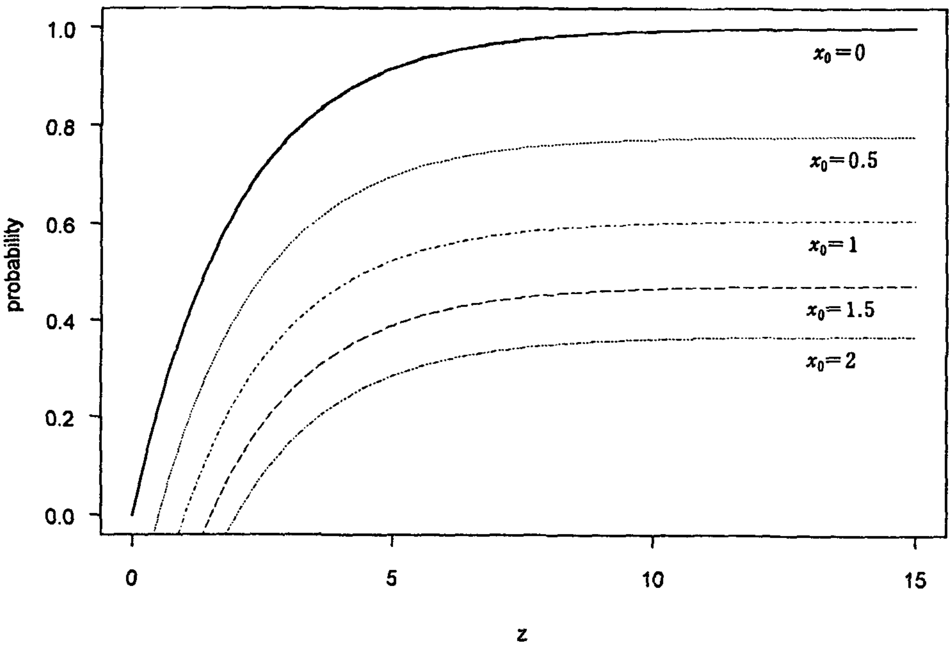

Figure 4 Limit distributions of μ-RBMPx_0

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