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The Counting Processes that the Number of Events in [0, t] has Generalized Poisson Distribution

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Abstract It is derived that the conditions of counting process $\{N(t) | t \geq 0\}$ in which the number of events in time interval [0, t] has a (n, n+1)-generalized Poisson distribution with parameters $(\theta t, \lambda)$ and a generalized inflated Poisson distribution with parameters $(\lambda t, \omega)$.

Keywords : (n, n+1)-generalized Poisson, generalized inflated Poisson

1. Introduction

Rao and Rubin (1964) introduced a generalized Poisson distribution with two parameters. This distribution has been attempted to take into account errors in recording a variable which in reality does have a Poisson distribution. Definitions of generalized Poisson distribution is the following:

Definition (Rao and Rubin) The discrete random variable X is said to be a generalized Poisson with parameters (θ, λ) if

$$\text{i) } P\{X = 0\} = e^{-\theta}(1 + \theta\lambda).$$

$$\text{ii) } P\{X = 1\} = \theta e^{-\theta}(1 - \lambda).$$

$$\text{iii) } P\{X = n\} = \frac{\theta^n}{n!} e^{-\theta}, \quad \text{for } n = 2,3,4,\cdots.$$

The generalized Poisson distribution is a Poisson distribution when $\lambda = 0$.

Cohen (1960a) introduced another generalized Poisson distribution which arises in a model representing a situation in which (with probability λ) a value (n+1) is classified as n. For convenience, this generalized Poisson distribution is called a (n,n+1)-generalized Poisson. The definition of a (n,n+1)-generalized Poisson is the following

Definition (Cohen) The discrete random variable X is said to be a (n,n+1)-generalized Poisson with parameters (θ, λ) if

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$$\text{i) } P\{X = k\} = e^{-\theta}\left(\frac{\theta^k}{k!}\right) \quad (k = 0,1,2,\cdots,n-1,n+2,n+3,\cdots)$$

$$\text{ii) } P\{X = n\} = e^{-\theta}\left(\frac{\theta^n}{n!}\right)\left\{1 + \frac{\lambda\theta}{(n+1)}\right\}.$$

$$\text{iii) } P\{X = n+1\} = e^{-\theta}\left(\frac{\theta^{n+1}}{(n+1)!}\right)(1-\lambda).$$

The generalized Poisson distribution defined by Rao and Rubin is (0,1)-generalized Poisson distribution and $(n,n+1)$-generalized Poisson distribution is Poisson distribution when $\lambda = 0$. Park (1995) introduced conditions that the number of events in time interval $[0,t]$ has (0,1)-generalized Poisson distribution with parameters $(\theta t, \lambda)$.

Singh (1966) introduced a generalized inflated Poisson distribution. Other names for this distribution are `pseudo-contagious' (Cohen, 1960b) and modified Poisson. The definition is the following;

Definition (Singh) *The discrete random variable X is said to be a generalized inflated* Poisson with parameters (λ, ω) if

$$\text{i) } P\{X = 0\} = \omega + (1-\omega)e^{-\lambda}$$

$$\text{ii) } P\{X = k\} = (1-\omega)e^{-\lambda}\left(\frac{\lambda^k}{k!}\right) \quad (k \geq 1).$$

2. Main results

The theorem for the conditions that the number of events in interval $[0,t]$ is a $(n,n+1)$-generalized Poisson distribution is developed as following:

Theorem 1 If the counting process $\{N(t) | t \geq 0\}$ is satisfying

1. $N(0) = 0$
2. $P\{N(t+h) - N(t) = 1 | N(t) = k\} = \theta h + o(h) \quad (k = 0,1,\cdots,n-1,n+2,\cdots)$
3. $P\{N(t+h) - N(t) = 1 | N(t) = n\} = \theta h + \dfrac{(n+1)\theta\lambda h}{n+1+\theta\lambda t} + o(h)$
4. $P\{N(t+h) - N(t) = 1 | N(t) = n+1\} = \dfrac{\theta h}{(1-\lambda)} + o(h)$
5. $P\{N(t+h) - N(t) \geq 2 | N(t) = y\} = o(h) \quad (y = 0,1,2,\cdots)$.

then the number of events in interval $[0,t]$ is a $(n,n+1)$-generalized Poisson distribution with parameters $(\theta t, \lambda)$.

Proof Let $P_k(t) = P\{N(t) = k\}$. Suppose $k \leq n-1$, then the differential equation for $P_k(t)$ is derived as following manner:

$$\begin{aligned}
P_k(t+h) &= P\{N(t+h)=k\} \\
&= P\{N(t)=k\}P\{N(t+h)-N(t)=0|N(t)=k\} \\
&\quad + P\{N(t)=k-1\}P\{N(t+h)-N(t)=1|N(t)=k-1\} \\
&\quad + \sum_{i=2}^{k} P\{N(t)=k-i\}P\{N(t+h)-N(t)=i|N(t)=k-i\} \\
&= P_k(t)\{1-\theta h+o(h)\} + P_{k-1}(t)\{\theta h+o(h)\} + o(h).
\end{aligned}$$

Hence,

$$\frac{P_k(t+h)-P_k(t)}{h} = -\theta P_k(t) + \theta P_{k-1}(t) + \frac{o(h)}{h}.$$

Letting $h \to 0$ yields

$$P_k'(t) = -\theta P_k(t) + \theta P_{k-1}(t) \tag{1}$$

When k=0, the differential equation provides

$$P_0'(t) = -\theta P_0(t).$$

The solution of the above differential equation is $P_0(t) = C_0 e^{-\theta t}$ and the condition $P_0(0) = 1$ implies that $C_0 = 1$. Thus

$$P_0(t) = e^{-\theta t} \tag{2}$$

When k=1, the differential equation (1) together with the result (2) provides

$$P_1'(t) = -\theta P_1(t) + \theta e^{-\theta t}.$$

whose general solution is $P_1(t) = \theta t e^{-\theta t} + C_1 e^{-\theta t}$. the boundary condition $P_1(0) = 0$ makes $C_1 = 0$. Thus

$$P_1(t) = \theta t e^{-\theta t}.$$

Now, by taking $k = 2,3,\cdots,n-1$ it can be shown by same method that the solution to these differential equation in (1) with the boundary conditions $P_0(k) = 0$ for $k = 2,3,\cdots,n-1$ are

$$P_k(t) = e^{-\theta t}\frac{(\theta t)^k}{k!}, \quad k = 2,3,\cdots,n-1.$$

Suppose k=n,

$$\begin{aligned}
P_n(t+h) &= P\{N(t+h)=n\} \\
&= P\{N(t)=n\}P\{N(t+h)-N(t)=0|N(t)=n\} \\
&\quad + P\{N(t)=n-1\}P\{N(t+h)-N(t)=1|N(t)=n-1\}
\end{aligned}$$

$$+\sum_{i=2}^{n} P\{N(t)=n-i\}P\{N(t+h)-N(t)=i|N(t)=n-i\}$$

$$=P_n(t)\{1-\theta h+\frac{(n+1)\theta\lambda h}{n+1+\theta\lambda t}+o(h)\}+P_{n-1}(t)\{\theta h+o(h)\}+o(h).$$

Hence,

$$\frac{P_n(t+h)-P_n(t)}{h}=\left(-\theta+\frac{(n+1)\theta\lambda}{n+1+\theta\lambda t}\right)P_n(t)+\theta P_{n-1}(t)+\frac{o(h)}{h}.$$

Letting $h\to 0$ yields

$$P_n'(t)=\left(-\theta+\frac{(n+1)\theta\lambda}{n+1+\theta\lambda t}\right)P_n(t)+\theta P_{n-1}(t). \tag{3}$$

From the differential equation (3),

$$P_n(t)=e^{-\int\left(\theta-\frac{(n+1)\theta\lambda}{n+1+\theta\lambda t}\right)dt}\int\theta\frac{(\theta t)^{n-1}}{(n-1)!}e^{\int\left(\theta-\frac{(n+1)\theta\lambda}{n+1+\theta\lambda t}\right)dt}dt$$

$$+C_n e^{-\int\left(\theta-\frac{(n+1)\theta\lambda}{n+1+\theta\lambda t}\right)dt}$$

$$=e^{-\theta t}(n+1+\theta\lambda t)^{n+1}\frac{\theta^n}{(n-1)!}\int t^{n-1}(n+1+\theta\lambda t)^{-(n+1)}dt$$

$$+C_n e^{-\theta t}(n+1+\theta\lambda t)^{n+1}$$

Since

$$\frac{\theta^n}{(n+1)!}\int t^{n-1}(n+1+\theta\lambda t)^{-(n+1)}dt$$

$$=-\frac{(\theta t)^{n-1}}{n!\lambda}(n+1+\theta\lambda t)^{-n}-\frac{(\theta t)^{n-2}}{n!\lambda^2}(n+1+\theta\lambda t)^{-n+1}$$

$$\cdots-\frac{1}{n!\lambda^n}(n+1+\theta\lambda t)^{-1}$$

$$=-\sum_{i=1}^{n}\frac{(\theta t)^{n-i}}{n!\lambda^i}(n+1+\theta\lambda t)^{-n-1+i},$$

then

$$P_n(t)=-e^{-\theta t}\sum_{i=1}^{n}\frac{(\theta t)^{n-i}}{n!\lambda^i}(n+1+\theta\lambda t)^i+C_n e^{-\theta t}(n+1+\theta\lambda t)^{n+1},$$

By boundary condition $P_n(0)=0$,

$$C_n = \frac{1}{(n+1)!\,\lambda^n}$$

Thus

$$P_n(t) = e^{-\theta t}\left(\frac{\theta^n t^n}{n!}\right)\left\{1 + \frac{\lambda\theta t}{(n+1)}\right\}.$$

Next, suppose $k=n+1$,

$$\begin{aligned}
P_{n+1}(t+h) &= P\{N(t+h) = n+1\} \\
&= P\{N(t) = n+1\}P\{N(t+h) - N(t) = 0 | N(t) = n+1\} \\
&\quad + P\{N(t) = n\}P\{N(t+h) - N(t) = 1 | N(t) = n\} \\
&\quad + \sum_{i=2}^{n+1} P\{N(t) = n+1-i\}P\{N(t+h) - N(t) = i | N(t) = n+1-i\} \\
&= P_{n+1}(t)\{1 - \frac{\theta h}{1-\lambda} + o(h)\} + P_n(t)\left\{\theta h - \frac{(n+1)\theta\lambda h}{n+1+\theta\lambda t}\right\} + o(h).
\end{aligned}$$

From

$$\begin{aligned}
P_n(t) &= e^{-\theta t}\left(\frac{\theta^n t^n}{n!}\right)\left\{1 + \frac{\lambda\theta t}{(n+1)}\right\} \\
P_{n+1}(t) &= e^{-\frac{\theta t}{1-\lambda}}\int\left\{\frac{(\theta t)^n}{n!} + \frac{\lambda(\theta t)^{n+1}}{(n+1)!}\right\}\left(\frac{(n+1)\theta(1-\lambda) + \theta^2\lambda t}{n+1+\theta\lambda t}\right)e^{\frac{\theta t}{1-\lambda}}dt \\
&\quad + C_{n+1}e^{-\frac{\theta t}{1-\lambda}} \\
&= \frac{(\theta t)^{n+1}}{(n+1)!}(1-\lambda)e^{-\theta t} + C_{n+1}e^{-\frac{\theta t}{1-\lambda}}.
\end{aligned}$$

By boundary condition $P_{n+1}(0) = 0, C_{n+1} = 0$, thus

$$P_{n+1}(t) = e^{-\theta t}\frac{(\theta t)^{n+1}}{(n+1)!}(1-\lambda)$$

Suppose $k=n+2$,

$$\begin{aligned}
P_{n+2}(t+h) &= P\{N(t+h) = n+2\} \\
&= P_{n+2}(t)\{1 - \theta t + o(h)\} + P_{n+1}(t)\left(\frac{\theta h}{1-\lambda t}\right) + o(h)
\end{aligned}$$

and

$$P_{n+2}(t) = e^{-\theta t}\int\frac{\theta(\theta t)^{n+1}}{(n+1)!}dt + C_{n+2}e^{-\theta t}.$$

By boundary condition $P_{n+2}(0) = 0$, we obtain $C_{n+2} = 0$. Thus

$$P_{n+2}(t) = e^{-\theta t}\frac{(\theta t)^{n+2}}{(n+2)!}.$$

Now, by taking k=n+3,n+4,$\cdots$ it can be shown by mathematical induction that the solution to equation (1) with boundary conditions $P_k(0) = 0$ for k=n+3,n+4,$\cdots$ are

$$P_k(t) = e^{-\theta t}\frac{(\theta t)^k}{k!}, \quad k = n+3, n+4, \cdots.$$

Hence the number of events in interval [0,t] has a (n,n+1)-generalized Poisson distribution with parameters $(\theta t, \lambda)$.

Next, we find conditions that the number of events in interval [0,t] is generalized inflated Poisson distribution.

Theorem 2 If the counting process $\{N(t) | t \geq 0\}$ is satisfied with

1. $N(0) = 0$
2. $P\{N(t+h) - N(t) = 1 | N(t) = 0\} = \left(\lambda - \dfrac{\lambda\omega}{\omega + (1-\omega)e^{-\lambda t}}\right)h + o(h)$
3. $P\{N(t+h) - N(t) = 1 | N(t) = n\} = \lambda h + o(h) \quad n \geq 1$
4. $P\{N(t+h) - N(t) \geq 2 | N(t) = y\} = o(h) \quad (y = 0,1,2,\cdots)$.

then the number of events in interval [0,t] is a generalized inflated Poisson distribution with parameter $(\lambda t, \omega)$.

Proof Let $P_n(t) = P\{N(t) = n\}$. The differential equation for $P_0(t)$ is derived as following manner:

$$\begin{aligned} P_0(t+h) &= P\{N(t+h) = 0\} \\ &= P\{N(t) = 0\}P\{N(t+h) - N(t) = 0 | N(t) = 0\} \\ &= P_0(t)\{1 - \lambda h + \frac{\lambda\omega h}{\omega + (1-\omega)e^{-\lambda t}} + o(h)\}. \end{aligned}$$

Hence,

$$\frac{P_0(t+h) - P_0(t)}{h} = -\left(\frac{\lambda(1-\omega)e^{-\lambda t}}{\omega + (1-\omega)e^{-\lambda t}}\right)P_0(t) + \frac{o(h)}{h}.$$

Letting $h \to 0$ yields

$$P_0'(t) = -\left(\frac{\lambda(1-\omega)e^{-\lambda t}}{\omega + (1-\omega)e^{-\lambda t}}\right)P_0(t).$$

which implies, by integration,

$$P_0(t) = C_0\{\omega + (1-\omega)e^{-\lambda t}\}.$$

Since $P_0(0) = P\{N(0) = 0\} = 1$, we obtain that $C_0 = 1$. Thus

$$P_0(t) = \omega + (1-\omega)e^{-\lambda t}.$$

When k=1, $P_1(t) = P\{N(t) = 1\}$.

$$\begin{aligned} P_1(t+h) &= P\{N(t+h) = 1\} \\ &= P\{N(t) = 1\}P\{N(t+h) - N(t) = 0 | N(t) = 1\} \\ &\quad + P\{N(t) = 0\}P\{N(t+h) - N(t) = 1 | N(t) = 0\} \\ &= P_1(t)\{1 - \lambda h + o(h)\} + P_0(t)\left(\lambda h - \frac{\lambda\omega h}{\omega + (1-\omega)e^{-\lambda t}} + o(h)\right). \end{aligned}$$

hence,

$$\frac{P_1(t+h) - P_1(t)}{h} = -\lambda P_1(t) + \lambda(1-\omega)e^{-\lambda t} + \frac{o(h)}{h}.$$

On taking the limit as $h \to 0$, we get the differential equation

$$P_1'(t) + \lambda P_1(t) = \lambda(1-\omega)e^{-\lambda t}. \tag{4}$$

The solution of the differential equation (4) is

$$P_1(t) = (1-\omega)\lambda t e^{-\lambda t} + C_1 e^{-\lambda t}.$$

and the condition $P_1(0) = 0$ implies that $C_1 = 0$. Thus

$$P_1(t) = (1-\omega)\lambda t e^{-\lambda t}. \tag{5}$$

Let $k \geq 2$,

$$\begin{aligned} P_k(t+h) &= P\{N(t+h) = k\} \\ &= P\{N(t) = k\}P\{N(t+h) - N(t) = 0 | N(t) = k\} \\ &\quad + P\{N(t) = k-1\}P\{N(t+h) - N(t) = 1 | N(t) = k-1\} \\ &\quad + \sum_{i=2}^{k} P\{N(t) = k-i\}P\{N(t+h) - N(t) = i | N(t) = k-i\} \\ &= (1-\lambda h)P_k(t) + \lambda h P_{k-1}(t) + o(h). \end{aligned}$$

Thus,

$$\frac{P_k(t+h) - P_k(t)}{h} = -\lambda P_k(t) + \lambda P_{k-1}(t) + \frac{o(h)}{h}.$$

On taking the limit as $h \to 0$, we get the differential equation

$$P_k'(t) + \lambda P_k(t) = \lambda P_{k-1}(t). \tag{6}$$

When k=2, the difference differential equation (6) with the result (5) provides

$$P_2'(t) + \lambda P_2(t) = (1-\omega)\lambda^2 t e^{-\lambda t}.$$

The solution of the above differential equation is

$$P_2(t) = (1-\omega)\frac{(\lambda t)^2}{2} e^{-\lambda t} + C_2 e^{-\lambda t}.$$

and the condition $P_2(0) = 0$ implies that $C_2 = 0$. Thus

$$P_2(t) = (1-\omega)\frac{(\lambda t)^2}{2} e^{-\lambda t}.$$

Now, by taking k=3,4,⋯ it can be shown by mathematical induction that the solution to the equation (6) with boundary conditions $P_k(0) = 0$ for k=3,4,⋯ are

$$P_k(t) = (1-\omega)\frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 3,4,\cdots.$$

Hence, the number of events in interval [0,t] has a generalized inflated Poisson distribution with parameters $(\lambda t, \omega)$.

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