⊠ 연구논문

An Evaluation of the Economic Design of Control Charts with Imprecise Information

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Abstract

This paper is concerned with designing $\overline{X}$-control charts when an estimate error may be inevitable. Estimate error often can not be avoided in estimating or measuring the parameter values of the cost model for the control charts. The bounded interval is a common practice to compensate for inherent estimating error. We introduce the propagation of error technique to deal with the economic design of the $\overline{X}$-control charts with imprecise information on the cost model parameters. A numerical example is presented to show its ability in the economic design of $\overline{X}$-control charts.

1. Introduction

Many useful control chart methodologies have been developed for quality control engineers to use for quality inspection. Most researchers developing these methodologies have paid primary attention to how to effectively improve a control chart only taking statistical criteria into account. This kind of control chart has the shortcoming of ignoring many of the related economic realities of the real world process. Therefore, the quality control engineers also need to look at the control chart from an economic point of view. In general, an economic control chart is constructed based on the concept of minimizing the expected total cost which is affected by the cost model parameters.

The problem when constructing an economic control chart is that the quality control engineers must collect data from the process being studied and estimate the parameter values. Let us take an economic design of an $\overline{X}$-control chart, specifically Duncan's model(1956). This model requires various parameter values such as the cost per unit of sampling and plotting that is independent of the sample size, the average cost per occasion of finding the assignable cause when it exists, the magnitude of the shift in the process mean when the assignable cause is present. None of these values are known with certainty, so they must be estimated. In estimating them the quality control engineers recognize that they are not precise and that their actual values will be somewhat different from the estimates. In order to reduce the impact of estimate errors, an allowance scheme is to make interval estimates such as ($c_j \pm \Delta c_j$) where Δc_j is the estimation error of an estimate c_j Though it is a quite common estimation approach, the problem is how to compute the aggregate error in the final economic merit due to individual estimate errors.

Many researchers such as Duncan(1956), Chiu and Wetherrill(1974), Goel, and Jain and Wu(1968) have studied the impact of estimate errors on the economic design of a control chart. Such studies have usually been made by investigating how the expected cost of constructing the control chart was affected by a change in one of the parameters in the cost model. One-at-a-time analyses are inappropriate to reflect the true realities of the real world. In the real world, more than one of the parameters in the cost model involve an estimate error. Therefore, quality control engineers need to know what the impact of composite estimate error is on the economic design of the control chart.

The purpose of this paper is to present an analytic error analysis for deriving the impact of composite estimate error on the economic design of a control chart when the parameters for the cost model involve estimate errors. First, we briefly review Duncans's model for the economic design of $\overline{X}$ control charts. Second, we introduce the propagation of error technique for the error analysis. Third, we present mathematical derivation for the impact of composite estimate error and a numerical example

2. Duncan's Economic Design Model

In this section we will briefly present the economic design of $\overline{X}$ control charts proposed by Duncan(1956).

Control charts are widely used in the manufacturing field to maintain statistical control of a process, which means differentiating between inevitable random causes and assignable causes of variability in the process. To construct the control charts quality control engineers must specify the combination of control chart design parameters (sample size n, width of the control limits k, and sampling frequency h). Depending on the scheme of the design parameters the design of the control charts has different economic consequences.

Some specific assumptions need to be made to formulate Duncan's economic model as follows:.

1) The production process is assumed to be in either one of two states: the "in-control" state ($E(X)=\mu=\mu_0$) or the "out-of-control" state defined by $\mu=\mu_0\pm\delta\sigma$ (note that σ is the true standard deviation and δ is the magnitude of shift in a process mean).
2) The length of time which the process remains in the "in-control" state is an exponential random variable with mean $1/\lambda$ given that it begins in the control state. (λ : average number of an assignable cause to occur).
3) There is only a single assignable cause for the shift in the process mean to the out-of-control state.
4) If the process is in the "out-of-control" state, it remains in operation during the search for the assignable cause.
5) The process is not self-correcting.

There are three categories of costs making up the expected total cost. First, the cost of sampling and testing is assumed to consist of both fixed and variable components. Owing to the difficulty of obtaining and evaluating cost information the use of more complex relationships is probably inappropriate. Second, the costs associated with producing defective items are covered by warranties or guarantees Third is the costs of investigating and possibly correcting the process following an action signal. The expected cost per unit time is computed as the ratio of expected cost incurred during a cycle to the expected cycle in unit time(1980).

For simplicity, we employ the result of the expected cost per hour incurred using the process proposed by Duncan(1982) as follows:

$$E(L)=(a_1+a_2n)/h+\{\lambda Ba_4+(\alpha a_3'/h)+\lambda a_3\}/(1+\lambda B) \qquad (1$$

where

$$B=\{1/(1-\beta)-1/2+\lambda h/12\}h+gn+D$$

We define

a_1 : the fixed cost of sampling.

a_2 : the variable cost of sampling.

a_3 : the cost of finding an assignable cause.

a_3' : the cost of investigating a false alarm.

a_4 : the hourly penalty cost for operating in the out-of-control state.

α : probability of looking for an assignable cause when it does not exist.

$$\cong 2\int_{k}^{\infty} 1/(2\pi)^{1/2}\exp(-z^2/2)dz$$

1-β : probability of looking for an assignable cause.

$$\cong \int_{k-\delta n^{1/2}}^{\infty} 1/(2\pi)^{1/2}\exp(-z^2/2)dz$$

λ : average number of an assignable cause to occur.

D : the time required to find an assignable cause.

g : the time required to sample one item and interpret the results.

It is noted that this economic cost function is the expected cost per hour incurred by the process. Equation (1) is a function of the control chart parameters n, k and h. The detail and simplified schemes of obtaining the optimum values of n, k and h by the minimization of $E(L)$ have been discussed by Chiu and Wetherrill (1977), Duncan (1956), and Goel, et al (1968). In this paper, we use the Chiu and Wetherrill's simplified scheme in which by constraining the power of the chart (1-β) to a specified value (0.90 or 0.95) the optimal n and k can be approximated by the solution to

$$(Z+k)/\phi(k)=\delta^2 a_3'/(a_2+\lambda a_4 g) \tag{2}$$

and

$$\delta(n^{1/2})-k=z \tag{3}$$

Where

$$z = 1.2826 \text{ if } 1-\beta = 0.90$$
$$= 1.6499 \text{ if } 1-\beta = 0.95$$
$$\phi(k)=1/(2\pi)^{1/2}\exp(-k^2/2).$$

Given n and k, the optimal value h is computed using the following equation.

$$h=[(a_3'-a_1+a_2 n)/\lambda a_4\{1/(1-\beta)-0.5\}]^{1/2} \tag{4}$$

The propagation of error technique will then be applied to equation (1), (2), (3) and (4) to derive the impact of estimation errors on the expected cost.

3. Propagation of Error Technique

In this section, we will present the different error analysis techniques and give some reason why we use the propagation of error technique instead of the others.

The main purpose of error analysis is to find Δy, most probable error of the function $y=f(c_1, .. , c_n)$ when each variable c_j has a bounded interval of $(c_j \pm \Delta c_j)$ where Δc_j is the estimation error of c_j. There are two approaches to obtain the composite error Δy due to individual errors Δc_js. One is the approximation approach using the total differential; the other is the statistical approach using the propagation of errors.

By means of the total differential, the composite error is defined as

$$\Delta y = \sum_{j=1}^{n} |\partial f / \partial c_j| \Delta c_j \tag{5}$$

But it is not certain whether the effect of each individual error is to increase or decrease the combined error, which is a matter of randomness. In other words, we are at a loss as to how to recognize when to use either $+\Delta c_j$ or $-\Delta c_j$. They may have equal chances of occurring. Therefore, the composite error Δy computed using the total differential may render a much larger amount than what it should be. While we would always like to quote the greatest error, we will usually find this hope frustrated. But even if such an error were found, it would be of little value if it were very large and of rare occurrence. What is of value is the shape of the distribution curve, particularly within some reasonable range of its peak which might include from 70 to 90% of the number of readings that are taken (1966). Consequently, the total differential will not be used in our analysis.

The propagation of error technique was proposed by Pugh and Winslow (1966). They point out that the purpose of the propagation of errors is to answer the question "Given some set of numbers and their errors, what are the errors in some prescribed function involving these numbers?" Since the interval of a distribution is proportional to its standard deviation, they obtained the propagation of errors of a function $y=f(c_1, \ldots, c_n)$ in the statistical fashion:

$$\sigma_y^2 = \sum_{j=1}^{n} \{(\partial f / \partial c_j) \sigma_{c_j}\}^2 \tag{6}$$

where σ_{c_j} is the standard deviation of variable c_j. The value of σ_y obtained from Equation (6) is an approximation in general Accordingly, if we use σ_y and Δy interchangeably, and use σ_{c_j} and Δc_j interchangeably, the propagation of errors equation is rewritten as

$$(\Delta y)^2 = \sum_{j=1}^{n} \{ (\partial f / \partial c_j) \Delta c_j \}^2 \quad (7)$$

Now it may be helpful to restate our reason to employ the propagation of error instead of the total differential in the error analysis: i) we do not have to consider the direction of individual errors (i.e., $+\Delta c_j, -\Delta c_j$) because the propagation of error equation (7) consists only of squares; ii) the composite error Δy which is computed by equation (7) is always smaller than that computed by the total differential equation (5) because the errors of opposite directions cancel each other internally (1966). For example, when we add two estimates ($a \pm \Delta a$) and ($b \pm \Delta b$), the composite error computed by equation (7) is $(\Delta a^2 + \Delta b^2)^{1/2}$, whereas it is ($\Delta a + \Delta b$) by equation (5). Further, the comment by Pugh and Winslow may be helpful (1966): "People not trained in statistical procedures often use equations like (5) instead of (7) when it is the latter that they should use. An equation like (5) always gives a more pessimistic statement of the propagated error than does (7) Although it is better to be too pessimistic than to be too optimistic in reporting the magnitude of the errors in one's experimental results, it is a disservice to one's colleagues to report a much larger error than is warranted."

4. Evaluation of a Total Cost with Imprecise Information

In this section, we will derive the composite errors associated with the expected cost function $E(L)$, a width of control limits k, a sample size n, and a sampling frequency h by applying a propagation of error technique. We will also present a simple illustrative example.

The parameter values used to construct control charts may be imprecisely measured in the real world due to several sources of estimation errors such as inherent variability in the process, and the measurement instruments. The imprecise parameter values have an impact on the derivation of the width of control limits, the sample size, and the sampling frequency, eventually causing the expected total cost to be unknown with certainty. Therefore, an evaluation of the expected total cost with consideration of the estimation errors is required for quality control engineers working in a practical field.

Looking at equation (1), we recognize that the expected cost is affected by the width of control limits, the sample size, and the sampling frequency, all of which are also dependent on the cost estimates. Thus, we need to derive the composite error with regard to each parameter(k, n, and h) and thereafter the expected cost function $E(L)$. Applying equation (7) to equation (2), (3), (4), and (1) results in equations (8), (9), (10), and (11), respectively: It is straightforward to derive the other equations except for equation (8). Since equation (2) is implicitly defined with respect to variable "k", we employed an implicit partial differentiation technique to derive equation (8) [see Appendix].

$$\Delta^2 k = \{\phi(z)\delta/C\}^2 [\{(2a_3\Delta\delta)^2 + (\delta\Delta a_3)^2\} + \{\delta a_3/(a_2+\lambda a_4 g)\}^2 \{\Delta^2 a_2 + (a_4 g\Delta\lambda)^2 + (\lambda g\Delta a_4)^2 + (\lambda a_4 \Delta g)^2\}] \quad (8)$$

where $C = (k(z+k)+1)(a_2+\lambda a_4 g)$

$$\Delta^2 n = \{2(z+k)/\delta\}^2 \{\Delta^2 k + (\Delta\delta/\delta)^2\} \quad (9)$$

$$\Delta^2 h = (\Delta^2 a_3' + \Delta^2 a_1 + n^2\Delta^2 a_2 + a_2^2\Delta^2 n)/\{4(a_3'+a_1+a_2 n)(\lambda a_4 P)\} + \{P^2(a_3'+a_1+a_2 n)(a_4^2\Delta^2\lambda + \lambda^2\Delta^2 a_4)\}/4(\lambda a_4 P)^3 \quad (10)$$

Where $P = (1/(1-\beta) - 0.5)$

When deriving the composite error with respect to the width of control limits, we employed an implicit partial differentiation technique together with the propagation of error technique. Also note that the z value is assumed to be fixed at 1.2826 as suggested in [Chiu, wetherill, 1974] and [Montgomery, 1982]

We can finally derive the composite error of the expected total cost with equation (8), (9) and (10). Then, the composite error equation is given by

$$\Delta^2 E(L) = (\Delta^2 a_1 + n^2\Delta^2 a_2 + a_2^2 + a_2^2\Delta n)/h^2 + \{1/(1/(1+\lambda B)^2\}\{B^2 a_4^2 \Delta^2 \lambda$$
$$+ \lambda^2 B^2 \Delta^2 a_4 + (\alpha/h)^2 \Delta^2 a_3' + (\alpha a_3'^{/h^2})^2 \Delta^2 h + a_3^2 \Delta^2 \lambda$$
$$+ \lambda^2 \Delta^2 a_3\} + \{1/(1+\lambda B)^4\}\{\lambda^2 a_4^2 (1/(1-\beta) - 0.5$$
$$+ \lambda h/6)^2 \Delta^2 h + (B + \lambda h^2/12)^2 \Delta^2 \lambda\} + \{A^2/(1+\lambda B)^4\}$$
$$(n^2 \Delta^2 g + g^2 \Delta^2 n + \Delta^2 D) \quad (11)$$

where $A = (\lambda B a_4 + \alpha a_3'/h + \lambda a_3)$

Now we will show a simple illustrative example to demonstrate how to calculate the impact of the estimate errors on the expected total cost function when designing control charts. The data for the example was taken from Montgomery's paper (1982). To solve the example, we modified the computer program by adding the composite error portion to it.

To construct control charts, quality control engineers need to estimate the values of the cost model as follows (See 1, 3, 7 and 9 for more detail):

a_1=\$1.00, a_2=\$0.10, a_3=\$25, a_3'=\$50, a_4=\$100,

λ=0.05, δ=2.0, g=0.0167 hour, D=1 hour.

The optimal values of k, n, and h are 2.99, 5 and 0.76 hours, respectively, with $E(L)$ of \$10.38/hour.

However, since a point estimation may not represent the true realities of the process under study, for illustrative purposes it is assumed in this paper that each estimated value involves 10% error. Then, the ranges of error for k, n, h and $E(L)$ were obtained as follows:

$\Delta k = 0.0439$, $2.9461 \le k \le 3.0339$

$\Delta n = 0.4667$ $4.5333 \le n \le 5.4667$

$\Delta h = 0.3633$ $0.3967 \le h \le 1.1233$

$\Delta E(L) = 1.5809$ $8.7991 \le E(L) \le 11.9609$

5. Conclusion

As preseneted in this paper, estimate is needed of the error of a quantity that is a function of several random variables. For this purpose, we employed the progation of errors technique to derive the composite error of an economic design model for $\overline{X}$-control charts assuming that information was imprecise. It should be noted that the resulting value for the composite error derived by using the technique suggested in this paper is not exact, but only approximate. If it is true that some error is involved in estimating parameters and the function to be handled is complex, this approach is useful for evaluating the expected cost when constructing $\overline{X}$-control charts and can be extensively used for other control charts. As a further research, we recommend to investigate what combination of the values of *n, k,* and *h* really minimizes the expected cost for an economic design of $\overline{X}$-control charts.

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APPENDIX

We employed an implicit partial differentiation technique to derive equation (8) because it is implicitly defined with respect to variable "*k*".

$$(z + k)/\phi(k) = \delta^2 a_3/(a_2 + \lambda a_4 g) \tag{1}$$

Equation (1) is rearranged like equation (2) regarding to variable "*k*".

$$F = (z + k)/\phi(k) = \delta^2 a_3/(a_2 + \lambda a_4 g) \tag{2}$$

where

$$\phi(k) = 1/(2\pi)^{1/2} e^{-(k^2/2)}$$

To derive the error value of k, we need to take a partial derivative of equation (2) with respect to all of the variables involved as followings:

i) $\partial k/\partial\delta = -F_\delta/F_k$

$$F_\delta = -2\delta a_3/a_2 + \lambda a_4 g$$

$$F_k = [\phi(k) - (z+k)\phi(k)']/\phi(k)^2$$

where

$$\begin{aligned}\phi(k)' &= (1/(2\pi)^{1/2} e^{-k^2/2})' \\ &= 1/(2\pi)^{1/2} e^{-k^2/2}(-2k/2) \\ &= -k/(2\pi)^{1/2} e^{-k^2/2} \\ &= -k\phi(k)\end{aligned}$$

$$F_k = [\ \phi(k) + k(z + k)\phi(k)]/\phi(k)^2$$

$$= [\ k(z + k) + 1]/\phi(k)$$

$$\therefore \quad \partial k/\partial \delta = 2\delta a_3\phi(k)/\ \{(a_2 + \lambda a_4 g)(k(z + k) + 1)\}$$

$$= 2\phi(k)\delta a_3/\ (\lambda a_4 g)$$

ii) $\partial k/\partial a_3 = -F_{a_3}/F_k$

$$F_{a_3} = -\delta^2/(a^2 + \lambda a_4 g)$$

$$F_k = \{k(z + k) + 1\}\ /\ \phi(k)$$

$$\therefore \quad \partial k/\partial a_3 = \phi(k)\delta^2/\ \{(k(z+k)+1)(a_2 + \lambda a_4 g)\}$$

iii) $\partial k/\partial a_2 = -F_{a_2}/F_k$

$$F_{a_2} = \delta^2 a_3/\ (a_2 + \lambda a_4 g)^2$$

$$F_k = \{k(z+k)+1\}\ /\ \phi(k)$$

$$\therefore \quad \partial k/\partial a_2 = -\ \phi(k)\delta^2 a_3/\ \{(k(z+k)+1)(a_2 + \lambda a_4 g)^2\}$$

iv) $\partial k/\partial \lambda = -F_\lambda/F_k$

$$F_\lambda = \delta^2 a_3 a_4 g/\ (a_2 + \lambda a_4 g)^2$$

$$F_k = \{k(z+k)+1\}\ /\ \phi(k)$$

$$\therefore \quad \partial k/\partial \lambda = -\ \phi(k)\delta^2 a_3 a_4 g\ /\ \{(k(z+k)+1)(a_2 + \lambda a_4 g)^2\}$$

v) $\partial k/\partial a_4 = -F_{a_4}/F_k$

$$F_{a_4} = \delta^2 a_3 \lambda g/\ (a_2 + \lambda a_4 g)^2$$

$$F_k = \{k(z+k)+1\}\ /\ \phi(k)$$

$$\therefore \quad \partial k/\partial a_4 = -(\ \phi(k)\delta^2 a_3 \lambda g)\ /\ \{(k(z+k)+1)(a_2 + \lambda a_4 g)^2\}$$

vi) $\partial k/\partial g = -F_g/F_k$

$$F_g = \delta^2 a_3 \lambda a_4/\ (a_2 + \lambda a_4 g)^2$$

$$F_k = \{k(z+k)+1\}\ /\ \phi(k)$$

$$\therefore \quad \partial k/\partial g = -\ \phi(k)\delta^2 a_3 \lambda a_4/\ (k(z+k)+1)(a_2 + \lambda a_4 g)^2$$

$$\therefore (\Delta k)^2 = (\partial k/\partial\delta)^2 + (\partial k/\partial a_3)^2 + (\partial k/\partial a_2)^2 + (\partial k/\partial\lambda)^2 + (\partial k/\partial a_4)^2 + (\partial k/\partial g)^2$$

$$= \{2\phi(k)\delta a_3\Delta\delta/(k(z+k)+1)(a_2+\lambda a_4 g)\}^2$$
$$+ \{\phi(k)\delta^2\Delta a_3/(k(z+k)+1)(a_2+\lambda a_4 g)\}^2$$
$$+ \{\phi(k)\delta^2 a_3\Delta a_2/(k(z+k)+1)(a_2+\lambda a_4 g)^2\}^2$$
$$+ \{\phi(k)\delta^2 a_3 a_4 g\Delta\lambda/(k(z+k)+1)(a_2+\lambda a_4 g)^2\}^2$$
$$+ \{\phi(k)\delta^2 a_3\lambda g\Delta a_4/(k(z+k)+1)(a_2+\lambda a_4 g)\}^2$$
$$+ \{\phi(k)\delta^2 a_3\lambda a_4\Delta g/(k(z+k)+1)(a_2+\lambda a_4 g)^2\}^2$$
$$= \{\phi^2(k)\delta^2(4\,a_3{}^2\Delta^2\delta+\delta^2\Delta^2 a_3)\}/c^2$$
$$+ \{\phi(k)\delta^4\,a_3{}^2(\Delta^2 a_2+\,a_4{}^2 g^2\Delta^2\lambda+\lambda^2 g^2\Delta^2 a_4+\lambda^2\,a_4{}^2\Delta g^2)\}/c'^2$$
$$= (\phi^2(k)\delta^2)/c^2\,[(4\,a_3{}^2\Delta^2\delta+\delta^2\Delta^2 a_3)+\{(\delta^2\,a_3{}^2)/(a_2+\lambda a_4 g)^2\}$$
$$\{\Delta^2 a_2+\,a_4{}^2 g^2\Delta^2\lambda+\lambda^2 g^2\Delta^2 a_4+\lambda^2\,a_4{}^2\Delta g^2\}\,]$$

where

$$c = (k(z+k)+1)(a_2+\lambda a_4 g)$$
$$c' = (k(z+k)+1)(a_2+\lambda a_4 g)^2$$