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The Useful Techniques to Determine the Prior Odds and the Likelihood Ratios for Bayesian Processor in Built-In-Test System+

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Abstract

It is very important to determine the likelihood ratios and the prior odds for designing a Bayesian processor in Built-In-Test system. Using traditional statistics, it is not difficult to determine the initial prior odds from the field data. For a newly designed system, development testing data or laboratory testing data could be used to replace field data. The likelihood ratios which play a key role in the Bayesian processor must be carefully determined, based on laboratory testing and statistical techniques. In this paper, expressing and determining the likelihood ratios by Geometric areas, Test, and Analytical method will be presented.

1. Introduction

1.1 General Concept of Built-In-Test and Bayesian Processor

The term "Built-In-Test" refers to a subsystem whose major purpose is to test the operating state of the primary system[Albert, 1986]. Briefly, BIT is the hardware and software that are integrated into a system to perform fault detection, diagnosis and isolation, and failure recording, along with possible reconfiguration or failure management[Lamberson, Shao, 1989].

The major shortcomings of BIT are false alarms and lack of fault coverage such as diagnostic problems. These shortcomings must be recognized as a very complex problem which involves many aspects[Malcolm, 1982].

The Bayesian processor allows us to use every piece of information we can get,

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and to obtain the probability of failure after the n^{th} test. The BIT false alarm probability could, therefore, be greatly reduced[15, 16, 17, 18].

1.2 The Major Function of Prior Odds and Likelihood Ratio in the Mathematical Models for the Bayesian Processor

In the general case of a test result (T: either positive T^+ or negative T^-), we can derive $P(F \mid T)$ and $P(\overline{F} \mid T)$ at one of the thresholds. From the definition of conditional probability and the multiplication rule, the probability that the primary system has failed at one of the thresholds given a test result can be expressed as

$$P(F \mid T) = \frac{P(F, T)}{P(T)} = \frac{P(T \mid F) \cdot P(F)}{P(T)}$$

and the probability that the primary system has not failed at one of the thresholds given a test result as

$$P(\overline{F} \mid T) = \frac{P(\overline{F}, T)}{P(T)} = \frac{P(T \mid \overline{F}) \cdot P(\overline{F})}{P(T)}$$

Hence,

$$\begin{aligned} \frac{P(F \mid T)}{P(\overline{F} \mid T)} &= \frac{P(T \mid F)}{P(T \mid \overline{F})} \cdot \frac{P(F)}{P(\overline{F})} \\ &= L \cdot O_0 \end{aligned}$$

Where

$$L = \frac{P(T \mid F)}{P(T \mid \overline{F})}$$

is called the likelihood ratio, and

$$O_0 = \frac{P(F)}{P(\overline{F})}$$

is called the prior initial failure odds[Kapur, Lamberson, 1977].

Therefore,

$$\left\{ \begin{matrix} \textit{Posterior} \\ \textit{Odds of Failure} \end{matrix} \right\} = \left\{ \begin{matrix} \textit{Likelihood} \\ \textit{Ratio} \end{matrix} \right\} \times \left\{ \begin{matrix} \textit{Prior} \\ \textit{Odds of Failure} \end{matrix} \right\}$$

The likelihood ratio, L, represents: (1) the ratio of the true positive rate to the false positive rate for a BIT failure at either the lower or upper threshold (L_l' or L_u); or (2) the ratio of the false negative rate to the true negative rate for a BIT pass at either the lower or upper threshold (L_l or L_u).(1981)

If the prior odds and the likelihood ratio $L(L_n = n^{th}$ *likelihood ratio*) are given, the posterior odds after the n^{th} test will be readily obtained as :

$$O_n = L_n \cdot O_{n-1}$$

$$\frac{P_n(F \mid T_n)}{P_n(\bar{F} \mid T_n)} = \frac{P_{n-1}(F \mid T_{n-1})}{P_{n-1}(\bar{F} \mid T_{n-1})}, \quad n = 1, 2, \cdots$$

Consequently, the probability that the system is faulty after the n^{th} test can be found.

2. Determining and Expressing the Prior Odds and the Likelihood Ratios

2.1 Prior Odds

For a given point of time t, it is not difficult, using traditional statistics, to determine the initial prior odds O_0 from the field data. For a newly designed system, the field data may not be available. Development testing data or laboratory testing data could be used to replace field data as an approximation[2, 3].

In references [9, 10], Malcolm treated the initial prior odds of failure, O_0, as a constant. This is not correct, because

$$O_0 = \frac{P(F)}{P(\bar{F})} = \frac{F(t_0)}{1-F(t_0)} = \frac{1-R(t_0)}{R(t_0)}, \quad t_0 > 0$$

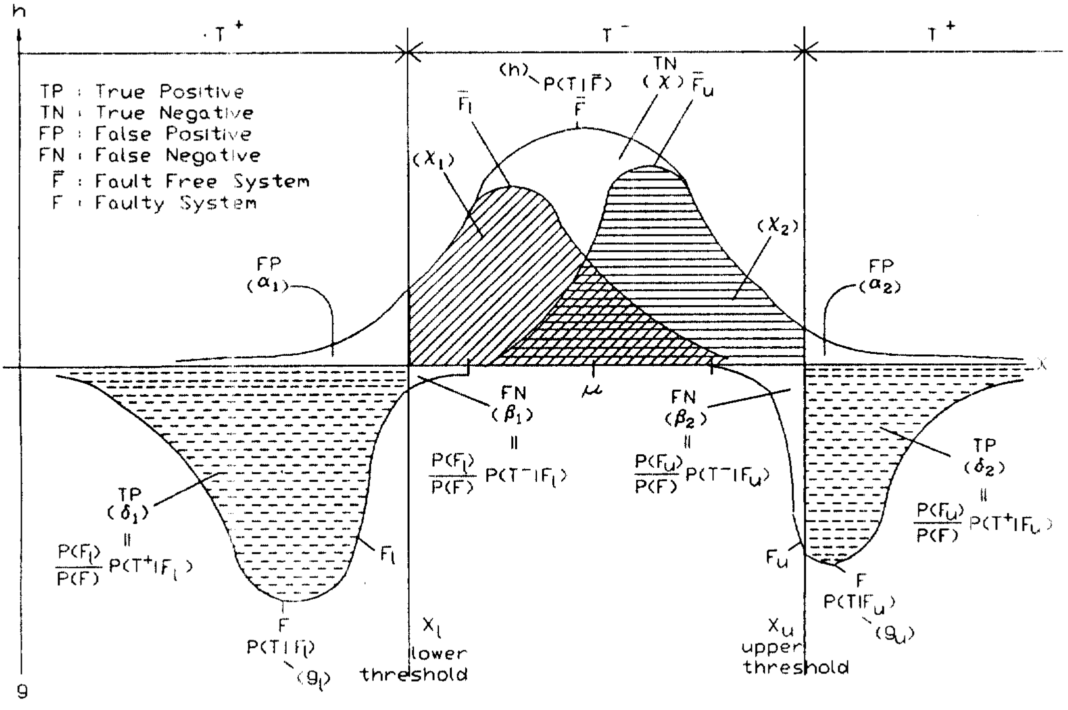
it is dependent on time t_0, where $R(t_0)$ is the reliability of the system at test start time t_0. Initial prior odds will not affect the final test result for a Bayesian processor if enough test times are taken. A plenty of test times will fine-tune the suspicious initial prior odds as posterior odds gains. However, the accuracy of the initial prior odds does affect the test times and the effectiveness of a Bayesian processor[13, 14].

2.2 Likelihood Ratios

The likelihood ratios which play a key role in the Bayesian processor must be carefully determined, based on laboratory testing and statistical techniques.

2.2.1 Expressing the Likelihood Ratios by Geometric Areas on BIT System

To deal with the likelihood ratios, we note that the total test result distribution for a fault-free system is composed of a test result distribution near the lower threshold ($\overline{F}_l$ case) and a test result distribution near the upper threshold ($\overline{F}_u$ case)[Lord, Gleason, 1981]. (For the notations that are not explicitly defined in this section you may refer to Figure 1.)



〈 Figure 1 〉 Test Result PDF and the Virtual Distributions for General Cases

Then we have the following identity :

$$\alpha_1 + \alpha_2 + x_1 + x_2 = 1 \qquad (1)$$

where

$$\alpha_1 = P(T^+ \mid \bar{F}_l) \cdot \frac{P(\bar{F}_l)}{P(\bar{F})}$$

and

$$\alpha_2 = P(T^+ \mid \bar{F}_u) \cdot \frac{P(\bar{F}_u)}{P(\bar{F})}$$

It is also reasonable to assume that

$$\frac{\alpha_1}{x_1} = \frac{\alpha_2}{x_2} \tag{2}$$

From (2) we can readily get the following relations :

$$\frac{\alpha_1}{\alpha_1 + x_1} = \frac{\alpha_2}{\alpha_2 + x_2} = \frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2 + x_1 + x_2} = \alpha_1 + \alpha_2 \tag{3}$$

$$\frac{x_1}{\alpha_1 + x_1} = \frac{x_2}{\alpha_2 + x_2} = \frac{x_1 + x_2}{\alpha_1 + \alpha_2 + x_1 + x_2} = x, \quad \text{where } x = x_1 + x_2 \tag{4}$$

Noting that

$$\begin{aligned} P(T^+ \mid \bar{F}) &= P(T^+ \mid \bar{F}_l) \cdot \frac{P(\bar{F}_l)}{P(\bar{F})} + P(T^+ \mid \bar{F}_u) \cdot \frac{P(\bar{F}_u)}{P(\bar{F})} \\ &= \alpha_1 + \alpha_2, \end{aligned}$$

We have

$$\begin{aligned} P(T^+ \mid \bar{F}_l) &= \alpha_1 \cdot \frac{P(\bar{F})}{P(\bar{F}_l)} \\ &= \alpha_1 \cdot \frac{1}{\alpha_1 + x_1} = \alpha_1 + \alpha_2 \end{aligned} \tag{5}$$

and

$$\begin{aligned} P(T^+ \mid \bar{F}_u) &= \alpha_2 \cdot \frac{P(\bar{F})}{P(\bar{F}_u)} \\ &= \alpha_2 \cdot \frac{1}{\alpha_2 + x_2} = \alpha_1 + \alpha_2 \end{aligned} \tag{6}$$

Also, noting that

$$P(T^+ \mid F) = P(T^+ \mid F_l) \cdot \frac{P(F_l)}{P(F)} + P(T^+ \mid F_u) \cdot \frac{P(F_u)}{P(F)}$$

$$\equiv \delta_1 + \delta_2$$

We have

$$P(T^+ \mid F_l) = \delta_1 \cdot \frac{P(F)}{P(F_l)} \tag{7}$$

and

$$P(T^+ \mid F_u) = \delta_2 \cdot \frac{P(F)}{P(F_u)} \tag{8}$$

Therefore

$$L_l^+ = \frac{P(T^+ \mid F_l)}{P(T^+ \mid \bar{F}_l)} = \left(\delta_1 \cdot \frac{P(F)}{P(F_l)} \right) / (\alpha_1 + \alpha_2) \tag{9}$$

and

$$L_u^+ = \frac{P(T^+ \mid F_u)}{P(T^+ \mid \bar{F}_u)} = \left(\delta_2 \cdot \frac{P(F)}{P(F_u)} \right) / (\alpha_1 + \alpha_2) \tag{10}$$

Let us further define L^+ as the likelihood ratio when the test result is positive. Then from ⟨Figure 1⟩,

$$L^+ = \frac{P(T^+ \mid F)}{P(T^+ \mid \bar{F})} = \frac{\delta_1 + \delta_2}{\alpha_1 + \alpha_2} \tag{11}$$

From (9) and (10), we find

$$L^+ = L_l^+ \cdot \frac{P(F_l)}{P(F)} + L_u^+ \cdot \frac{P(F_u)}{P(F)} \tag{12}$$

Similarly, we will have

$$P(T^- \mid \bar{F}_l) = x_1 \cdot \frac{P(\bar{F})}{P(\bar{F}_l)}$$

$$= x_1 \cdot \frac{1}{\alpha_1 + x_1} = x \tag{13}$$

$$P(T^- \mid \bar{F}_u) = x_2 \cdot \frac{P(\bar{F})}{P(\bar{F}_u)}$$

$$= x_2 \cdot \frac{1}{\alpha_2 + x_2} = x \tag{14}$$

and

$$P(T^- \mid F_l) = \beta_1 \cdot \frac{P(F)}{P(F_l)} \tag{15}$$

$$P(T^- \mid F_u) = \beta_2 \cdot \frac{P(F)}{P(F_u)} \tag{16}$$

Therefore

$$L_l^- = \frac{P(T^- \mid F_l)}{P(T^- \mid \bar{F}_l)} = \left(\beta_1 \cdot \frac{P(F)}{P(F_l)} \right) / x \tag{17}$$

and

$$L_u^- = \frac{P(T^- \mid F_u)}{P(T^- \mid \bar{F}_u)} = \left(\beta_2 \cdot \frac{P(F)}{P(F_u)} \right) / x \tag{18}$$

Let us further define L^- as the likelihood ratio when the test result is negative. Then from ⟨Figure 1⟩,

$$L^- = \frac{P(T^- \mid F)}{P(T^- \mid \bar{F})} = (\beta_1 + \beta_2)/x \tag{19}$$

From (17) through (19), we find

$$L^- = L_l^- \cdot \frac{P(F_l)}{P(F)} + L_u^- \cdot \frac{P(F_u)}{P(F)} \tag{20}$$

2.2.2 Determining the Likelihood Ratios by Test

In (9) to (11), (17) to (19), and ⟨Figure 1⟩, we must figure out the areas $(\alpha_1+\alpha_2)$, β_1, β_2, δ_1 and δ_2. Generally speaking, the areas $(\alpha_1+\alpha_2)$, β_1, β_2, δ_1 and δ_2 can be obtained from exhaustive laboratory tests.

First, we must set the upper and lower thresholds for a system, and then repeat BIT a large number of times on a system that is known to be fault-free under varying environmental conditions, and record the percentage of times that a BIT fail indication is generated. The percentage recorded is actually the area $\alpha_1+\alpha_2$, or the probability of a false positive.

More difficult procedures are needed to obtain the area β_1, β_2, or the probability of a false negative at the lower and upper thresholds respectively. We must introduce as many various faults as possible into a system, one by one, under varying environmental conditions, and then record the percentage of BIT passes found and the total numbers of tests at the lower and upper thresholds, respectively.[Luthra, 1990]

From these data, it is easy to generate β_1, β_2, and the fractions $\frac{P(F_l)}{P(F)}$ and $\frac{P(F_u)}{P(F)}$. Then δ_1 and δ_2 will be readily obtained:

$$\begin{aligned}\delta_1 &= \frac{P(F_l)}{P(F)} \cdot P(T^{+} \mid F_l) \\ &= \frac{P(F_l)}{P(F)} \cdot \{1-P(T \mid F_l)\} \\ &= \frac{P(F_l)}{P(F)} - \frac{P(F_l)}{P(F)} \cdot P(T^{-} \mid F_l) \\ &= \frac{P(F_l)}{P(F)} - \beta_1 \qquad (21)\end{aligned}$$

In a same way, we have

$$\delta_2 = \frac{P(F_u)}{P(F)} - \beta_2 \qquad (22)$$

2.2.3 Determining the Likelihood Ratios by Analytical Method

We will now develop analytical models for the areas α_1, α_2, β_1, β_2, δ_1, and δ_2. Denote the probability density functions for $P(T \mid \overline{F})$, $P(T \mid F_l)$ and $P(T \mid F_u)$ as $h(x)$, $g_l(x)$ and $g_u(x)$, respectively, where $P(T \mid \overline{F})$ is the probability of the test result when the primary system is fault-free, $P(T \mid F_l)$ is the probability of the test result when the primary system fails at the lower threshold, and $P(T \mid F_u)$ is the probability of the test result when the primary system fails at the upper threshold.

According to the total probability theorem, we have

$$P(T) = \sum_i P(T \mid S_i) \cdot P(S_i)$$

Where S_i have $\{\overline{F}, F_l, F_u\}$ as mutually exclusive subsets, Then

$$P(T) = P(T \mid \overline{F}) \cdot P(\overline{F}) + P(T \mid F_u) \cdot P(F_u) + P(T \mid F_l) \cdot P(F_l) = 1$$

as we assume any failure modes will be detected by BIT.

The areas α_1, α_2, β_1, β_2, δ_1 and δ_2 will be given by:

$$\alpha_1 = \int_{-\infty}^{X_l} h(x)dx \tag{23}$$

$$\alpha_2 = \int_{X_u}^{\infty} h(x)dx \tag{24}$$

$$x = 1 - \alpha_1 - \alpha_2 = \int_{X_l}^{X_u} h(x)dx \tag{25}$$

$$\delta_1 = \frac{P(F_l)}{P(F)} \int_{-\infty}^{X_l} g_l(x)dx \tag{26}$$

$$\delta_2 = \frac{P(F_u)}{P(F)} \int_{X_u}^{+\infty} g_u(x)dx \tag{27}$$

$$\beta_1 = \frac{P(F_l)}{P(F)} - \delta_1 = \frac{P(F_l)}{P(F)} \left(1 - \int_{-\infty}^{X_l} g_l(x)dx \right) \tag{28}$$

$$\beta_2 = \frac{P(F_u)}{P(F)} - \delta_2 = \frac{P(F_u)}{P(F)} \left(1 - \int_{X_u}^{+\infty} g_u(x)dx \right) \tag{29}$$

Note that in (26) and (27) there is a fraction before the integral, because there are two test result *pdf* curves which are mutually exclusive for a faulty primary system.

On substitution of (23) through (29) into (9) to (11) and (17) to (19), we will get analytical formulas for the likelihood ratios L_l^+, L_u^+, L_l^-, and L_u^- as follows:

$$\begin{aligned} L_l^+ &= \left(\delta_1 \cdot \frac{P(F)}{P(F_l)} \right) / (\alpha_1 + \alpha_2) \\ &= \left(\int_{-\infty}^{X_l} g_l(x)dx \right) / \left(\int_{-\infty}^{X_l} h(x)dx + \int_{X_u}^{+\infty} h(x)dx \right) \end{aligned} \qquad (30)$$

$$\begin{aligned} L_u^+ &= \left(\delta_2 \cdot \frac{P(F)}{P(F_u)} \right) / (\alpha_1 + \alpha_2) \\ &= \left(\int_{X_u}^{+\infty} g_u(x)dx \right) / \left(\int_{-\infty}^{X_l} h(x)dx + \int_{X_u}^{+\infty} h(x)dx \right) \end{aligned} \qquad (31)$$

$$\begin{aligned} L^+ &= \frac{\delta_1 + \delta_2}{\alpha_1 + \alpha_2} = L_l^+ \cdot \frac{P(F_l)}{P(F)} + L_u^+ \cdot \frac{P(F_u)}{P(F)} \\ &= \left(\frac{P(F_l)}{P(F)} \int_{-\infty}^{X_l} g_l(x)dx + \frac{P(F_u)}{P(F)} \int_{X_u}^{+\infty} g_u(x)dx \right) / \\ &\quad \left(\int_{-\infty}^{X_l} h(x)dx + \int_{X_u}^{+\infty} h(x)dx \right) \end{aligned} \qquad (32)$$

$$\begin{aligned} L_l^- &= \left(\beta_1 \cdot \frac{P(F)}{P(F_l)} \right) / x \\ &= \left(1 - \int_{-\infty}^{X_l} g_l(x)dx \right) / \left(\int_{X_l}^{X_u} h(x)dx \right) \end{aligned} \qquad (33)$$

$$\begin{aligned} L_u^- &= \left(\beta_2 \cdot \frac{P(F)}{P(F_u)} \right) / x \\ &= \left(1 - \int_{X_u}^{+\infty} g_u(x)dx \right) / \left(\int_{X_l}^{X_u} h(x)dx \right) \end{aligned} \qquad (34)$$

and

$$
\begin{aligned}
L^{-} &= \frac{\beta_l^{-} + \beta_u^{-}}{x} = L_l^{-} \cdot \frac{P(F_l)}{P(F)} + L_u^{-} \cdot \frac{P(F_u)}{P(F)} \\
&= \left(\frac{P(F_l)}{P(F)} \cdot \left(1 - \int_{-\infty}^{x_l} g_l(x)dx \right) + \frac{P(F_u)}{P(F)} \cdot \left(1 - \int_{x_u}^{+\infty} g(x)dx \right) \right) \\
&\quad / \left(\int_{x_l}^{x_u} h(x)dx \right) \qquad (35)
\end{aligned}
$$

If the likelihood ratios have not been obtained directly from the BIT development tests, these will be provided by the micro-software based on the mathematical models given in this section. Therefore, these analytical models can be used for programming on computer simulations and examining the unknown or new-born system which is related and/or controlled by BIT.

3. Conclusion

More definite suggestions to get the Prior Odds are introduced in section 2.1. Section 2.2.1 also present that Likelihood Ratios (L , L) by Geometric Areas.

In the real world, experience has taught us that it is impossible to extrapolate BIT laboratory performance to field performance, because the system and BIT parameter variabilities are much more extreme under field conditions. Therefore the likelihood ratios should be updated when field testing and data collection are possible. With proper planning, realistic values of likelihood ratios under field conditions can be ascertained and, theoretically, the Bayesian processor can be fine-tuned.[Rosin, 1990]

Based on the above analytical models, we will have no problem in determining L_l^{+}, L_u^{+}, L_l^{-}, L_u^{-}, L^{+} and L^{-} as long as the *pdf*'s of the test results for a fault-free and a faulty system are given.[19, 20]

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