▩ 연구논문

Bootstrap Confidence Bounds for $P(X>Y)$ in 1-Way Random Effect Model with Equal Variances

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Abstract

We construct bootstrap confidence bounds for reliability, $R=P(X>Y)$, where X and Y are independent normal random variables. 1-way random effect models with equal variances are assumed for the populations of X and Y. We compare the accuracy of the proposed bootstrap confidence bounds and classical confidence bound for small samples via Monte Carlo simulation.

1. Introduction

The random effect model is appropriate for measurements when the number of batches in a population is large. In statistical quality control and reliability analysis sometimes one is interested in $R=P(X>Y)$. For example, suppose that a quality control engineer decides to compare failure times (X and Y) of automobile batteries of types A and B, respectively. He randomly selects n_1 batteries from each of l_1 batches of type A and n_2 from each of l_2 batches of type B batteries. Then he tests the batteries under a specific condition and records failure times. The aim of the engineer is to find a confidence bound for the reliability R. For simplicity, we assume that 1-way random effect models for the populations of X and Y have equal variances.

Reisser and Guttman(1986) examined statistical inference for R in stress strength model with normal distribution. Guttman, Johnson, Bhattacharyya and Reisser(1988) obtained confidence limits for R in stress strength models with explanatory variables. Aminzadeh(1991) derived confidence bounds based on the approximate distributions for R under 1-way random effect model. Since the true distribution of the estimator for R is often skewed and biased for a small sample, the interval based on the asymptotic normal distribution may deteriorate the accuracy. We will use the bootstrap method to rectify these problems. Efron(1979) initially introduced the bootstrap method to assign the accuracy for an estimator. To construct approximate confidence interval for a parameter, Efron(1981, 1982, 1987) and Hall(1988) proposed the percentile method, the bias correct method(BCa method), the bias correct acceleration method(BCa method), and the percentile-t method, etc.

In this paper, we propose several bootstrap confidence bounds for R based on percentile, BC, BCa and percentile-t methods under 1-way random effect model with equal variances. Also we investigate the accuracy of the proposed bootstrap confidence intervals and confidence interval based on Aminzadeh(1991)'s method through Monte Carlo simulation. In particular, we observe the accuracy of these intervals for small sample and/or large value of R.

2. Preliminaries

We assume that n_1 measurements from each of l_1 batches of population 1 and n_2 measurements from each of l_2 batches of population 2 are selected. Let μ_x and μ_y are overall means for populations 1 and 2. And let A and B are batch effects for populations 1 and 2. Then 1-way random effect models for X and Y are defined as follows:

$$X_{ij} = \mu_x + A_j + e_{ij}, \quad i = 1, 2, \cdots, n_1, \quad j = 1, 2, \cdots, l_1 \tag{2.1}$$

and

$$Y_{qr} = \mu_y + B_r + \varepsilon_{qr}, \quad q = 1, 2, \cdots, n_2, \quad r = 1, 2, \cdots, l_2 \tag{2.2}$$

where A_j, e_{ij}, B_r, ε_{qr} are stochastically independent normal random variables with means zero and standard deviations, σ_A, σ_e, σ_B, σ_ε, respectively.

From (2.1) and (2.2) we can see that $X_{ij} \sim N(\mu_x, \sigma_x^2)$ and $Y_{qr} \sim N(\mu_y, \sigma_y^2)$, where $\sigma_x^2 = \sigma_A^2 + \sigma_e^2$ and $\sigma_y^2 = \sigma_B^2 + \sigma_\varepsilon^2$. For equal variances $\sigma_x^2 = \sigma_y^2 = \sigma^2$, the reliability is computed as $R = \Phi(\delta)$, where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable and $\delta = (\mu_x - \mu_y)/\sqrt{2\sigma^2}$. Let $\underline{X} = (X_{11}, X_{12}, \cdots, X_{n_1 l_1})$ and $\overline{Y} = (Y_{11}, Y_{12}, \cdots, Y_{n_2 l_2})$ be vecrors of measurements for X and Y, respectively. And let $N_1 = n_1 l_1$ and $N_2 = n_2 l_2$. By Aminzadeh(1991), the estimator $\hat{R}$ of R is given by

$$\hat{R} = \Phi(\hat{\delta}) = \Phi((\overline{X}_{..}-\overline{Y}_{..}) / \sqrt{2\hat{\sigma}^2}), \tag{2.3}$$

where $\overline{X}_{..}=\sum_{i=1}^{n_1}\sum_{j=1}^{l_1}X_{ij}/N_1$, $\overline{Y}_{..}=\sum_{i=1}^{n_2}\sum_{j=1}^{l_2}Y_{ij}/N_2$, $\hat{\sigma}^2=(f\hat{\sigma}_x^2+g\hat{\sigma}_y^2)/(f+g)$

$$\hat{\sigma}_x^2=(S_e^2(n_1-1)+S_A^2)/n_1,\quad \hat{\sigma}_y^2=(S_\varepsilon^2(n_2-1)+S_B^2)/n_2,$$

$$f = (k_1+1)^2(l_1-1)N_1/\{N_1(k_1+n_1^{-1})^2+(1-n_1^{-1})(l_1-1)\},$$

$$g = (k_2+1)^2(l_2-1)N_2/\{N_2(k_2+n_2^{-1})^2+(1-n_2^{-1})(l_2-1)\},$$

$$k_1=\sigma_A^2/\sigma_e^2,\quad k_2=\sigma_B^2/\sigma_\varepsilon^2.$$

Note that S_e^2, S_ε^2 and S_A^2, S_B^2 are mean squares within and between batches for populationl 1 and 2, respectively.

In order to construct approximate confidence interval for R based on the normal approximation, Aminzadeh(1991) proved that $\hat{\delta}=(\overline{X}_{..}-\overline{Y}_{..})/\sqrt{2}\,\hat{\sigma}$ has asymptotic normal distribution with mean δ and variance $\sigma_\delta^2=K/2+\delta^2/\{2(f+g)\}$, where

$$K = (n-1)k_1/\{N_1(k_1+1)\}+(n_2-1)k_2/\{N_2(k_2+1)\}+1/N_1+1/N_2.$$

The asymptotic variance of δ is estimated by $\hat{\sigma}_\delta^2=\hat{K}/2+\hat{\delta}^2/\{2(\hat{f}+\hat{g})\}$, where $\hat{K}$, $\hat{f}$ and $\hat{g}$ are computed by using $\hat{\sigma}_A^2=(S_A^2-S_e^2)/n_1$, $\hat{\sigma}_B^2=(S_B^2-S_\varepsilon^2)/n_2$, $\hat{k}_1=(s_A^2/S_e^2-1)/n_1$ and $\hat{k}_2=(S_B^2/S_\varepsilon^2-1)/n_2$ instead of σ_A^2, σ_B^2, k_1 and k_2, respectively. Hence, $100(1-2\alpha)$ % Aminzadeh's confidence interval for R is given by

$$(\Phi(\hat{\delta}+z^{(\alpha)}\cdot\hat{\sigma}_\delta),\ \Phi(\hat{\delta}+z^{(1-\alpha)}\cdot\hat{\sigma}_\delta)), \tag{2.4}$$

where $z^{(\alpha)}$ is the 100 α percentile of standard normal distribution.

3. Bootstrap Confidence Bounds for Reliability

The bootstrap procedure is a resampling scheme that one attempts to learn the sampling properties of a statistic by recomputing its value on the basis of a new sample realized from the original one. The bootstrap procedure for construction of bootstrap estimators for R can be described as follows:

(1) Compute the plug-in estimates of μ_x, μ_y and σ^2 given by $\overline{X}_{..}$, $\overline{Y}_{..}$ and $S_p^2 = (\hat{f}\hat{\sigma}_x^2 + \hat{g}\hat{\sigma}_y^2)/(\hat{f}+\hat{g})$ from $\underline{X}$ and $\underline{Y}$, respectively.

(2) Construct the sampling distribution $\hat{F}$ and $\hat{G}$ (from $\underline{X}$ and $\underline{Y}$) based on $\overline{X}_{..}$, $\overline{Y}_{..}$ and S_p^2, respectively. That is, $\hat{F} \sim N(\overline{X}_{..}, S_p^2)$ and $\hat{G} \sim N(\overline{Y}_{..}, S_p^2)$.

(3) Generate B random samples of size N_1 and N_2 from fixed $\hat{F}$ and $\hat{G}$, respectively. The corresponding samples called the *bootsrrap samples* are denoted by $\underline{X}^{*b} = (X^{*b}_{11}, X^{*b}_{12}, \cdots, X^{*b}_{n_1 l_1})$ and $\underline{Y}^{*b} = (Y^{*b}_{11}, Y^{*b}_{12}, \cdots, Y^{*b}_{n_2 l_2})$, $b = 1, 2, \cdots, B$.

(4) Compute $\hat{R}^{*b} = \Phi(\hat{\delta}^{*b})$, where $\hat{\delta}^{*b} = (\overline{X}^{*b}_{..} - \overline{Y}^{*b}_{..})/\sqrt{2\, S_p^{2*b}}$. We call $\overline{X}^{*b}$, $\overline{Y}^{*b}$, S_p^{2*b} and $\hat{R}^{*b}$ by *bootstrap estimators* for μ_x, μ_y, σ_p^2 and R, respectively.

3.1 Percentile method

The confidence interval by the bootstrap percentile method (percentile interval) is obtained by percentiles of the empirical bootstrap distribution of $\hat{R}^*$. Let $\hat{H}^*$ be the empirical cumulative distribution function of $\hat{R}^*$. Then it is constructed by $\hat{H}^*(s) = B^{-1}\sum_{b=1}^{B} I(\hat{R}^{*b} \le s)$, where s is arbitrary real value and $I(\cdot)$ is an indicator function. And let $\hat{H}^{*-1}(\alpha)$ be the 100α empirical percentile of $\hat{R}^*$ given by

$$\hat{H}^{*-1}(\alpha) = \inf\{\, s : \hat{H}^*(s) \ge \alpha \,\} \tag{3.1}$$

That is, $\hat{H}^{*-1}(\alpha)$ is the $B\alpha$th value in the ordered list of the B replications of $\hat{R}^{*b}$. If $B\alpha$ is not an integer, we can take the largest integer that is less than or equal to $(B+1)\alpha$. Then $100(1-2\alpha)\%$ percentile interval for R is approximated by

$$(\,\hat{H}^{*-1}(\alpha),\ \hat{H}^{*-1}(1-\alpha)) \tag{3.2}$$

3.2 Bias correct method

The *BC* method adjusts a possible bias in estimating R. The bias correction is given by

$$\hat{z}_0 = \Phi^{-1}(\hat{H}^*(\hat{R})) = \Phi^{-1}\left[B^{-1}\sum_{b=1}^{B} I(\hat{R}^{*b} \le \hat{R}) \right], \tag{3.3}$$

where $\Phi^{-1}(\cdot)$ indicates the inverse function of the standard normal cumulative distribution function. That is, $\hat{z}_0$ is the discrepancy between the medians of $\hat{R}^*$ and $\hat{R}$ in normal unit. Therefore, we have $100(1-2\alpha)\%$ approximate *BC* interval for R given by

$$(\hat{H}^{*-1}(\alpha_1), \hat{H}^{*-1}(\alpha_2)), \tag{3.4}$$

where $\alpha_1 = \Phi(2\hat{z}_0 + z^{(\alpha)})$ and $\alpha_2 = \Phi(2\hat{z}_0 + z^{(1-\alpha)})$.

3.3 Bias correct acceleration method

The *BCa* method corrects both the bias and standard error for $\hat{R}$. The confidence interval by *BCa* method(*BCa* interval) requires to calculate the bias-correction constant $\hat{z}_0$ and the acceleration constant $\hat{a}$. In fact, the bias-correction constant $\hat{z}_0$ is the same as that of *BC* method. And $\hat{a}$, measured on a normalized scale, refers to the rate of change of the standard error of $\hat{R}$ with respect to the true reliability R.

For the parametric bootstrap method, all calculations relate only to the sufficient statistic $\overline{X}_{..}$, $\overline{Y}_{..}$ and S^2 for μ_x, μ_y and σ^2, respectively, where $S^2 = (\sum_{i=1}^{n_1}\sum_{j=1}^{l_1}(X_{ij} - \overline{X}_{..})^2 + \sum_{q=1}^{n_2}\sum_{r=1}^{l_2}(Y_{qr} - \overline{Y}_{..})^2) / (N_1+N_2)$. Of course, $\overline{X}_{..}$, $\overline{Y}_{..}$ and S^2 are distributed $N(\mu_x, \sigma^2/N_1)$, $N(\mu_y, \sigma^2/N_2)$ and $\{\sigma^2/(N_1+N_2)\}\cdot\chi^2(N_1+N_2-2)$, respectively. Also, $\overline{X}_{..}$, $\overline{Y}_{..}$ and S^2 are stochastically independent. Let $\hat{\underline{\eta}}' = (\overline{X}_{..}, \overline{Y}_{..}, S^2)$ and $\underline{\eta}' = (\mu_x, \mu_y, \sigma^2)$. Then the joint probability density function of $\hat{\underline{\eta}}'$ can be written as

$$f_\eta(\hat{\underline{\eta}}') = f_0(\hat{\underline{\eta}}')\exp[g_0(\hat{\underline{\eta}}', \underline{\eta}') - \Psi_0(\underline{\eta}')], \tag{3.5}$$

where

$$f_0(\hat{\underline{\eta}}') = [2\pi\Gamma((N_1+N_2-2)/2)\cdot 2^{(N_1+N_2-2)/2}]^{-1} \cdot \sqrt{N_1 N_2}\cdot(S^2)^{(N_1+N_2-2)/2-1},$$

$$g_0(\underline{\eta}', \hat{\underline{\eta}}') = \{-N_1\overline{X}_{..}^2 - 2N_1\mu_x\overline{X}_{..} + N_2\overline{Y}_{..}^2 - 2N_2\mu_y\overline{Y}_{..} - (N_1+N_2)S^2\}/2\sigma^2$$
$$+(N_1-3)/2\cdot\log(S_x^2) + (N_2-3)/2\cdot\log(S_y^2)$$

and $\Psi_0(\underline{\eta}') = (N_1\mu_x^2 + N_2\mu_y^2)/2\sigma^2 + \log(\sigma^2)$.

For multiparameter family case, we will find $\hat{a}$ following Stein's construction (1956). That is, we replace the multiparameter family $\Im = \{f_{\underline{\eta}}(Z)\}$ by the least favorable one parameter family $\hat{\Im} = \{\widehat{f_{\hat{\underline{\eta}}}(Z)} \equiv f_{\hat{\underline{\eta}}+\lambda\hat{\underline{\omega}}}(Z)\}$, where $Z = (X, Y)$. Then we first obtain $\hat{\underline{\omega}}$ such that the least favorable direction at $\underline{\eta} = \hat{\underline{\eta}}$ is defined to be $\hat{\underline{\omega}} = (\mathfrak{L}_{\hat{\underline{\eta}}})^{-1}\hat{\nabla}_{\hat{\underline{\eta}}}$, where $\mathfrak{L}_{\hat{\underline{\eta}}}$ is Fisher information matrix and $\hat{\nabla}_{\hat{\underline{\eta}}}$ is the gradient of δ given by $\hat{\nabla}_{\hat{\underline{\eta}}} = \frac{\partial\delta}{\partial\underline{\eta}}|_{\underline{\eta}=\hat{\underline{\eta}}}$. By some algebraic calculation, we have

$$\hat{\mathcal{C}}_{\hat{\underline{\eta}}} = \begin{bmatrix} N_1/S^2 & 0 & 0 \\ 0 & N_2/S^2 & 0 \\ 0 & 0 & (N_1+N_2)/(2S^4) \end{bmatrix}$$

and

$$\hat{\nabla}_{\hat{\underline{\eta}}} = \begin{bmatrix} 1/\sqrt{2S^2} \\ -1/\sqrt{2S^2} \\ -(\overline{X}..-\overline{Y}..)/(2\sqrt{2}(S^2)^{3/2}) \end{bmatrix}$$

Hence, we have $\underline{\omega}' = (W_1, W_2, W_3)$, where $W_1 = \sqrt{S^2}/(\sqrt{2}N_1)$, $W_2 = \sqrt{S^2}/(\sqrt{2}N_2)$ and $W_3 = -(\overline{X}..-\overline{Y}..)\sqrt{S^2}/(\sqrt{2}N_1+N_2))$. By the method of Efron(1987), $\hat{a}$ can be obtained as

$$\hat{a} = \frac{1}{6}\cdot\frac{\hat{\Psi}^{(3)}(0)}{(\hat{\Psi}^{(2)}(0))^{3/2}}, \tag{3.6}$$

where $\hat{\Psi}^{(j)}(0) = \frac{\partial\ \Psi_0(\hat{\eta}+\lambda\hat{\underline{\omega}})}{\partial\lambda^j}\Big|_{\lambda=0}$. Calculating $\hat{\Psi}^{(j)}(\cdot)$ and $\hat{\underline{\omega}}$, we can obtain

$$\begin{aligned}\hat{\Psi}^{(2)}(0) &= -W_3^2/(S^2)^2 + N_1(S^2W_1-\overline{X}..W_3)^2/(S^2)^3 \\ &\quad + N_2(S^2W_2-\overline{Y}..W_3)^2/(S^2)^3\end{aligned}$$

and

$$\begin{aligned}\hat{\Psi}^{(3)}(0) &= 2W_3^3/S^6 + 3N_1\cdot\{2\overline{X}..S^2W_1W_3^2 - \overline{X}..^2W_3^3 - S^4W_1^2W_3\}/S^8 \\ &\quad + 3N_2\cdot\{2\overline{Y}..S^2W_2W_3^2 - \overline{Y}..^2W_3^3 - S^4W_2^2W_3\}/S^8.\end{aligned}$$

Therefore, we have $100(1-2\alpha)\%$ approximate *BCa* interval for R by

$$(\Phi(\hat{H}^{*-1}(\alpha_3)), \Phi(\hat{H}^{*-1}(\alpha_4))), \tag{3.7}$$

where $\alpha_3 = \Phi[\hat{z}_0 + (\hat{z}_0+z^{(\alpha)})/\{1-\hat{a}(\hat{z}_0+z^{(\alpha)})\}]$ and

$\alpha_4 = \Phi[\hat{z}_0 + (\hat{z}_0+z^{(1-\alpha)})/\{1-\hat{a}(\hat{z}_0+z^{(1-\alpha)})\}]$.

3.4 Percentile-t method

The confidence interval by the percentile-t method (percentile-t interval) is constructed by using the bootstrap distribution of an approximately pivotal quantity for $\hat{\delta}$ instead of the bootstrap distribution of $\hat{\delta}$. We define an approximate bootstrap pivotal quantity for $\hat{\delta}$ by

$$\hat{\delta}^*_{STUD} = (\hat{\delta}^* - \hat{\delta}) / \hat{\sigma}^*_{\delta}, \tag{3.8}$$

where $\hat{\sigma}^*_{\delta}$ is the bootstrap estimator of σ_{δ}, that is,

$$\hat{\sigma}^*_{\delta} = (\hat{K}^*/2 + \hat{\delta}^{2*}/\{2(\hat{f}^* + \hat{g}^*)\})^{1/2} \tag{3.9}$$

We compute the empirical distribution function $\hat{H}^*_{STUD}$ of $\hat{\delta}^*_{STUD}$ by

$$\hat{H}^*_{STUD}(s) = B^{-1} \sum_{b=1}^{B} I(\hat{\delta}^{*b}_{STUD} \leq s), \tag{3.10}$$

for all s. Let $\hat{H}^{*-1}_{STUD}(\alpha)$ denote 100α empirical percentile of $\hat{\delta}^*_{STUD}$. Actually we compute $\hat{H}^{*-1}_{STUD}(\alpha)$ by

$$\hat{H}^{*-1}_{STUD}(\alpha) = \inf\{s : \hat{H}^{*-1}_{STUD}(s) \geq (\alpha)\} \tag{3.11}$$

That is, $\hat{H}^{*-1}_{STUD}(\alpha)$ is the $B\alpha$th value in the ordered list of the B replications of $\hat{\delta}^*_{STUD}$. Then we have $100(1-2\alpha)\%$ approximate percentile-t interval for R by

$$(\Phi(\hat{\delta} + \hat{\sigma}_{\delta} \cdot \hat{H}^{*-1}_{STUD}(\alpha)),\ \Phi(\hat{\delta} + \hat{\sigma}_{\delta} \cdot \hat{H}^{*-1}_{STUD}(1-\alpha))) \tag{3.12}$$

4. Monte Carlo Simulation Studies

To compare the approximate bootstrap confidence intervals with the confidence interval based on asymptotic normal distribution, we compute the results obtained in Section 2 and Section 3. The methods are compared mainly based on coverage probability and interval length. The normal random numbers were generated by IMSL subroutine RNNOF. We use the true reliabilities $R = 0.3,\ 0.5,\ 0.7,\ 0.9$ and batch sizes $l_1 = l_2 = 3,\ 5,\ 10$ with fixed $n_1 = n_2 = 3$. We also use the confidence level $(1-2\alpha) = 0.90$. For given independent random samples, the approximate confidence intervals were constructed by each method with bootstrap replications $B = 2000$ times. And the Monte Carlo samplings were repeated 2000 times. The coverage probability(CP) for all cases and the length(IL) of all intervals are reported in 〈Table 1〉 and 〈Table 2〉, respectively.

We can summarize the following properties based on an inspection of 〈Table 1〉 and 〈Table 2〉.

(1) For small batch size, the values of coverage probability for the proposed approximate bootstrap confidence intervals work better than that of the interval based on Aminzadeh(AMIN)'s method for all R.
(2) As batch size increases, the values of coverage probability for all approximate intervals converge to true confidence level $(1-2\alpha)$, as a whole.
(3) The values of interval length for all approximate confidence intervals tend to decrease as R deviates from 0.5. For small batch size, the value of interval length for the interval based on Aminzadeh's method is slightly shorter than those of the intervals based on bootstrap methods, except for large value of R.
(4) As a whole, the values of interval length for the approximate intervals based on all methods converge to true interval length as batch size increase.

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〈 Table 1 〉 Results of coverage probability for $1-2\alpha=0.9$

BATCH SIZE	R	AMIN	Percentile	BC	BCa	Percentile-t
3	0.3	0.8375	0.8690	0.8845	0.8715	0.8760
	0.5	0.8500	0.8750	0.8950	0.8795	0.8855
	0.7	0.8650	0.8960	0.9050	0.8980	0.9030
	0.9	0.8860	0.8815	0.9055	0.8965	0.9115
5	0.3	0.8770	0.8845	0.8910	0.8830	0.8995
	0.5	0.8740	0.8960	0.9005	0.8900	0.8980
	0.7	0.8765	0.8850	0.8935	0.8985	0.8910
	0.9	0.8860	0.8890	0.8960	0.8985	0.8980
10	0.3	0.8900	0.9015	0.9015	0.9010	0.8990
	0.5	0.8905	0.9025	0.9045	0.8995	0.9030
	0.7	0.8825	0.8895	0.8930	0.8905	0.8930
	0.9	0.9050	0.9005	0.9005	0.9035	0.9075

〈 Table 2 〉 Results of interval length for $1-2\alpha=0.9$

BATCH SIZE	R	AMIN	Percentile	BC	BCa	Percentile-t
3	0.3	0.3450	0.3733	0.3773	0.3737	0.3885
	0.5	0.3797	0.4205	0.4221	0.4208	0.4380
	0.7	0.3509	0.3770	0.3809	0.3778	0.3953
	0.9	0.2324	0.2166	0.2270	0.2213	0.2453
5	0.3	0.2866	0.2956	0.2972	0.2957	0.3061
	0.5	0.3127	0.3306	0.3313	0.3310	0.3385
	0.7	0.2865	0.2965	0.2982	0.2968	0.3059
	0.9	0.1804	0.1708	0.1749	0.1723	0.1857
10	0.3	0.2107	0.2130	0.2136	0.2131	0.2174
	0.5	0.2300	0.2366	0.2369	0.2369	0.2387
	0.7	0.2089	0.2127	0.2134	0.2129	0.2154
	0.9	0.1256	0.1225	0.1240	0.1230	0.1272