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FUZZY QUASICOMPONENTS

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ABSTRACT. We define fuzzy quasicomponents and prove some properties related to fuzzy components.

1. Introduction

Z. Chong-you [2] introduced the notions of the O-components and the fuzzy path components of a fuzzy set in a fuzzy topological space, and proved some fundamental properties related to these notions. In particular, he showed that, if the fuzzy set considering is locally fuzzy path connected and open in the fuzzy subspace topology on the support of itself, O-component and fuzzy path component of the fuzzy set are the same.

In this paper, we define the Q-components, the O-quasicomponents and the Q-quasicomponents of a fuzzy set in a fuzzy topological space, and discuss some of their properties. In section 3 we prove each Q-[resp. O-]component of a fuzzy set lies in a fuzzy Q-[resp. O-]quasicomponent of the fuzzy set. In section 4 we prove the O-component and the fuzzy O-quasicomponent of a locally fuzzy O-connected open set are the same.

2. Preliminaries

Throughout this paper X will denote a non-empty set.

DEFINITION 2.1. ([1],[2]) A *fuzzy set* in X is a function from X to the unit interval $[0, 1]$. The fuzzy set which always takes the value 1 on X is denoted by X and the fuzzy set which always takes the value 0 on X is denoted by $\emptyset$.

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For fuzzy sets A, B and $A_\alpha (\alpha \in J)$, we define

(1) $A = B$ whenever $A(x) = B(x)$ for all $x \in X$,

(2) $A \subset B$ whenever $A(x) \leq B(x)$ for all $x \in X$,

(3) $\cup A_\alpha(x)$=sup$\{A_\alpha(x) | \alpha \in J\}$ for all $x \in X$,

(4) $\cap A_\alpha(x)$=inf$\{A_\alpha(x) | \alpha \in J\}$ for all $x \in X$.

Let a be a point of X. A fuzzy set a_λ, defined by

$$a_\lambda(x) = \begin{cases} \lambda \ (0 < \lambda \leq 1) & \text{for } x = a \\ 0 & \text{for } x \neq a, \end{cases}$$

is called a *fuzzy point.* A fuzzy point a_λ is said to belong to a fuzzy set A, denoted by $a_\lambda \in A$, if $a_\lambda \subset A$.

DEFINITION 2.2. ([1],[2]) A collection τ of fuzzy sets in X is called a *fuzzy topology* on X, if

(1) $X, \emptyset \in \tau$,

(2) $\cap A_i \in \tau$ for any finite subcollection $\{A_i | i = 1, \cdots, n\}$ of τ,

(3) $\cup A_\alpha \in \tau$ for any subcollection $\{A_\alpha | \alpha \in J\}$ of τ.

The pair (X, τ) is called a *fuzzy topological space.* Every element of τ is called a *fuzzy open set* in X. The complement of a fuzzy open set is called a *fuzzy closed set* in X.

DEFINITION 2.3. ([2]) Let D be a fuzzy set in X. The set $\{x \in X | D(x) > 0\}$, denoted by D_0, is called the *support* of D. If τ is a fuzzy topology on X, then the collection $\tau_{D_0} = \{A \cap \chi_{D_0} | A \in \tau\}$ is called the *fuzzy subspace topology* on D_0, and the pair (D_0, τ_{D_0}) is called the fuzzy subspace of (X, τ).

In the sequel, (D_0, τ_{D_0}) is the fuzzy subspace of (X, τ).

DEFINITION 2.4. ([2]) Two fuzzy sets A and B in a fuzzy topological space (X, τ) are said to be Q-[resp. O-]*separated* if there is a pair of fuzzy closed [resp. fuzzy open] sets F_1 and F_2 in (X, τ) such that $A \subset F_1, B \subset F_2, F_1 \cap B = \emptyset$ and $F_2 \cap A = \emptyset$. A fuzzy set D in a fuzzy topological space (X, τ) is called Q-[resp. O-]*disconnected* if there exist non-empty fuzzy sets A and B in the fuzzy subspace (D_0, τ_{D_0}) such that A and B are Q-[resp. O-]separated and $A \cup B = D$. A fuzzy set is called Q-[resp. O-]*connected* if it is not Q-[resp. O-]disconnected. The maximal Q-[resp. O-]connected fuzzy set in (X, τ) which is contained in a fuzzy set D is called fuzzy Q-[resp. O-]*component* of D.

LEMMA 2.5. *Let D be a Q-[resp. O-]disconnected fuzzy set in (X,τ) and let A and B be non-empty fuzzy sets in (D_0,τ_{D_0}) such that A and B are Q-[resp. O-]separated and $A\cup B=D$. If C is a Q-[resp. O-]connected fuzzy set in (X,τ) which is contained in D, then either $C\subset A$ or $C\subset B$.*

PROOF. Let F_1 and F_2 be two fuzzy closed [resp. fuzzy open] sets in (X,τ) satisfying $A\subset F_1, B\subset F_2, A\cap F_2=\emptyset$ and $F_1\cap B=\emptyset$. Assume that $C\subset A$ and $C\subset B$. Then $C\cap A\neq\emptyset$ and $C\cap B\neq\emptyset$. Furthermore, we have $C\cap A\subset F_1, C\cap B\subset F_2, F_1\cap(C\cap A)=\emptyset, F_2\cap(C\cap B)=\emptyset$ and $(C\cap A)\cup(C\cap B)=C$. This contradicts to the hypothesis. □

COROLLARY 2.6. *Let $\mathcal{D}=\{D_\alpha|\alpha\in J\}$ be a collection of Q-[resp. O-]connected fuzzy sets in (X,τ). If $\cap\{D_\alpha|\alpha\in J\}\neq\emptyset$, then $\cup\{D_\alpha|\alpha\in J\}$ is a Q-[resp. O-]connected fuzzy set in (X,τ).*

PROOF. Assume that $\cup D_\alpha$ is Q-[resp. O-]disconnected in (X,τ). Then there is a pair of non-empty fuzzy sets A, B in $((\cup D_\alpha)_0,\tau_{(\cup D_\alpha)_0})$ such that A and B are Q-[resp. O-]separated and $A\cup B=\cup D_\alpha$. Let $a_\lambda\in\cap D_\alpha$. Since $A\cap B=\emptyset$, either $a_\lambda\in A$ or $a_\lambda\in B$. Suppose that $a_\lambda\in A$. By Lemma 2.5, $D_\alpha\subset A$ for all α. Thus $B=\emptyset$, contrary to the choice of B. □

3. Fuzzy Quasicomponents

PROPOSITION 3.1. *Let D be a fuzzy set in (X,τ). For arbitrary two fuzzy points a_λ and b_μ belonging to D, define $a_\lambda\sim_Q b_\mu$ [resp. $a_\lambda\sim_O b_\mu$] if there is no pair of fuzzy closed [resp. fuzzy open] sets F_1 and F_2 in (D_0,τ_{D_0}) satisfying $F_1\cap F_2=\emptyset, F_1\cup F_2=D, a_\lambda\in F_1$ and $b_\mu\in F_2$. Then this is an equivalence relation on the collection of all fuzzy points belonging to D.*

PROOF. Reflexivity and symmetry are trivial. To show transitivity, assume that there exist fuzzy points a_λ, b_μ and c_ν belonging to D such that $a_\lambda\sim_Q b_\mu$ and $b_\mu\sim_Q c_\nu$, but $a_\lambda\nsim_Q c_\nu$ [resp. $a_\lambda\sim_O b_\mu$ and $b_\mu\sim_O c_\nu$, but $a_\lambda\nsim_O c_\nu$]. Then there is a pair of fuzzy closed [resp. fuzzy open] sets F_1 and F_2 in (D_0,τ_{D_0}) such that $F_1\cap F_2=\emptyset, F_1\cup F_2=D, a_\lambda\in F_1$ and $c_\nu\in F_2$. Since $max\{F_1(b),F_2(b)\}=D(b)\geq b_\mu(b)$, we have either $b_\mu \in F_1$ or $b_\mu \in F_2$. This is an obvious contradiction. Thus the relation is transitive. □

DEFINITION 3.2. Let D be a fuzzy set in (X,τ) and let a_λ be a fuzzy point belonging to D. We will call the collection $\{b_\mu \in D | a_\lambda \sim_Q b_\mu\}$ [resp. $\{b_\mu \in D | a_\lambda \sim_O b_\mu\}$] the *fuzzy* Q-[resp. O-]*quasicomponent* of a_λ in D.

LEMMA 3.3. *Let D be a fuzzy set in (X,τ), let a_λ be a fuzzy point belonging to D and let Q be the fuzzy Q-[resp. O-]quasicomponent of a_λ in D. If $\mathcal{A}$ is the non-empty family of all fuzzy closed [resp. fuzzy open] sets in (D_0, τ_{D_0}) which contain Q, then Q is equal to $\cap\{A | A \in \mathcal{A}\}$.*

PROOF. Clearly, $Q \subset \cap\{A | A \in \mathcal{A}\}$.

To show that $\cap\{A|A \in \mathcal{A}\} \subset Q$, assume there exists $b \in D_0$ such that $\mu = \inf\{A(b) | A \in \mathcal{A}\} > Q(b)$. Then $b_\mu \in A$ for all $A \in \mathcal{A}$ but $b_\mu \notin Q$. Choose a pair of fuzzy closed [resp. fuzzy open] sets F_1 and F_2 in (D_0, τ_{D_0}) such that $F_1 \cap F_2 = \emptyset, F_1 \cup F_2 = D, a_\lambda \in F_1$ and $b_\mu \in F_2$. Note that $Q \subset F_1$. Thus $F_1 \in \mathcal{A}$, contrary to the fact that $F_1 \cap F_2 = \emptyset$. □

REMARK. In the above lemma, the condition $\mathcal{A} \neq \emptyset$ is necessary. See Example 3.6.

COROLLARY 3.4. *Let D be a fuzzy set in (X,τ) and let Q be a fuzzy Q-quasicomponent of D. If there exists a fuzzy closed set in (D_0, τ_{D_0}) which contains Q, then Q is a fuzzy closed set in (D_0, τ_{D_0}).*

THEOREM 3.5. *Let D be a fuzzy set in (X,τ), let a_λ be a fuzzy point belonging to D and let Q be a fuzzy Q-[resp. O-]quasicomponent of a_λ in D. If C is a Q-[resp. O-]component of D which contains a_λ, then $C \subset Q$.*

PROOF. Assume that there exists a point $b \in D_0$ such that $\mu = C(b) > Q(b)$. Then $b_\mu \not\sim a_\lambda$, so there exists a pair of fuzzy closed [resp. fuzzy open] sets F_1 and F_2 in (D_0, τ_{D_0}) such that $F_1 \cap F_2 = \emptyset, F_1 \cup F_2 = D, a_\lambda \in F_1$ and $b_\mu \in F_2$. Since $F_1 \cap C \subset F_1, F_2 \cap C \subset F_2, (F_1 \cap C) \cap F_2 = \emptyset$ and $(F_2 \cap C) \cap F_1 = \emptyset$, the non-empty fuzzy sets $F_1 \cap C$ and $F_2 \cap C$ are Q-[resp. O-]separated in (C_0, τ_{C_0}). By noting that $(F_1 \cap C) \cup (F_2 \cap C) = C$, we have the fuzzy set C is Q-[resp. O-]disconnected, contrary to the choice of C. □

In general, the fuzzy Q-[resp. O-]components are not equal to fuzzy Q-[resp. O-]quasicomponents as is shown from the following example.

EXAMPLE 3.6. Let $X = \{a, b, c\}$, let $A_{\lambda\mu\nu}$ $(0 \leq \lambda, \mu, \nu \leq 1)$ be a fuzzy set in X defined by

$$A_{\lambda\mu\nu}(x) = \begin{cases} \lambda & \text{for } x = a \\ \mu & \text{for } x = b \\ \nu & \text{for } x = c, \end{cases}$$

and let $\tau = \{X, \emptyset, A_{\frac{1}{3}11}, A_{1\frac{1}{3}\frac{1}{2}}, A_{\frac{1}{3}\frac{1}{3}\frac{1}{2}}\}$. Clearly, (X, τ) is a fuzzy topological space. Consider the fuzzy set $D = A_{\frac{1}{2}\frac{1}{3}\frac{1}{4}}$. It is easy to show that D is a fuzzy Q-quasicomponent in D. Now, let $A = A_{\frac{1}{2}00}, B = A_{0\frac{1}{3}\frac{1}{4}}, F_1 = A_{\frac{2}{3}00}$ and $F_2 = A_{0\frac{2}{3}\frac{1}{2}}$. Since $A \subset F_1, B \subset F_2, F_1 \cap B = \emptyset, F_2 \cap A = \emptyset$ and $A \cup B = D$, we have D is Q-disconnected.

4. Local Fuzzy Connectedness

DEFINITION 4.1. Let D be a fuzzy set in (X, τ) and let a_λ be a fuzzy point belonging to D. If for every neighborhood V of a_λ in the fuzzy subspace (D_0, τ_{D_0}) there is a Q-[resp. O-]connected neighborhood N of a_λ in (D_0, τ_{D_0}) such that $a_\lambda \in N \subset (D \cap V)$, then D is said to be *locally fuzzy Q-[resp. O-]connected at the fuzzy point a_λ*. If D is locally fuzzy Q-[resp. O-]connected at each of its points, then it is said to be *locally fuzzy Q-[resp. O-]connected* in (X, τ).

LEMMA 4.2. *Let D be a fuzzy open set in (X, τ). If for each fuzzy open set G in (D_0, τ_{D_0}) which is contained in D each Q-[resp. O-]component of G is a fuzzy open set in (D_0, τ_{D_0}), then D is locally fuzzy Q-[resp. O-]connected.*

PROOF. Let a_λ be a fuzzy point in D and let V be a neighborhood of a_λ in (D_0, τ_{D_0}). Since D is a fuzzy open set in (X, τ), $D \cap V$ is a neighborhood of a_λ in (D_0, τ_{D_0}). Let C be a Q-[resp. O-]component of $D \cap V$ which contains a_λ. By the hypothesis, C is a fuzzy open set in (D_0, τ_{D_0}). Thus D is locally fuzzy Q-[resp. O-]connected in (X, τ). □

THEOREM 4.3. *Let D be a fuzzy open set in (X,τ). If for every fuzzy point a_λ and every neighborhood U of a_λ in (D_0,τ_{D_0}) there exists a Q-[resp. O-]connected fuzzy set A containing a neighborhood of a_λ in (D_0,τ_{D_0}) and contained in D such that $a_\lambda \in A \subset U$, then D is locally fuzzy Q-[resp. O-]connected in (X,τ).*

PROOF. Let C be a Q-[resp. O-]component of a fuzzy open set U in (D_0,τ_{D_0}) and let a_λ be a fuzzy point belonging to C. By the given hypothesis, there is a Q-[resp. O-]connected fuzzy set A in (D_0,τ_{D_0}) containing a neighborhood V of a_λ such that $a_\lambda \in A \subset U$. Since $a_\lambda \in A$ and $a_\lambda \in C$, we have $A \subset C$. This implies that $V \subset C$. Thus C is a fuzzy open set in (D_0,τ_{D_0}). By Lemma 4.2, D is locally fuzzy Q-[resp. O-]connected. □

If we use Corollary 2.6, a proof of the following lemma can be obtained by applying the same method used in the proof of Theorem 5.1 of [2].

LEMMA 4.4. *If a fuzzy set D in (X,τ) is locally fuzzy Q-[resp. O-]connected, then for each fuzzy open set G in (D_0,τ_{D_0}) which is contained in D, each Q-[resp. O-]component of G is a fuzzy open set in (D_0,τ_{D_0}).*

THEOREM 4.5. *If D is a locally fuzzy O-connected open set in (X,τ), then the O-components of D and the fuzzy O-quasicomponents in D are the same.*

PROOF. Let Q be a fuzzy O-quasicomponent of a fuzzy point a_λ in D and let C be an O-component of D containing a_λ. By Theorem 3.5, $C \subset Q$. Assume that $Q \neq C$, and let R be the union of all the fuzzy O-components of D that are different from C and intersect Q. Since D is locally fuzzy O-connected in (X,τ) and D is a fuzzy open set in (D_0,τ_{D_0}), we have two fuzzy open sets C and R in (D_0,τ_{D_0}) such that $C \cap R = \emptyset$ and $C \cup R = Q$. This implies that every fuzzy point belonging to R is not equivalent to a_λ. This is a contradiction. Thus $Q = C$. □

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