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ASYMPTOTIC STABILITY OF COMPETING SPECIES

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ABSTRACT. Large-time asymptotic behavior of the solutions of interacting population reaction-diffusion systems are considered. Polynomial stability was proved.

1. Introduction

In this paper we consider a system of two competing species with Dirichlet boundary conditions. The system of equations are:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \sigma_1 \Delta u_1 = u_1[a + \tilde{f}_1(t, u_1, u_2)], \\ \dfrac{\partial u_2}{\partial t} - \sigma_2 \Delta u_2 = u_2[a + \tilde{f}_2(t, u_1, u_2)] \end{cases} \tag{1.1}$$

for $x \in \Omega,\ t > 0$. Here $u_i(x,t),\ i = 1, 2$, represents the concentration of two species at position x and time t. The parameters a, b, σ_1, σ_1 are positive constants, with a and b representing growth rates when no interaction occurs, σ_1 and σ_2 representing diffusion rates. The functions $\tilde{f}_i : \mathbb{R}^2 \to \mathbb{R},\ i = 1, 2$ have Hölder continuous partial derivatives up to second order in compact sets. Further, we assume that

$$\tilde{f}_1(t, 0, 0) = \tilde{f}_2(t, 0, 0) = 0. \tag{1.2}$$

For (u_1, u_2) in the first open quadrant, the first partial derivatives of $\tilde{f}_1$, $\tilde{f}_2$ satisfy:

$$\frac{\partial \tilde{f}_i}{\partial u_j} < 0 \quad \text{for each } i, j = 1, 2. \tag{1.3}$$

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We assume here that there are two functions f_1, f_2 such that

$$|\tilde{f}_i(t,u_1,u_2) - f_i(u_1,u_2)| \le K(1+t)^{-\gamma}, \quad i=1,2 \tag{1.4}$$

for some positive constants K and γ and

$$1 < \min_{x,t} \left| \frac{\overline{u}_i(x)}{\overline{u}_j(x)} \cdot \frac{(\partial \tilde{f}_i/\partial u_i)(t,\overline{u}_1(x),\overline{u}_2(x))}{(\partial \tilde{f}_i/\partial u_j)(t,\overline{u}_1(x),\overline{u}_2(x))} \right| \tag{1.5}$$

for each $x \in \overline{\Omega}, t > 0,\ i \neq j,\ 1 \le i,\ j \le 2$. Here $\overline{u}_i,\ i=1,2$, are the equilibrium solution of (1.1). Concerning the stability of the equilibrium solution of (1.1) when the reaction functions $\tilde{f}_i(t,u_1,u_2),\ i=1,2$, are independent of time we have the following result.

THEOREM 1.1([L]). *Let $(\overline{u}_1(x),\overline{u}_2(x))$ be an equilibrium solution to (1.1) when*
$\tilde{f}_i(t,u_1,u_2) \equiv f_i(u_1,u_2),\ i=1,2$. Suppose that the conditions (1.2) and (1.3) hold and that
(1.6)

$$\left| \frac{\overline{u}_i(x)}{\overline{u}_j(x)} \frac{(\partial f_j/\partial u_i)(\overline{u}_1(x),\overline{u}_2(x))}{(\partial f_j/\partial u_j)(\overline{u}_1(x),\overline{u}_2(x))} \right| < \min \left| \frac{\overline{u}_i(x)}{\overline{u}_j(x)} \frac{(\partial f_i/\partial u_i)(\overline{u}_1(x),\overline{u}_2(x))}{(\partial f_i/\partial u_j)(\overline{u}_1(x),\overline{u}_2(x))} \right|$$

for each $x \in \overline{\Omega},\ 1 \le i,\ j \le 2,\ i \neq j$, then $(\overline{u}_1(x),\overline{u}_2(x))$ is asymptotically stable.

Hence, it is our purpose to study the stability of the equilibrium solution of (1.1) under the conditions stated above.

2. Main Result

We use the standard upper and lower solution methods([L], [P]) to prove the following stability result.

THEOREM 2.1. *Let $(\overline{u}_1(x),\overline{u}_2(x))$ be an equilibrium solution to (1.1) under the following boundary conditions;*

$$\overline{u}_i(x,t) \equiv 0,\ i=1,2 \text{ for all } t>0, x \in \partial\Omega.$$

Suppose that $\tilde{f}_1, \tilde{f}_2$ satisfy the conditions (1.2)–(1.5). Then $(\overline{u}_1,\overline{u}_2)$ is asymptotically stable.

PROOF. Let

$$w_2 = (1+p(t))\overline{u}_2(x),$$
$$v_1 = (1-p(t))\overline{u}_1(x)$$

Then

$$\begin{aligned}
&\frac{\partial w_2}{\partial t} - \sigma_2 \Delta w_2 - w_2[b + \tilde{f}_2(t, v_1, w_2)] \\
&\quad = p'\overline{u}_2 \\
&\quad\quad - (1+p(t))\sigma_2\Delta\overline{u}_2 - (1+p(t))[b + \tilde{f}_2(t, v_1, w_2)] \\
&\quad = p'\overline{u}_2 + \\
&\quad\quad (1+p(t))\big(-\sigma_2\Delta\overline{u}_2 - \overline{u}_2\tilde{f}_2(t, v_1, w_2)\big) \\
&\quad = p'\overline{u}_2 + (1+p(t))\big(-\sigma_2\Delta\overline{u}_2 - \overline{u}_2 f_2(\overline{u}_1, \overline{u}_2)\big) - \\
&\quad\quad (1+p(t))\overline{u}_2[\tilde{f}_2(t, v_1, w_2) - f_2(\overline{u}_1, \overline{u}_2)] \\
&\quad = p'\overline{u}_2 + \\
&\quad\quad (1+p(t))\overline{u}_2[\tilde{f}_2(t, v_1, w_2) - f_2(\overline{u}_1, \overline{u}_2)] \\
&\quad = p'\overline{u}_2 + \\
&\quad\quad (1+p(t))\overline{u}_2[\tilde{f}_2(t, v_1, w_2) - \tilde{f}_2(t, \overline{u}_1, \overline{u}_2)] \\
&\quad\quad - (1+p(t))\overline{u}_2[\tilde{f}_2(t, \overline{u}_1, \overline{u}_2) - f_2(\overline{u}_1, \overline{u}_2)] \\
&\geq \overline{u}_2 \times \Big(p' - (1+p(t))\,p(t)\,\overline{u}_1 |\frac{\partial \tilde{f}_2}{\partial u_1}(t, \eta_1, \eta_2)| + \\
&\quad (1+p(t))\,p(t)\,\overline{u}_2 |\frac{\partial \tilde{f}_2}{\partial u_2}(t, \eta_1, \eta_2)| \\
&\quad - (1+p(t))[\tilde{f}_2(t, \overline{u}_1, \overline{u}_2) - f_2(\overline{u}_1, \overline{u}_2)]\Big)
\end{aligned}$$

At this point we choose $p(t)$ to be

$$p(t) := (1 + (1+t)^{-\gamma}) \qquad \gamma < c.$$

Then, since

$$|\tilde{f}_2(t, \overline{u}_1, \overline{u}_2) - f_2(\overline{u}_1, \overline{u}_2)| \leq | \leq K(1+t)^{-c},$$

We have

$$\frac{\partial w_2}{\partial t} - \sigma_2 \Delta w_2 - w_2[b + \tilde{f}_2(t, v_1, w_2)] \geq$$

$$(1+t)^{-\gamma}\overline{u}_2 \times \big(-p(1+t)^{-1} - [1+(1+t)^{-\gamma}]\overline{u}_1|\frac{\partial \tilde{f}_2}{\partial u_1}(t, \eta_1, \eta_2)| +$$

$$+ [1+(1+t)^{-\gamma}]\overline{u}_2|\frac{\partial \tilde{f}_2}{\partial u_2}(t, \eta_1, \eta_2)| - c[1+(1+t)^{-\gamma}](1+t)^{-c+\gamma}\big).$$

Therefore

$$\frac{\partial w_2}{\partial t} - \sigma_2 \Delta w_2 - w_2[b + \tilde{f}_2(t, v_1, w_2)] \geq 0$$

if

$$\overline{u}_2(x)|\frac{\partial \tilde{f}_2}{\partial u_2}(t, \eta_1, \eta_2)| \geq \frac{p}{(1+t)[1+(1+t)^{-\gamma}]} + \frac{K}{(1+t)^{c-\gamma}} + \overline{u}_1(x)|\frac{\partial \tilde{f}_2}{\partial u_1}(t, \eta_1, \eta_2)|.$$

Similarly, we have

$$\begin{aligned}
\frac{\partial v_1}{\partial t} &- \sigma_1 \Delta v_1 - v_1[a + \tilde{f}_1(t, v_1, w_2)] \\
&= -\gamma(1+t)^{-\gamma-1}\,\overline{u}_1 \\
&\quad - \sigma_1[1-(1+t)^{-\gamma}]\Delta \overline{u}_1 \\
&\quad\quad - [1-(1+t)^{-\gamma}]\overline{u}_1[a + \tilde{f}_1(t, v_1, w_2)] \\
&= (1+t)^{-\gamma}\overline{u}_1(x) \times \quad \big(-\gamma(1+t)^{-1} \\
&\quad - [1-(1+t)^{-\gamma}]\overline{u}_1(x)|\frac{\partial \tilde{f}_1}{\partial u_2}(t, \eta_1, \eta_2)| \\
&\quad + [1-(1+t)^{-\gamma}]\overline{u}_2(x)|\frac{\partial \tilde{f}_1}{\partial u_1}(t, \eta_1, \eta_2)| \\
&\quad - K[1+(1+t)^{-\gamma}](1+t)^{-c+\gamma}\big).
\end{aligned}$$

Therefore

$$\frac{\partial v_1}{\partial t} - \sigma_1 \Delta v_1 - v_1[a + \tilde{f}_1(t, v_1, w_2)] \geq 0$$

if

$$\overline{u}_1(x)|\frac{\partial \tilde{f}_1}{\partial u_1}(t,\eta_1,\eta_2)| \geq \frac{\gamma}{(1+t)[1-(1+t)^{-\gamma}]} + \frac{K}{(1+t)^{c-\gamma}} + \overline{u}_2(x)|\frac{\partial f_1}{\partial u_2}|.$$

Hence it suffices to find constant γ satisfying the following two inequalities;

$$\begin{cases} \overline{u}_2(x)|\frac{\partial \tilde{f}_2}{\partial u_2}(t,\eta_1,\eta_2)| \geq \gamma + c + \overline{u}_1(x)|\frac{\partial \tilde{f}_2}{\partial u_1}(t,\eta_1,\eta_2)|, \\ \overline{u}_1(x)|\frac{\partial \tilde{f}_1}{\partial u_1}(t,\eta_1,\eta_2)| \geq \gamma + c + \overline{u}_2(x)|\frac{\partial \tilde{f}_1}{\partial u_2}(t,\eta_1,\eta_2|, \end{cases}$$

which is possible if we can choose positive constants α and β such that

$$\begin{cases} \gamma + c \leq \overline{u}_2(x)|\frac{\partial \tilde{f}_2}{\partial u_2}(t,\overline{u}_1,\overline{u}_2)| - \frac{\beta}{\alpha}\overline{u}_1(x)|\frac{\partial \tilde{f}_2}{\partial u_1}(t,\overline{u}_1,\overline{u}_2)|, \\ \gamma + c \leq \overline{u}_1(x)|\frac{\partial \tilde{f}_1}{\partial u_1}(t,\overline{u}_1,\overline{u}_2)| - \frac{\alpha}{\beta}\overline{u}_2(x)|\frac{\partial \tilde{f}_1}{\partial u_2}(t,\overline{u}_1,\overline{u}_2)| \end{cases}$$

Hence we can choose a small positive number γ provided we have the inequalities (2.4). This completes the proof. □

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