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ON THE IDEAL CLASS GROUPS OF REAL ABELIAN FIELDS

JAE MOON KIM

ABSTRACT. Let F_0 be the maximal real subfield of $\mathbb{Q}(\zeta_q + \zeta_q^{-1})$ and $F_\infty = \cup_{n\geq 0} F_n$ be its basic $\mathbb{Z}_p$-extension. Let A_n be the Sylow p-subgroup of the ideal class group of F_n. The aim of this paper is to examine the injectivity of the natural map$A_n \to A_m$ induced by the inclusion $F_n \to F_m$ when $m > n \geq 0$. By using cyclotomic units of F_n and by applying cohomology theory, one gets the following result: If p does not divide the order of A_1, then $A_n \to A_m$ is injective for all $m > n \geq 0$.

1. Introduction

Let q be an odd prime and $F_0 = \mathbb{Q}(\zeta_q + \zeta_q^{-1})$ be the maximal real subfield of $\mathbb{Q}(\zeta_q)$, where ζ_q is a primitive qth root of 1. For each integer $n \geq 1$, we choose a primitive nth root ζ_n of 1 so that $\zeta_m^{\frac{m}{n}} = \zeta_n$ if $n|m$. For each odd prime p satisfying $p \equiv 1 \pmod q$, we consider the basic $\mathbb{Z}_p$-extension $F_\infty = \cup_{n\geq 0} F_n$ of F_0, *i.e.*, $F_n = F_0\mathbb{Q}_n$, where $\mathbb{Q}_n$ is the unique subfield of $\mathbb{Q}(\zeta_{p^{n+1}})$ of degree p^n over $\mathbb{Q}$.

Let A_n be the Sylow p-subgroup of the ideal class group of F_n. It is known that there exist integers $\nu \geq 0$, $\lambda \geq 0$ and μ such that $e_n = \mu p^n + \lambda n + \nu$ for $n \gg 0$, where e_n is the exact power of the order of A_n dividing p(see [7]). In 1979, L. Washington and B. Ferrero proved that $\mu = 0$ in our situation (see [1]), thus $e_n = \lambda n + \nu$ for $n >> 0$. Note that the class field theory says that the norm map $A_m \to A_n$ is surjective for $m > n \geq 0$. However it is unknown whether or not the natural map $A_n \to A_m$ induced by the inclusion $F_n \to F_m$ is injective.

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The aim of this paper is to discuss the injectivity of $A_n \to A_m$ for $m > n \geq 0$. We will study this by means of the behaviors of units and cyclotomic units in the $\mathbb{Z}_p$-extension. Let E_n be the group of units of the ring of integers of F_n and C_n be the subgroup of E_n consisting of cyclotomic units of F_n. By the analytic class number formula, we have $[E_n : C_n] = h_n$, where h_n is the class number of F_n. The natural inclusion $C_m \to E_m$ induces a homomorphism $H^1(G_{m,n}, C_m) \to H^1(G_{m,n}, E_m)$ between the cohomology groups, where $G_{m,n} = \mathrm{Gal}(F_m/F_n)$. In [5], it is proved that this induced map is injective if $\prod_{\substack{\chi\in\hat{\Delta}^+\\ \chi\neq 1}} B_{1,\chi\omega^{-1}} \not\equiv 0 \bmod p$, where Δ^+ is the Galois group $\Delta^+ = \mathrm{Gal}(F_0/\mathbb{Q})$, ω is the Teichmüller character on $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and $B_{1,\chi\omega^{-1}}$ is the first generalized Bernoulli number attached to the character $\chi\omega^{-1}$. In section 2, we will show that this map is actually an isomorphism under certain conditions. And in section 3, we discuss the injectivity of $A_n \to A_m$ for $m > n \geq 0$ by using results from section 2.

2. Induced homomorphism

In this section, we examine when the induced homomorphism

$$H^1(G_{m,n}, C_m) \to H^1(G_{m,n}, E_m)$$

is an isomorphism. As was already mentioned, this map is known to be injective if $p \nmid \prod_{\substack{\chi\in\hat{\Delta}^+\\ \chi\neq 1}} B_{1,\chi\omega^{-1}}$.

THEOREM 1. *Suppose* $p \nmid \prod_{\substack{\chi\in\hat{\Delta}^+\\ \chi\neq 1}} B_{1,\chi\omega^{-1}}$. *The induced homomorphism* $H^1(G_{m,n}, C_m) \to H^1(G_{m,n}, E_m)$ *is an isomorphism if* $p \nmid h_0$, *where* h_0 *is the class number of* F_0.

PROOF. Let $B_m = E_m/C_m$ for each $m \geq 0$. From the short exact sequence $0 \to C_m \to E_m \to B_m \to 0$, we have the following long exact sequence of cohomology groups

$$0 \to C_m^{G_{m,n}} \to E_m^{G_{m,n}} \to B_m^{G_{m,n}} \to H^1(G_{m,n}, C_m) \\ \to H^1(G_{m,n}, E_m) \to H^1(G_{m,n}, B_m) \to \cdots,$$

where $A^{G_{m,n}} = \{a \in A \mid \sigma a = a \quad {}^{\forall}\sigma \in G_{m,n}\}$ for a G-module A. Clearly $E_m^{G_{m,n}} = E_n$. In [2], it is shown that $C_m^{G_{m,n}} = C_n$. Since $p \nmid \prod_{\substack{\chi\in\hat{\Delta}^+\\ \chi\neq 1}} B_{1,\chi\omega^{-1}}$, $H^1(G_{m,n}, C_m) \to H^1(G_{m,n}, E_m)$ is an injection. Hence we have

$$(*)\quad 0 \to C_n \to E_n \to B_M^{G_{m,n}} \xrightarrow{0} H^1(G_{m,n}, C_m) \\ \to H^1(G_{m,n}, E_m) \to H^1(G_{m,n}, B_m) \to \cdots .$$

Therefore we get

$$B_m^{G_{m,n}} \simeq E_n/C_n = B_n \text{ for all } m > n \geq 0.$$

In particular $B_m^{G_{m,0}} = E_0/C_0$, hence the Tate cohomology group $H^0(G_{m,0}, B_m) = B_m^{G_{m,0}}/NB_m = \{0\}$ since $p \nmid h_0 = [E_0 : C_0]$. Since B_m is a finite group, its Herbrand quotient is equal to 1. Thus

$$H^1(G_{m,0}, B_m) = \{0\}.$$

Now consider the following inflation-restriction sequence:

$$0 \to H^1(G_{n,0}, B_m^{G_{m,n}}) \xrightarrow{\text{inf}} H^1(G_{m,0}, B_m) \\ \xrightarrow{\text{res}} H^1(G_{m,n}, B_m)^{G_{m,n}} \xrightarrow{\text{trans}} H^2(G_{n,0}, B_m^{G_{m,n}}) \to \cdots$$

Since $H^2(G_{n,0}, B_m^{G_{m,n}}) \simeq H^0(G_{n,0}, B_n) = \{0\}$, every term in the above exact sequence except $H^1(G_{m,n}, B_m)^{G_{m,n}}$ is trivial. Therefore so is $H^1(G_{m,n}, B_m)^{G_{m,n}}$. Since both $G_{m,n}$ and $H^1(G_{m,n}, B_m)$ are p-groups, $H^1(G_{m,n}, B_m)$ must be trivial. Hence from (*) we get an isomorphism

$$H^1(G_{m,n}, C_m) \xrightarrow{\sim} H^1(G_{m,n}, E_m).$$

3. Injectivity of $A_n \to A_m$

In this section we examine various situations when the natural map $A_n \to A_m$ is injective.

THEOREM 2. *Suppose* $p \nmid \prod_{\substack{\chi \in \hat{\Delta}^+ \\ \chi \neq 1}} B_{1,\chi\omega^{-1}}$ *and* $p \nmid h_0$. *Then* $A_n \to A_m$ *is injective.*

PROOF. Since $H^1(G_{m,n}, C_m) \simeq H^1(G_{m,n}, E_m)$ by Theorem 1, we have

$$H^1(\Gamma_n, C_\infty) \simeq H^1(\Gamma_n, E_\infty) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^l,$$

where $\Gamma_n = \varprojlim \mathrm{Gal}(F_m/F_n) = \mathrm{Gal}(F_\infty/F_0)$ and $l = \frac{1}{2}\varphi(q)$, the number of primes ideals of F_0 above p (see [4]). Let E'_m be the group of p-units of F_m and let $E'_\infty = \cup_{m \geq 0} E'_m$. Then the natural inclusion $E_m \to E'_m$ induces a homomorphism $H^1(G_{m,n}, E_m) \to H^1(G_{m,n}, E'_m)$. Then by taking direct limits under the inflation maps, we have a homomorphism $H^1(\Gamma_n, E_\infty) \to H^1(\Gamma_n, E'_\infty)$. Since $H^1(\Gamma_n, E'_\infty)$ is finite (see [3]) and since $H^1(\Gamma_n, E_\infty)$ is p-divisible, $H^1(\Gamma_n, E_\infty) \to H^1(\Gamma_n, E'_\infty)$ is a zero map. Hence $H^1(G_{m,n}, E_m) \to H^1(G_{m,n}, E'_m)$ is also a zero map.

Suppose that a fractional $\mathfrak{a}_n$ of F_n becomes principal in F_m, say $\mathfrak{a}_n = (\alpha_m)$ for some $\alpha_m \in F_m$. Let σ be a generator of $G_{m,n}$. Since $\mathfrak{a}_n^\sigma = \mathfrak{a}_n$, we have $(\alpha_m)^\sigma = (\alpha_m)$. Thus $\alpha_m^{\sigma-1} = \eta_m$ is a unit in F_m whose norm to F_n equals 1. Since $H^1(G_{m,n}, E_m) \to H^1(G_{m,n}, E'_m)$ is a zero map $\eta_m = \beta_m^{\sigma-1}$ for some p-unit β_m of F_m. So we have $\alpha_m^{\sigma-1} = \beta_m^{\sigma-1}$. Therefore $\alpha_m = \alpha_n \beta_m$ for some $\alpha_n \in F_n$ and thus $\mathfrak{a}_n = (\alpha_m) = (\alpha_n)(\beta_m)$. In [6], it is proved that prime ideals of F_n above p are principal. Hence the ideal $\mathfrak{a}_n(\alpha_n^{-1})$ is a principal ideal (γ_n) for some $\gamma_n \in F_n$. Therefore $\mathfrak{a}_n = (\alpha_n \gamma_n)$.

COROLLARY. *If* $p \nmid h_1$, *then* $A_n \to A_m$ *is injective.*

PROOF. By class field theory, we have $p \nmid h_0$ since $p \nmid h_1$. By theorem 2 of [6], we also have $p \nmid \prod_{\substack{\chi \in \hat{\Delta}^+ \\ \lambda \neq 1}} B_{1,\chi\omega^{-1}}$. Therefore the result follows from theorem 2.

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Department of Mathematics
Inha University
Inchon 402-751, Korea
E-mail address: jmkim@munhak.inha.ac.kr