Kangweon–Kyungki Math. Jour. **4** (1996), No. 1, pp. 71–76

# PERTURBATION AND JUMP OF A SEMI-FREDHOLM OPERATOR

DONG HARK LEE AND PHIL UNG CHUNG

ABSTRACT. The purpose of the present paper is to derive the perturbations and jumps of semi-Fredholm operators.

## 1. Introduction

Throughout this paper, we suppose that $X$ is a Banach space and write $BL(X)$ for the set of all bounded linear operators on $X$.

A linear operator $T \in BL(X)$ is called upper semi-Fredholm if it has closed range with finite dimensional null space, and lower semi-Fredholm if it has closed range with its range of finite codimension.

If $T$ is upper or lower semi-Fredholm, we call it semi-Fredholm.

We shall introduce $T^{\infty}(X) = \cap_{n=1}^{\infty} T^n(X)$ for the hyperrange and $T^{-\infty}(0) = \cup_{n=1}^{\infty} T^{-n}(0)$ for the hyperkernel of $T \in BL(X)$.

$T \in BL(X)$ is hyperexact if $T^{-1}(0) \subseteq T^{\infty}(X)$.

We shall say that $T$ has ascent $\leq k$ if there exists a positive integer $k$ such that $T^{-\infty}(0) = T^{-k}(0)$ and $T$ has descent $\leq k$ if there exists a positive integer $k$ for which $T^{\infty}(X) = T^k(X)$.

The punctured neighborhood theorem says that if $T \in BL(X)$ is semi-Fredholm then there is $\epsilon > 0$ for which $n(T-\lambda)$ and $d(T-\lambda)$ are both constant for $0 < |\lambda| < \epsilon$, $\lambda \in C$, where $n(T) = \dim(T^{-1}(0))$, $d(T) = \text{codim}(T(X))$. We define the jump $j(T)$ of a semi-Fredholm operator $T \in BL(X)$ ;

$$j(T) \stackrel{\text{def}}{=} \begin{cases} n(T) - n(T-\lambda) \text{ if } T \text{ is upper semi-Fredholm,} \\ d(T) - d(T-\lambda) \text{ if } T \text{ is lower semi-Fredholm.} \end{cases}$$

Received December 21, 1995.

1991 *Mathematics Subject Classification.* 47A53.

*Key words and phrases.* semi-Fredholm operator, jump, perturbation.

T.T. West ([5]) has shown that if $j(T) \neq 0$, then there is the smallest integer $t$ for which $T^{-1}(0) \subseteq T^{t-1}(X)$ but $T^{-1}(0) \not\subseteq T^{t}(X)$.

The purpose of this paper is to derive the perturbation and jumps of semi-Fredholm operators.

LEMMA 1. *Let $T$, $T_i \in BL(X), i \in N$ be semi-Fredholm operators. Then*

(1.1)
$$\begin{aligned}(T_1T_2\cdots T_m)^{-1}(0) &\subseteq (T_1T_2\cdots T_m)^{\infty}(X)\\ &\text{iff } (T_1T_2\cdots T_m)^{-\infty}(0) \subseteq (T_1T_2\cdots T_m)^{\infty}(X)\\ &\text{iff } j(T_1T_2\cdots T_m) = 0\end{aligned}$$

*and*

$$T^{-m}(0) \subseteq (T^m)^{\infty}(X) \quad \text{iff} \quad j(T^m) = 0 \tag{1.2}$$

*for each $m \in N$.*

PROOF. Since each $T_i$ is semi-Fredholm, $T_1T_2\cdots T_m$ is also semi-Fredholm. Hence we have

$$\begin{aligned}(T_1T_2\cdots T_m)^{-\infty}(0) &\subseteq (T_1T_2\cdots T_m)^{\infty}(X)\\ &\Longleftrightarrow (T_1T_2\cdots T_m)^{-1}(0) \subseteq (T_1T_2\cdots T_m)^{n}(X) \text{ for each } n \in N\\ &\Longleftrightarrow (T_1T_2\cdots T_m)^{-1}(0) \subseteq (T_1T_2\cdots T_m)^{\infty}(X).\end{aligned}$$

and

$$j(T_1T_2\cdots T_m) = 0 \Longleftrightarrow (T_1T_2\cdots T_m)^{-\infty}(0) \subseteq (T_1T_2\cdots T_m)^{\infty}(X).$$

Using the above argument, we have

$$T^{-m}(0) \subseteq (T^m)^{\infty}(X) \Longleftrightarrow j(T^m) = 0,$$

for each $m \in N$. □

## 2. Main Results

THEOREM 2. *Let each $T_i \in BL(X), i \in N$ be semi-Fredholmand let $k$ and $t$ be the smallest integers for which , for each $m \in N$,*

$$(T_1T_2\cdots T_m)^{-1}(0) \cap (T_1T_2\cdots T_m)^{\infty}(X) = (T_1T_2\cdots T_m)^{-1}(0) \cap (T_1T_2\cdots T_m)^{k}(X),$$

*and*

$$(T_1T_2\cdots T_m)^{-1}(0) \subseteq (T_1T_2\cdots T_m)^{t-1}(X),$$
$$(T_1T_2\cdots T_m)^{-1}(0) \not\subseteq (T_1T_2\cdots T_m)^{t}(X)$$

*respectively. If either $k < t$ or $k = t$ and $T_1T_2\cdots T_m$ is hyperexact, then we have*

$$d((T_1T_2\cdots T_m)^k) = kd(T_1T_2\cdots T_m) \tag{2.1}$$

*and*

$$j((T_1T_2\cdots T_m)^k) = kj(T_1T_2\cdots T_m). \tag{2.2}$$

PROOF. Since each operator of the form $(T_1T_2\cdots T_m)^n$, $n \in N$ is semi-Fredholm, we have

$$X/((T_1T_2\cdots T_m)^n(X) + (T_1T_2\cdots T_m)^{-1}(0)) \cong (T_1T_2\cdots T_m)(X)/(T_1T_2\cdots T_m)^{n+1}(X).$$

In particular, if $k < t$, then

$$(T_1T_2\cdots T_m)^{-1}(0) \subseteq (T_1T_2\cdots T_m)^{k}(X).$$

Hence we have

$$X/(T_1T_2\cdots T_m)^k(X) \cong (T_1T_2\cdots T_m)(X)/(T_1T_2\cdots T_m)^{k+1}(X).$$

By the inductive steps, we have

$$X/(T_1T_2\cdots T_m)^k(X) \cong k(X/(T_1T_2\cdots T_m)(X)).$$

Since $(T_1 \cdots T_m)$, $(T_1 \cdots T_m)^k$ are semi-Fredholm, $X/(T_1T_2 \cdots T_m)(X)$ and $X/(T_1T_2 \cdots T_m)^k(X)$ are finite dimensional normed spaces. Then,

$$d((T_1T_2 \cdots T_m)^k) = kd(T_1T_2 \cdots T_m).$$

By the duality of $d((T_1T_2 \cdots T_m)^k)$, we have

$$n((T_1T_2 \cdots T_m)^k) = kn(T_1T_2 \cdots T_m).$$

If $k = t$ and $T_1T_2 \cdots T_m$ is hyperexact, then

$$(T_1T_2 \cdots T_m)^{-1}(0) \subseteq (T_1T_2 \cdots T_m)^{k=t}(X) = (T_1T_2 \cdots T_m)^{\infty}(X).$$

Since $j(T_1T_2 \cdots T_m) = 0$ by Lemma 1, we have

$$kn(T_1T_2 \cdots T_m) = kn(T_1T_2 \cdots T_m - \lambda) = n((T_1T_2 \cdots T_m)^k)$$

for sufficiently small $\lambda$. Using the punctured neighborhood theorem,

$$\begin{aligned} n((T_1T_2 \cdots T_m - \lambda)^l) &- n((T_1T_2 \cdots T_m)^l - \mu) \\ &= l[n(T_1T_2 \cdots T_m)] - l[n(T_1T_2 \cdots T_m)] = 0 \end{aligned}$$

for each $l \in N$ and for a sufficiently small $\mu \in C$. Hence

$$n((T_1T_2 \cdots T_m)^k) = kn(T_1T_2 \cdots T_m),$$

and

$$j((T_1T_2 \cdots T_m)^k) = kn(T_1T_2 \cdots T_m).$$

By the duality of $n((T_1T_2 \cdots T_m)^k)$,

$$d((T_1T_2 \cdots T_m)^k) = kd(T_1T_2 \cdots T_m).$$

Thus, we have the required results. □

COROLLARY 3. *Let $T \in BL(X)$ be semi-Fredholm and let $k$ and $t$ be the smallest integer such that $T^{-1}(0) \cap T^{\infty}(X) = T^{-1}(0) \cap T^k(X)$ and $T^{-1}(0) \subseteq T^{t-1}(X)$ but $T^{-1}(0) \nsubseteq T^t(X)$ respectively. If either $k < t$ or $k = t$ and $T$ is hyperexact, then we have*

(3.1) $$d(T^k) = kd(T),$$

(3.2.) $$j(T^k) = kj(T).$$

PROOF. Using Theorem 2, our assertion can be easily proved. □

THEOREM 4. *Let $S$, $T \in BL(X)$ commute.*

*(4.1) If $T$ is an upper semi-Fredholm operator with finite ascent, and $S$ is a compact operator with finite ascent, then*

$$j(T+S) = n(T+S).$$

*(4.2) If $T$ is a lower semi-Fredholm operator with finite descent and $S$ is a compact operator with finite descent, then*

$$j(T+S) = d(T+S).$$

*(4.3) If $S+T$ is a semi-Fredholm operator with finite ascent and $t$ is the smallest integer for which $(T+S)^{-1}(0) \subseteq (T+S)^{t-1}(X)$ but $(T+S)^{-1}(0) \not\subseteq (T+S)^{t}(X)$ and either $k < t$ or $k = t$ and $T+S$ is hyperexact, then*

$$j((T+S)^k) = kj(T+S) = kn(T+S).$$

PROOF. Suppose that $T$ is upper semi-Fredholm with finite ascent and $S$ is compact operator with finite ascent for (4.1). Then $T+S$ is upper semi-Fredholm and it has finite ascent. If $T+S$ has ascent $k$, then we have

$$(T+S)^{-1}(0) \cap (T+S)^k(X) = \{0\}.$$

and

$$\dim(T+S-\lambda)^{-1}(0) = \dim((T+S)^{-1}(0) \cap (T+S)^{\infty}(X))$$

for sufficiently small $\lambda$. Since

$$(T+S)^{-1}(0) \cap (T+S)^{\infty}(X) = (T+S)^{-1}(0) \cap (T+S)^{k}(X) = \{0\},$$

we have

$$\dim((T+S-\lambda)^{-1}(0)) = 0.$$

Hence

$$j(T+S) = n(T+S) - n(T+S-\lambda) = n(T+S)$$

for sufficiently small $\lambda$. And we have $d(T+S-\lambda) = 0$ for sufficiently small $\lambda$([1],[3]). Thus $j(T+S) = d(T+S)$. Let $t$ be the smallest integer for which $(T+S)^{-1}(0) \subseteq (T+S)^{t-1}(X)$ but $(T+S)^{-1}(0) \not\subseteq (T+S)^{t}(X)$ and $k \leq t$ for (4.3). If $T+S$ has finite ascent, then, (4.3) follows at once from Corollary 3 and (4.1). □

## References

1. Dong Hark Lee and Woo Yong Lee, *The Jump of a Semi-Fredholm Operator*, Comm. Korean Math. Soc. 9, No.3 (1994), 593–598.
2. M. O'SEARCOLD and T.T. West, *Continuity of the generalized kernel and range of semi-Fredholm operators*, Math. Proc. Camb. Phil. Soc. 105 (1989), 513-522.
3. R.E. Harte, Invertibility and Singularity for Bounded Linear Operators (1988), Dekker, New York, 174–297.
4. ______, *Taylor Exactness and Kato's Jump*, Proc. A.M.S, Vol. 119, No. 3 (1993), 793–801.
5. T.T. West, M.R.I.A, *A Riesz-Schauder Theorem for semi-Fredholm Operators*, Proc. R. Ir. Acad. Vol.87A. 2 (1987), 137–146.
6. T.T. West, *Removing the Jump-kato's Decomposition*, Rocky Mountain Journal of Mathematics, Vol. 20, No. 2, Spring (1990), 603–612.

Dong Hark Lee
Department of Mathematics Education
Kangwon National University
Chuncheon 200-701, Korea

Phil Ung Chung
Department of Mathematics
Kangwon National University
Chuncheon 200-701, Korea