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ITERATIVE APPROXIMATION TO M-ACCRETIVE OPERATOR EQUATIONS IN BANACH SPACES

Jong An Park and Yang Seob Park

Abstract. In 1994 Z.Liang constructed an iterative method for the solution of nonlinear equations involving m-accretive operators in uniformly smooth Banach spaces. In this paper we apply the slight variants of Liang's iterative methods and generalize the results of Z.Liang. Moreover our proof is more simple than Liang's proof.

1. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space. A Banach space $(X, \|\cdot\|)$ is called smooth if the norm of X is Gâteaux differentiable on $X - \{0\}$. The normalized duality mapping J is defined by $J(x) = \{x^* \in X^* | (x, x^*) = \|x\|^2, \|x^*\| = \|x\|\}$, where X^* is the dual of X and $(\ ,\ )$ is the dual pairing. In a smooth Banach space J is single-valued. A Banach space $(X, \|\cdot\|)$ is called uniformly smooth if X^* (the dual of X) is uniformly convex. In a uniformly smooth Banach space the duality mapping J is uniformly continuous on any bounded subset of X.

Denote the closed ball $\{x \in X : \|x - y\| \le r\}$ by $B(x, r)$. And the domain and range of a operator A is denoted by $D(A), R(A)$ respectively.

An operator $A : D(A) \subset X \longrightarrow X$ is said to be accretive if for any x, y in $D(A)$ there exists $j \in J(x - y)$ such that

$$(Ax - Ay,\ j) \ge 0.$$

An accretive operator A is called m-accretive if $R(A + \lambda I) = X$ for all $\lambda > 0$, where I denotes the identity operator. If $A : D(A) \subset X \longrightarrow X$

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is m-accretive, then for any $f \in X$ the equation

$$x + Ax = f \cdots\cdots\cdots (*)$$

has unique solution q in $D(A)$. The problem is to construct an iteration which converges strongly or weakly to the unique solution q. If X is a Hilbert space, the problem has been studied by Bruck[1], Chidume[2], and Dotson[4]. Chidume[3], and Weng[7] also studied it in L_p spaces. In 1994 Z.Liang[5] generalized the results of Chidume[3]. Our theorems generalized Liang's results.

We need the following lemmas.

Lemma A. *(Park[6]). Let $\{\beta_n\}$ be a nonnegative real sequence and suppose $\{\beta_n\}$ satisfies the following inequality*

$$\beta_{n+1} \leq (1-\alpha_n)\beta_n + \epsilon\alpha_n,$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty, \alpha_n \in (0,1), \epsilon > 0$. Then $0 \leq \limsup_{n\to\infty} \beta_n \leq \epsilon$.

Lemma B. *(Park[6]). Let $(X, \|\cdot\|)$ be a smooth Banach space. Suppose one of the followings holds.*

(1) *J is uniformly continuous on any bounded subsets of X.*
(2) *$(x-y, J(x)-J(y)) \leq \|x-y\|^2$ for all x, y in X.*
(3) *For any bounded subset B of X there is a real function $c : R^+ \longrightarrow R^+$ such that*

$$(x-y, J(x)-J(y)) \leq c(\|x-y\|)$$

for all x, y in B where c satisfies $\lim_{t\to 0^+} c(t)/t = 0$.

Then for any $\epsilon > 0$ and any bounded subset B there is $\delta > 0$ such that

$$\|tx + (1-t)y\|^2 \leq 2(x, J(y))t + 2\epsilon t + (1-2t)\|y\|^2$$

for all $x, y \in B$ and $t \in [0, \delta)$.

Remark 1. If X is uniformly smooth, then (1) in Lemma B holds. If X is a Hilbert space, then (2) in Lemma B holds.

2. Main results

In this section we construct an iteration which converges to a unique solution of $(*)$. The convergence can be shown by the inequality in Lemma B. Furthermore we obtain Liang's result in more generalized Banach spaces.

THEOREM 1. *Let* $(X, \|\cdot\|)$ *be a smooth Banach spaces satisfying one of the assumptions in Lemma B. Let* $A : D(A) \subset X \longrightarrow X$ *be an m-accretive and Lipschitzian operator with Lipschitz constant* L *where* $D(A)$ *is open. Let* $q \in D(A)$ *be a unique solution of* $x + Ax = f$ *for given* f *in* X. *Define* $S : D(A) \longrightarrow X$ *by* $Sx = -Ax + f$ *and let* $\{c_n\} \subset [0,1]$ *satisfy*

(i) $\sum_{n=0}^{\infty} c_n = \infty$,
(ii) $\lim_{n\to\infty} c_n = 0$.

Then there exists a closed ball B *of* q *such that* $B \subset D(A)$ *and for any* x_0 *in* B *we can construct* $\{x_n\}$ *in* B *and* $\{p_n\}$ *in* X *such that for all* $n \geq 0$,

(i) $p_n = (1 - c_n)x_n + c_n Sx_n$,
(ii) $\|x_{n+1} - q\| \leq \|p_n - q\|$.

And all iterations $\{x_n\}, \{p_n\}$ *satisfying (i),(ii) converge strongly to* q.

Proof. Since A is accretive, $(Sx - Sy, J(x - y)) \leq 0$ for all x, y in $D(A)$. And q is a fixed point of S. Since $D(A)$ is open, there exists $d > 0$ such that $B(q, d) \subset D(A)$. We may suppose that the Lipschitz constant L is greater than 1. Then we denote $B(q, \frac{d}{2L})(\subset B(q, d))$ by B. For any $x_0 \in B$,

$$\|Sx_0 - q\| = \|Sx_0 - Sq\| \leq L\|x_0 - q\| \leq \frac{d}{2} < d.$$

So $p_0 = (1 - c_0)x_0 + c_0 Sx_0 \in B(q, d)$ and if $p_0 \in B$, then we define $x_1 = p_0$ and if $p_0 \in B(q, d) - B$, then we define $x_1 = q + \frac{d}{2L\|p_0 - q\|}(p_0 - q)$. In both case $x_1 \in B$ and

$$\|x_1 - q\| \leq \|p_0 - q\|.$$

Since $x_1 \in B, p_1 = (1 - c_1)x_1 + c_1 Sx_1 \in B(q, d)$. Inductively we can define $x_2, p_2, x_3, p_3, \cdots\cdots, x_n, p_n$ such that

(i) $x_n \in B$ and $p_n \in B(q, d)$,

(ii) $p_n = (1 - c_n)x_n + c_n S x_n$,
(iii) $\|x_{n+1} - q\| \le \|p_n - q\|$.

Now we show that p_n converges strongly to q. By Lemma B, for any $\epsilon > 0$ and $B(0, d)$ we can choose $\delta > 0$ such that

$$\|tx + (1-t)y\|^2 \le 2(x, J(y))t + 2\epsilon t + (1-2t)\|y\|^2$$

for all $x, y \in B(0, d)$ and $t \in [0, \delta)$. Since $c_n \to 0$, there exists N such that $c_n < \delta$ for all $n \ge N$. Here $x_n - q, Sx_n - q$ are in $B(0, d)$. For such n

$$\begin{aligned}\|p_n - q\|^2 &= \|(1-c_n)x_n + c_n S x_n - q\|^2 \\ &= \|(1-c_n)(x_n - q) + c_n(Sx_n - q)\|^2 \\ &\le 2(Sx_n - q, J(x_n - q))c_n + 2\epsilon c_n + (1-2c_n)\|x_n - q\|^2.\end{aligned}$$

Since $(Sx_n - q, J(x_n - q)) = (Sx_n - Sq, J(x_n - q)) \le 0$,

$$\|p_n - q\|^2 \le 2\epsilon c_n + (1-2c_n)\|x_n - q\|^2.$$

Since $\|x_{n+1} - q\|^2 \le \|p_n - q\|^2$,

$$\|p_n - q\|^2 \le 2\epsilon c_n + (1-2c_n)\|p_{n-1} - q\|^2.$$

We let $\beta_n = \|p_n - q\|$ and $\alpha_n = 2c_{n+1}$ in Lemma A. Then $0 \le \limsup_{n\to\infty} \|p_n - q\| \le \epsilon$. Since ϵ is arbitrary, p_n converges strongly to q. □ □

Remark 2. Z.Liang [5] chose x_{n+1} such that

$$\|p_n - x_{n+1}\| = \inf\{\|p_n - x\| : x \in B\}$$

by using the reflexivity of a uniformly smooth Banach space X. But we don't know the reflexivity of our Banach spaces. So our choice of x_{n+1} which is really the same choice in Liang [5] is possible without the reflexivity of X.

REMARK 3. If the Lispchitz constant L in Theorem 1 is equal to(or less than) 1, then we can choose $x_{n+1} = (1 - c_n)x_n + c_n Sx_n$ and $x_n(\in B(q, d))$ converges strongly to q. Furthermore if $D(A) = X$ and the range of A is bounded, then Theorem 1 can be deduced by the same way in its proof without the Lipschitz conditions of A. And in this case the iterations are really the method of steepest descent in [8].

DEFINITION. An operator $A : D(A) \subset X \longrightarrow X$ is called locally Lipschitzian with constant L if for each $q \in D(A)$ there is an $r > 0$ such that

$$\|Ax - Ay\| \le L\|x - y\| \text{ for all } x, y \in B(q, r).$$

We also obtain the following theorem.

THEOREM 2. *Let $(X, \|\cdot\|)$ be a smooth Banach space satisfying one of the assumptions in Lemma B. Let $A : D(A) \subset X \longrightarrow X$ be an m-accretive and locally Lipschitzian operator with Lipschitz constant L where $D(A)$ is open. Then we can construct $\{p_n\}, \{x_n\}$ as in Theorem 1 so that p_n converges strongly to q, which is a unique solution of $(*)$.*

Proof. We can choose $d_1, d_2 > 0$ such that $B(q, d_1) \subset D(A)$ and A is a Lipschitzian operator on $B(q, d_2)$. We set $d = \min\{d_1, d_2\}$ and the same argument as in the proof of Theorem 1 holds in $B(q, d)$. □ □

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Department of Mathematics
Kangwon University
Chunchon 200-701, Korea
E-mail: jongan@cc.kangwon.ac.kr