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ON THE COMPARISON OF COMPLEXES OF MODULES OF GENERALIZED FUNCTIONS AND GENERALIZED HUGHES COMPLEXES

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0. Introduction

This paper continues the development of the theroy of generalized Hughes complexes for a module M over a commutative ring R which is not necessarily noetherian. These complexes, introduced by R.Y. Sharp and M. Yassi in [**8**] provide an 'umbrella concept' which covers all the complexes of modules of generalized fractions of type described by L.O' Carroll in [**5**], and all the algebraic Cousin complexes (in Noetherian case) previously studied by R.Y. Sharp.

One of the main results of [**8**] is Theorem 3.5, which shows that, given a chain $\mathcal{U}$ of triangular subsets on R(see [**5**, p.420]), there is a family $\mathcal{S}(\mathcal{U})$ of systems of ideals of R(see [**8**, 2.6]), such that when R is Noetherian, the complex of modules of generalized fractions $C(\mathcal{U}, M)$ is isomorphic over Id_M to the generalized Hughes complex $\mathcal{H}(\mathcal{S}(\mathcal{U}), M)$. (we say that a morphism of complexes $\Psi = (\Psi^i)_{i\geq -2} : C_1^\bullet \longrightarrow C_2^\bullet$ is over Id_M if $\Psi^{-1} : M \longrightarrow M$ is the identity mapping Id_M). At the end of [**8**], it was asked whether there is any analogue of that theorem in the case when R is not necessarily Noetherian.

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The purpose of [7] is to address that question. In [7] it has been proved that, in general, there is a morphism of complexes (over Id_M)

$$\Theta = (\theta^i)_{i\geq -2} : \mathcal{H}(\mathcal{S}(\mathcal{U}), M) \longrightarrow C(\mathcal{U}, M)$$

which is isomorphism when R is N-ring (and, in particular, when R is Noetherian). They also gave an example to show that Θ is not always an isomorphism. (Note that this example does not provide a negative answer to above mentioned question).

In this paper we intend to respond to that question more completely. We show that if Θ is not an isomorphism, then there is no isomorphism of complexes over Id_M from $\mathcal{H}(\mathcal{S}(\mathcal{U}), M)$ to $C(\mathcal{U}, M)$. This gives a negative answer to Sharp and Yassi's question in general case (see 4.4). We also establish a necessary and sufficient condition for the above complexes to be isomorphic.

In section 2 we introduce the notion of divisibility in certain generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$. This notion can be used when one wishes to work with elements in the terms of the complex $\mathcal{H}(\mathcal{S}, M)$.

In section 3, by using the above notion, we define explicity a morphism of complexes $\mathcal{H}(\mathcal{S}, M) \to C(\mathcal{U}, M)$ over Id_M and we prove that this morphism is unique over Id_M. Moreover, in 3.3 we establish equivalent conditions for Θ to be an isomorphism.

In the final section, which is devoted to applications of divisibility, we prove an stronger form of the main result of [8] and [7] (see 4.3). Also, in this section we provide a necessary and sufficient condition for the exactness of the certain generalized Hughes complexes $\mathcal{H}(\mathcal{S}, M)$.

1. Preliminaries

Throughout this paper, R will denote a commutative ring (with non-zero identity) and M will denote an R-module; $\mathcal{C}(R)$ will denote the category of all R-modules and R-homomorphisms. We use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers. For any $n \in \mathbb{N}$, $D_n(R)$ denotes the set of $n \times n$ lower triangular matrices over R. Incidently we denote the entries of each element of $D_n(R)$ by the corresponding small letters. For $H = (h_{ij}) \in D_n(R)$, the determinant of H is denoted by $|H|$ and h_{ii} is denoted by h_i for $1 \leq i \leq n$. We use $^{\mathrm{T}}$ to denote matrix transpose. Given $H \in D_n(R)$ with $n > 1$ we also use H^* to denote the $(n-1)\times(n-1)$ submatrix of H obtained by deletion of the nth row and nth column of H.

1.1 REMINDER: COMPLEXES OF MODULES OF GENERALIZED FRACTIONS.

The concept of a chain of triangular subsets on R is explained in [**5**, p.420] and [**8**, 2.3]. Such a chain $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ determines a complex of modules of generalized fractions

$$0 \to M \xrightarrow{e^0} U_1^{-1}M \xrightarrow{e^1} U_2^{-2}M \to \cdots \to U_n^{-n}M \xrightarrow{e^n} U_{n+1}^{-n-1}M \to \cdots$$

in which $e^0(m) = m/(1)$ for all $m \in M$ and

$$e^n\left(\frac{m}{(u_1,\cdots,u_n)}\right) = \frac{m}{(u_1,\cdots,u_n,1)}$$

for all $n \in \mathbb{N}$, $m \in M$ and $(u_1,\cdots,u_n) \in U_n$. We shall denote this complex by $C(\mathcal{U},\ M)$. We shall need to use many of the properties of modules of generalized fractions reviewed in [**8**, section 2].

1.2 A REVIEW ON THE CONSTRUCTION OF GENERALIZED HUGHES COMPLEXES.

A system of ideals of R [**1**] is a non-empty set Φ of ideals of R such that, whenever $\mathfrak{a}, \mathfrak{b} \in \Phi$, there exists $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{ab}$.

Note that (see [**8**, 1.2]) Φ gives rise to an additive, left exact functor

$$D_\Phi := \varinjlim_{\mathfrak{b}\in\Phi} \mathrm{Hom}_R(\mathfrak{b}, .)$$

from $\mathcal{C}(R)$ to itself. Given $\mathfrak{a} \in \Phi$ and an R-module G, we shall assume that $[\ ] : \mathrm{Hom}_R(\mathfrak{a}, G) \to D_\Phi(G)$ is the canonical homomorphism, and that, for each $x \in G$, the homomorphism $\lambda_{\mathfrak{a},x} : \mathfrak{a} \to G$ is such that $\lambda_{\mathfrak{a},x}(r) = rx$ for all $r \in \mathfrak{a}$. Now there is an R-homomorphism $\eta_\Phi(G) : G \to D_\Phi(G)$ which is such that, for each $g \in G$, $\eta_\Phi(G)(g) = [\lambda_{\mathfrak{b},g}]$ (for any $\mathfrak{b} \in \Phi$).

Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ be a family of systems of ideals of R. The generalized Hughes complex for M with respect to $\mathcal{S}$ has the form

$$0 \to M \xrightarrow{h^{-1}} K^0 \xrightarrow{h^0} K^1 \xrightarrow{h^1} K^2 \to \cdots \to K^i \xrightarrow{h^i} K^{i+1} \to \cdots$$

and is denoted by $\mathcal{H}(\mathcal{S}, M)$. This complex, which is a generalization of one constructed by K.R. Hughes in [**3**], is described as follows. Write $K^{-2} = 0, K^{-1} = M$, and use $h^{-2} : K^{-2} \to K^{-1}$ to denote the zero homomorphism. Then, for all $n \in \mathbb{N}_0, K^n := D_{\Phi_{n+1}}(\text{Coker}h^{n-2})$, while $h^{n-1} : K^{n-1} \to K^n$ is the composition of the natural epimorphism $\pi_{n-1} : K^{n-1} \to \text{Coker } h^{n-2}$ and the homomorphism

$$\eta_{\Phi_{n+1}}(\text{Coker } h^{n-2}) : \text{Coker } h^{n-2} \longrightarrow D_{\Phi_{n+1}}(\text{Coker } h^{n-2}) = K^n.$$

1.3 NOTATION.

Throughout this paper, $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ will denote a chain of triangular subsets on R and $\mathcal{S} = (\Phi_n)_{n\in\mathbb{N}}$ will denote a family of systems of ideals of R such that, for each $n \in \mathbf{N}$, the set

$$\Phi(U_n) := \{\sum_{j=1}^{n} Ru_j : (u_1, \cdots, u_n) \in U_n\}$$

is a subset of Φ_n. Note that, by [**8**, 2.5], $\Phi(U_n)$ is a system of ideals of R. We shall also maintain the notations of 1.1 and 1.2 for $C(\mathcal{U}, M)$ and $\mathcal{H}(\mathcal{S}, M)$ respectivly without further command.

2. Divisibility in generalized Hughes complexes

In this section, we introduce the concept of divisibility in $\mathcal{H}(\mathcal{S}, M)$ which is, in certain situation, similar to the concept of repeated division in Cousin complexes introduced by R.Y. Sharp in [**6**, § 2]. We shall also establish some properties of this concept.

LEMMA 2.1. *Let $n \in \mathbb{N}_0$, $m \in M$ and $u = (u_1, \cdots, u_{n+1}) \in U_{n+1}$. Assume that there exist $H \in D_{n+1}(R)$, $w = (w_1, \cdots, w_{n+1})$*

$\in U_{n+1}$ and $f_i \in \mathrm{Hom}_R\,(\sum_{j=1}^{i+1} Rw_j,\ Coker\, h^{i-2})$ *for all* $i = 0, \cdots, n$ *such that*

$$(a) \quad Hu^T = w^T \quad \text{and}$$

$$(b) \quad f_i(\sum_{j=1}^{i+1} a_j w_j) = \begin{cases} a_1 h_1 m & i = 0 \\ a_{i+1} h_{i+1} \pi_{i-1}([f_{i-1}]) & i \neq 0 \end{cases}$$

for any choices of $a_1, \cdots, a_{n+1} \in R$. *Then*
(i) $u_j[f_i] = 0$ *for all integers* i, j *with* $1 \leq j \leq i \leq n$ *and*
(ii) $\pi_i(u_{i+1}[f_i]) = 0$ *for all* $i = 0, 1, \cdots, n$.

Proof. It follows from (b) that $u_1 f_0 = \lambda_{Rw_1,m}$. Hence the claim is true in the case when $n = 0$.

Now suppose, inductively, that $n \in \mathbf{N}$ and the result has been proved for all non-negative integers less than n. It immediately follows from the inductive hypothesis that $u_j[f_i] = 0$ for all i, j with $1 \leq j \leq i \leq n-1$ and that $\pi_i(u_{i+1}[f_i]) = 0$ for all $0 \leq i \leq n-1$. In order to complete the inductive step, it is enough to show that $u_j[f_n] = 0$ for $j \leq n$ and $\pi_n(u_{n+1}[f_n]) = 0$. Let $1 \leq k \leq n$ and $a_1, \cdots, a_{n+1} \in R$. Then, by (b) and the inductive hypothesis,

$$u_k f_n(\sum_{j=1}^{n+1} a_j w_j) = a_{n+1} h_{n+1} \pi_{n-1}(u_k[f_{n-1}]) = 0.$$

Also, it follows from inductive hypothesis and $(a), (b)$ that,

$$u_{n+1} f_n(\sum_{j=1}^{n+1} a_j w_j) = a_{n+1} w_{n+1} \pi_{n-1}([f_{n-1}]) = \lambda_{\mathfrak{a},x}(\sum_{j=1}^{n+1} a_j w_j),$$

where $\mathfrak{a} = \sum_{j=1}^{n+1} Rw_j$, $x = \pi_{n-1}([f_{n-1}])$. Therefore $u_{n+1}[f_n] = [u_{n+1} f_n] = [\lambda_{\mathfrak{a},x}]$ and consequently $\pi_n(u_{n+1}[f_n]) = 0$. The inductive step is therefore complete.

LEMMA 2.2. *Let the situation and notation be as in 2.1. Moreover, assume that there exist* $H' \in D_{n+1}(R), w' = (w'_1, \cdots, w'_{n+1}) \in U_{n+1}$ *and* $f'_i \in \mathrm{Hom}_R(\sum_{j=1}^{i+1} Rw'_j,\ \mathit{Coker}\ h^{i-2})$ *for all* $i = 0, 1, \cdots, n$ *such that*

$$(a') \quad H'u^T = w'T \quad \text{and}$$

$$(b') \quad f'_i(\sum_{j=1}^{i+1} a_j w'_j) = \begin{cases} a_1 h'_1 m & i = 0 \\ a_{i+1} h'_{i+1} \pi_{i-1}([f_{i-1}]) & i \neq 0 \end{cases}$$

for all $a_1, \cdots, a_{n+1} \in R$.

Then $[f_i] = [f'_i]$, *in* K^i, *for all* $i = 0, 1, \cdots, n$.

Proof. In the case when $n = 0$ the proof is a straightforward adaptation of following. So suppose, inductively, that $n \in \mathbb{N}$ and the result has been proved for $n-1$. There are $L, L' \in D_{n+1}(R)$ and $z = (z_1, \cdots, z_{n+1}) \in U_{n+1}$ such that $Lw^T = z^T = L'w'^T$. It therefore follows that $f_n(z_j) = 0$ for $j = 1, \cdots, n$ and

$$f_n(z_{n+1}^2) = z_{n+1} f_n(z_{n+1}) = z_{n+1} l_{n+1} h_{n+1}(\pi_{n-1}([f_{n-1}])).$$

Also, since $z^T = L'H'u^T$, it follows from 2.1 that

$$f_n(z_{n+1}^2) = l'_{n+1} h'_{n+1} l_{n+1} h_{n+1} u_{n+1}(\pi_{n-1}([f_{n-1}])).$$

Similarly, we can show that $f'_n(z_j) = 0$ for all $j = 1, \cdots, n$ and

$$f'_n(z_{n+1}^2) = l_{n+1} h_{n+1} l'_{n+1} h'_{n+1} u_{n+1}(\pi_{n-1}([f'_{n-1}])).$$

So, by inductive hypothesis, the restriction of f_n and f'_n to $(\sum_{j=1}^{n} Rz_j + Rz_{n+1}^2)$ are equal and consequently $[f_n] = [f'_n]$ *in* K^n.

Now we can define the concept of divisibility in the complex $\mathcal{H}(\mathcal{S}, M)$.

DIFINITION 2.3. Let $n \in \mathbf{N}_0, m \in M$ and $u = (u_1, \cdots, u_{n+1}) \in U_{n+1}$. We say that m is divisible by u or that u divides m with respect to $\mathcal{S}$ (briefly u divides m), if there exist $H \in D_{n+1}(R)$, $w = (w_1, \cdots, w_{n+1}) \in U_{n+1}$ and $f_i \in \mathrm{Hom}_R(\sum_{j=1}^{i+1} Rw_j, \text{ Coker } h^{i-2})$ for all $i = 0, 1, \cdots, n$ such that

$$(a) \quad Hu^T = w^T \quad \text{and}$$

$$(b) \quad f_i(\sum_{j=1}^{i+1} a_j w_j) = \begin{cases} a_1 h_1 m & i = 0 \\ a_{i+1} h_{i+1} \pi_{i-1}([f_{i-1}]) & i \neq 0 \end{cases}$$

for any choices of $a_1, \cdots, a_{n+1} \in R$.

Let the situation be as above. Then the class $[f_i] \in K^i$ is denoted by $m \div (u_1, \cdots, u_{i+1})$ for all $i = 0, 1, \cdots, n$. Note that if m is divisible by $(u_1, \cdots, u_{n+1})$, then m is divisible by $(u_1, \cdots, u_{i+1})(0 \leq i \leq n)$. In the rest of this paper, we interpret $m \div (u_1, \cdots, u_n)$ as m whenever $n = 0$.

In the following theorem we provide a charactrization of divisibility.

THEOREM 2.4. *Let $n \in \mathbf{N}_0, m \in M$ and $u = (u_1, \cdots, u_{n+1}) \in U_{n+1}$. Then the following statements are equivalent.*

(i) u divides m.

(ii) $(u_1, \cdots, u_n)$ divides m and there are $w = (w_1, \cdots, w_{n+1}) \in U_{n+1}$, $H \in D_{n+1}(R)$ and $f_n \in \mathrm{Hom}_R(\sum_{j=1}^{n+1} Rw_j, \text{Coker } h^{n-2})$ such that $Hu^T = w^T$ and

$$f_n(\sum_{j=1}^{n+1} a_j w_j) = a_{n+1} h_{n+1} \pi_{n-1}(m \div (u_1, \cdots, u_n)).$$

(iii) $(u_1, \cdots, u_n)$ divides m and there are $w = (w_1, \cdots, w_{n+1}) \in U_{n+1}$, $H \in D_{n+1}(R)$ such that $Hu^T = w^T$ and

$$(\sum_{j=1}^{n} Rw_j :_R w_{n+1}) \subseteq ann_R(\pi_{n-1}(h_{n+1}(m \div (u_1, \cdots, u_n)))). \qquad (*)$$

Moreover, if (ii) is satisfied, then $m \div u = [f_n]$.

Proof. The implications (i)$\Longrightarrow$(ii) and (ii)$\Longrightarrow$(iii) are clear.

(iii) $\Longrightarrow$ (i) It follows from $(*)$ that there is an R-homomorphism $f_n : \sum_{j=1}^{n+1} Rw_j \to \text{Coker } h^{n-2}$ which is such that for $a_1, \cdots, a_n \in R$

$$f_n(\sum_{j=1}^{n+1} a_j w_j) = a_{n+1} h_{n+1} \pi_{n-1}(m \div (u_1, \cdots, u_n)).$$

Also, by 2.3, there exist $v = (v_1, \cdots, v_n) \in U_n, K \in D_n(R)$ and $f_i \in \text{Hom}_R(\sum_{j=1}^{i+1} Rv_j, \text{ Coker } h^{i-2})$ such that $K(u_1, \cdots, u_n)^T = v^T$ and

$$f_i(\sum_{j=1}^{i+1} a_j v_j) = a_{i+1} k_{i+1} \pi_{i-1}([f_{i-1}])$$

for all $i = 0, 1, \cdots, n-1$ and $a_1, \cdots, a_n \in R$. Next, there are $P, Q \in D_{n+1}(R)$ and $z = (z_1, \cdots, z_{n+1}) \in U_{n+1}$ such that $P(v,1)^T = z^T = Qw^T$. Let $L = (l_{ij}) \in D_{n+1}(R)$ be such that $L^* = P^* K$, and $l_{n+1j} = h'_{n+1j}$ for each $1 \leq j \leq n+1$, where $(h'_{ij}) = QH$. Then $Lu^T = z^T$ and so, since $[f_i] = [f_i|_{\sum_{j=1}^{i+1} Rz_j}]$ and $f_i|_{\sum_{j=1}^{i+1} Rz_j}$ satisfy the condition (b) of 2.3 for all $i = 0, \cdots, n$, u divides m.

The final claim follows from the same arguments used in the proof of the implication (iii)$\Longrightarrow$(i).

Now, we prove some properties of divisibility which will be used later.

LEMMA 2.5. *Let* $n \in \mathbb{N}_0, m \in M$ *and* $u = (u_1, \cdots, u_{n+1}) \in U_{n+1}$. *Assume that* u *divides* m. *Then*

(i) For each $a \in R$, u *divides* am *and* $a(m \div u) = am \div u$

(ii) $u_j(m \div (u_1, \cdots, u_{i+1})) = 0$ *for all integers* i, j *with* $1 \leq j \leq i \leq n$ *and* $\pi_i(u_{i+1}(m \div (u_1, \cdots, u_{i+1}))) = 0$ *for all* $i = 0, 1, \cdots, n$.

(iii) If $m' \in M$ and $u' = (u'_1, \cdots, u'_{n+1}) \in U_{n+1}$ are such that u' divides m', then, for all $H, H' \in D_{n+1}(R)$ and each $w = (w_1, \cdots, w_{n+1}) \in U_{n+1}$ with $Hu^T = w^T = H'u'^T$, w divides $|H|m + |H'|m'$ and

$$(m \div u) + (m' \div u') = (|H|m + |H'|m') \div w.$$

In particular $m \div u = |H|m \div w$.

Proof. Part (i) is a straightforward calculation, while part (ii) is an immidiate consequence of 2.1 and 2.3.

(iii) In the case when $n = 0$ the proof is a straightforward adaptation of the following. So suppose, inductively, that $n \in \mathbb{N}$ and the claim has been proved for $n - 1$. By 2.4, there exist $K, K' \in D_{n+1}(R)$, $v = (v_1, \cdots, v_{n+1})$, $v' = (v'_1, \cdots, v'_{n+1}) \in U_{n+1}$, $f_n \in \mathrm{Hom}_R(\sum_{j=1}^{n+1} Rv_j, \text{Coker } h^{n-2})$ and $f'_n \in \mathrm{Hom}_R(\sum_{j=1}^{n+1} Rv'_j, \text{Coker } h^{n-2})$ such that $Ku^T = v^T$, $K'u'^T = v'^T$ and that, for all $a_1, \cdots, a_{n+1} \in R$,

$$f_n(\sum_{j=1}^{n+1} a_j v_j) = a_{n+1} k_{n+1} \pi_{n-1}(m \div (u_1, \cdots, u_n)),$$

$$f'_n(\sum_{j=1}^{n+1} a_j v'_j) = a_{n+1} k'_{n+1} \pi_{n-1}(m' \div (u'_1, \cdots, u'_n)).$$

Next, since $v, w, v' \in U_{n+1}$, there are $P, P', Q, Q' \in D_{n+1}(R)$ and $z = (z_1, \cdots, z_{n+1})$, $z' = (z'_1, \cdots, z'_{n+1}) \in U_{n+1}$ such that $Pw^T = z^T = Qv^T$ and $P'w^T = z'^T = Q'v'^T$. Also there exist $L, L' \in D_{n+1}(R)$ and $y = (y_1, \cdots, y_{n+1}) \in U_{n+1}$ such that $Lz^T = y^T = L'z'^T$. Thus $y^T = LQv^T$, and so

$$f_n(y_{n+1}^2) = y_{n+1} f_n(y_{n+1}) = y_{n+1} l_{n+1} q_{n+1} k_{n+1} \pi_{n-1}(m \div (u_1, \cdots, u_n)).$$

Therefore, since $y^T = LPHu^T$, we have, in view of (ii),

$$f_n(y_{n+1}^2) = l_{n+1} p_{n+1} h_{n+1} l_{n+1} q_{n+1} k_{n+1} u_{n+1} \pi_{n-1}(m \div (u_1, \cdots, u_n)).$$

Now, it follows from (ii) and the facts that $v_{n+1} = \sum_{j=1}^{n+1} k_{n+1j} u_j$,

$z_{n+1} = \sum_{j=1}^{n+1} q_{n+1j} v_j$ and $y_{n+1} = \sum_{j=1}^{n+1} l_{n+1j} z_j$ that

$$\begin{aligned} f_n(y_{n+1}^2) &= l_{n+1} p_{n+1} h_{n+1} l_{n+1} q_{n+1} v_{n+1} \pi_{n-1}(m \div (u_1, \cdots, u_n)) \\ &= l_{n+1} p_{n+1} h_{n+1} l_{n+1} z_{n+1} \pi_{n-1}(m \div (u_1, \cdots, u_n)) \\ &= l_{n+1} p_{n+1} h_{n+1} y_{n+1} \pi_{n-1}(m \div (u_1, \cdots, u_n)). \end{aligned}$$

Consequently, by (i) and inductive hypothesis,

$$f_n(y_{n+1}^2) = l_{n+1} p_{n+1} y_{n+1} \pi_{n-1}(|H| m \div (w_1, \cdots, w_n)).$$

Hence, since $y_{n+1} = \sum_{j=1}^{n+1} l'_{n+1j} z'_j$ and $z'_{n+1} = \sum_{j=1}^{n+1} p'_{n+1j} w_j$, we have, in view of (ii),

$$f_n(y_{n+1}^2) = l_{n+1} p_{n+1} l'_{n+1} p'_{n+1} w_{n+1} \pi_{n-1}(|H| m \div (w_1, \cdots, w_n)).$$

Similarly, we have

$$f'_n(y_{n+1}^2) = l'_{n+1} p'_{n+1} l_{n+1} p_{n+1} w_{n+1} \pi_{n-1}(|H'| m' \div (w_1, \cdots, w_n)).$$

Hence if the restriction of f_n and f'_n to $(\sum_{j=1}^{n} Ry_j + Ry_{n+1}^2)$ are denoted by g_n and g'_n respectively, then, by inductive hypothesis,

$$\begin{aligned} &(g_n + g'_n)(y_{n+1}^2) \\ &\quad = l_{n+1} p_{n+1} l'_{n+1} p'_{n+1} w_{n+1} \pi_{n-1}(|H| m + |H'| m' \div (w_1, \cdots, w_n)) \\ &\quad = l_{n+1} p_{n+1} y_{n+1} \pi_{n-1}(|H| m + |H'| m' \div (w_1, \cdots, w_n)). \end{aligned}$$

Now, let $D = diag\ (1, \cdots, 1, y_{n+1}) \in D_{n+1}(R)$. Then $DLP \in D_{n+1}(R)$ and $DLPw^T = (y_1, \cdots, y_n, y_{n+1}^2)^T$. Therefore, since the restriction of $(g_n + g'_n)$ to $\sum_{j=1}^{n} Ry_j$ is zero, the inductive step is complete by 2.4.

EXAMPLE 2.6. Let R be an N-ring ([**2**, p.115]) and let $n \in \mathbb{N}_0, m \in M$ and $u = (u_1, \cdots, u_{n+1}) \in U_{n+1}$. We show that u divides m. Suppose, inductively, that each element of M is divisible by $(u_1, \cdots, u_n)$. It follows from [**2**, 2.3] that there exists $t \in \mathbb{N}$ such that

$$(\sum_{j=1}^{n} Ru_j :_R u_{n+1}^t) = (\sum_{j=1}^{n} Ru_j :_R u_{n+1}^{t+1}).$$

Hence, if $\alpha \in (\sum_{j=1}^{n} Ru_j :_R u_{n+1}^{t+1})$, then $\alpha u_{n+1}^t = \sum_{j=1}^{n} r_j u_j$ for some $r_1, \cdots, r_n \in R$. So, by 2.5 and inductive hypothesis,

$$\alpha \pi_{n-1}(u_{n+1}^t m \div (u_1, \cdots, u_n)) = \sum_{j=1}^{n} r_j \pi_{n-1}(u_j m \div (u_1, \cdots, u_n)) = 0.$$

Therefore $(\sum_{j=1}^{n} Ru_j :_R u_{n+1}^{t+1}) \subseteq ann_R(\pi_{n-1}(u_{n+1}^t m \div (u_1, \cdots, u_n)))$. Also, if $D = diag\ (1, \cdots, 1, u_{n+1}^t) \in D_{n+1}(R)$, then $Du^T = (u_1, \cdots, u_{n+1}^{t+1})^T$. Hence, in view of 2.4, the claim follows.

THEOREM 2.7. *For all* $i \in \mathbb{N}_0$, *let*

$$\Phi_{i+1}^{-i-1}(M) = \{m \div (u_1, \cdots, u_{i+1}) : m \in M, (u_1, \cdots, u_{i+1}) \in U_{i+1}\}.$$

Then, for all $i \in \mathbb{N}_0$, $\Phi_{i+1}^{-i-1}(M)$ *is a submodule of* K^i *and so, if* d^{i+1} *denotes the restriction of* h^i *to* $\Phi_{i+1}^{-i-1}(M)$ *for each* $i \geq -2$ (*interpret* $\Phi_{-1}^{+1}(M) = 0$, $\Phi_0^0(M) = M$), *then*

$$d^{i+1}(m \div (u_1, \cdots, u_{i+1})) = m \div (u_1, \cdots, u_{i+1}, 1)$$

and

$$0 \to M \xrightarrow{d^0} \Phi_1^{-1}(M) \xrightarrow{d^1} \Phi_2^{-2}(M) \to \cdots \to \Phi_i^{-i}(M) \xrightarrow{d^i} \Phi_{i+1}^{-i-1}(M) \to \cdots$$

is a subcomplex of $\mathcal{H}(S, M)$; *we denote this subcomplex by* $C(S, M)$.

Proof. It follows from 2.5 (i), (iii) that $\Phi_{i+1}^{-i-1}(M)$ is a submodule of K^i.

Let $i \in \mathbf{N}_0$, $m \in M$ and $u = (u_1, \cdots, u_{i+1}) \in U_{i+1}$ be such that $m \div u \in \Phi_{i+1}^{-i-1}(M)$. Then, by 2.5 (ii), there is (a well defined) R-homomorphism $f_{i+1} : (\sum_{j=1}^{i+1} Ru_j + R) \to \text{ Coker } h^{i-1}$ which is such that, for all $a_1, \cdots, a_{i+2} \in R$,

$$f_{i+1}(\sum_{j=1}^{i+1} a_j u_j + a_{i+2}) = a_{i+2}\pi_i(m \div u).$$

Let I_{i+2} be the identity matrix of order $i+2$. Then $I_{i+2}(u,1)^T = (u,1)^T$. Hence, in view of 2.4, $(u,1)$ divides m and also $m \div (u,1) = [f_{i+1}]$. Since, for each $\mathfrak{a} \in \Phi_{i+2}$, $\lambda_{\mathfrak{a},\pi_i(m\div u)}$ is the restriction of f_{i+1} to $\mathfrak{a}$, we have

$$h^i(m \div u) = \eta_{\Phi_{i+2}}(\text{ Coker } h^{i-2})(\pi_i(m \div u)) = [\lambda_{\mathfrak{a},\pi_i(m\div u)}] = [f_{i+1}],$$

as required.

3. A unique morphism of complexes

In this section, we prove that there is exactly one morphism of complexes

$$\Theta = (\theta^i)_{i \geq -2} : \mathcal{H}(\mathcal{S}, M) \longrightarrow C(\mathcal{U}, M)$$

over Id_M, and we describe each $\theta^i : K^i \to U_{i+1}^{-i-1}M$ explicity. We also provide a necessary and sufficent condition for the complex $\mathcal{H}(\mathcal{S}, M)$ to be isomorphic to the complex $C(\mathcal{U}, M)$.

PROPOSITION 3.1. *Assume that, for each $i \in \mathbf{N}$ and $\mathfrak{b} \in \Phi_i$, there exists $(v_1, \cdots, v_i) \in U_i$ such that $\sum_{j=1}^{i} Rv_j \subseteq \mathfrak{b}$. Then, for all $n \in \mathbf{N}_0$ and $\beta \in K^n$, there exist $m \in M$ and $u \in U_{n+1}$ such that u divides m and $\beta = m \div u$.*

Proof. Let $n \in \mathbf{N}_0$ and $\beta \in K^n$. Suppose, inductively, that the claim has been proved for smaller values of n. There exist $v =$

$(v_1, \cdots, v_{n+1}) \in U_{n+1}$ and $f \in \mathrm{Hom}_R(\sum_{j=1}^{n+1} Rv_j,\ \mathrm{Coker}\ h^{n-2})$ such that $\beta = [f]$. It therefore follows from the inductive hypothesis that, for all $j = 1, \cdots, n+1$, there exists $m_j \div (w_{1j}, \cdots, w_{nj}) \in K^{n-1}$ such that $f(v_j) = \pi_{n-1}(m_j \div (w_{1j}, \cdots, w_{nj}))$. So, by 2.5 (iii), there is $t \in U_n$ such that $\mathrm{Im} f \subseteq \{\pi_{n-1}(m \div t) : m \in M\}$. Next, there exist $H, K \in D_{n+1}(R)$ and $w' = (w'_1, \cdots, w'_{n+1}) \in U_{n+1}$ such that $Hv^T = w'^T = K(t,1)^T$. It therefore follows from 2.5 (iii) that

$$\mathrm{Im} f \subseteq \{\pi_{n-1}(m \div (w'_1, \cdots, w'_n)) : m \in M\}.$$

Hence, there is $m_0 \in M$ such that $f(w'_{n+1}) = \pi_{n-1}(m_0 \div (w'_1, \cdots, w'_n))$ and, by 2.5 (ii), $f(w'^2_j) = 0 (1 \leq j \leq n)$. Also, if we let $D = diag\ (w'_1, \cdots, w'_n, 1)$. Then $Dw'^T = (w'^2_1, \cdots, w'^2_n, w'_{n+1})^T$. Hence, in view of 2.4, w' divides m_0 and

$$\beta = [f|_{\sum_{j=1}^{n} Rw'^2_j + Rw'_{n+1}}] = m_0 \div (w'_1, \cdots, w'_{n+1}).$$

THEOREM 3.2. *Let $\theta^{-1} = Id_M$. Let the situation be as in 3.1. Then, for all $n \in \mathbf{N}_0$, there is an R-homomorphism $\theta^n : \Phi_{n+1}^{-n-1}(M) \longrightarrow U_{n+1}^{-n-1}M$ which is such that, for each $m \div (u_1, \cdots, u_{n+1}) \in \Phi_{n+1}^{-n-1}(M)$,*

$$\theta^n(m \div (u_1, \cdots, u_{n+1})) = \frac{m}{(u_1, \cdots, u_{n+1})}.$$

Moreover, if $n \in \mathbf{N}_0$, and $\psi_n : \Phi_{n+1}^{-n-1}(M) \rightarrow U_{n+1}^{-n-1}M$ is an R-homomor- phism such that the diagram

$$\begin{array}{ccc} \Phi_n^{-n}(M) & \xrightarrow{d^n} & \Phi_{n+1}^{-n-1}(M) \\ {\scriptstyle \theta^{n-1}}\downarrow & & \downarrow{\scriptstyle \psi_n} \\ U_n^{-n}M & \xrightarrow{e^n} & U_{n+1}^{-n-1}M \end{array}$$

commutes, then $\psi_n = \theta^n$. *Consequently, the family* $\Theta = (\theta^i)_{i \geq -2}$ *is a unique morphism of complexes over* Id_M *from* $C(S, M)$ *to* $C(\mathcal{U}, M)$.

Proof. To prove the first part of claim, in view of 2.5, it is enough to show that, for all $n \in \mathbb{N}_0$, if θ^{n-1} is a well defined map with the required property, then so is θ^n.

Let $n \in \mathbb{N}_0$ and let $m \div (u_1, \cdots, u_{n+1}) = 0$. Then, by 2.4 and the fact that $\Phi(U_{n+1})$ is a cofinal subset of Φ_{n+1}, there exist $w = (w_1, \cdots, w_{n+1}) \in U_{n+1}, H \in D_{n+1}(R)$ and $f \in \mathrm{Hom}_R(\sum_{j=1}^{n+1} Rw_j,$ Coker $h^{n-2})$ such that $Hu^T = w^T$ and $f(\sum_{j=1}^{n+1} a_j w_j) = a_{n+1} h_{n+1} \pi_{n-1}(m \div (u_1, \cdots, u_n)) = 0$ for all $a_1, \cdots, a_{n+1} \in R$. Therefore, by 2.5,

$$0 = f(w_{n+1}) = \pi_{n-1}(|H|m \div (w_1, \cdots, w_n)).$$

That is, $|H|m \div (w_1, \cdots, w_n) \in \mathrm{Im}\, h^{n-2}$. It therefore follows from 3.1 and 2.7 that there exist $m' \in M$ and $v = (v_1, \cdots, v_{n-1}) \in U_{n-1}$ such that

$$|H|m \div (w_1, \cdots, w_n) = m' \div (v_1, \cdots, v_{n-1}, 1).$$

Next, there are $K, L \in D_n(R)$ and $w' = (w'_1, \cdots, w'_n) \in U_n$ such that
$K(w_1, \cdots, w_n)^T = w'^T = L(v, 1)^T$. So, by 2.5 (iii) and the well defindness of θ^{n-1},

$$0 = \theta^{n-1}((|K||H|m - |L|m') \div (w'_1, \cdots, w'_n)) = \frac{|K||H|m - |L|m'}{(w'_1, \cdots, w'_n)},$$

in $U_n^{-n}M$. Consequently in $U_n^{-n}M$

$$\frac{|H|m}{(w_1, \cdots, w_n)} = \frac{|K||H|m}{(w'_1, \cdots, w'_n)} = \frac{|L|m'}{(w'_1, \cdots, w'_n)} = \frac{m'}{(v_1, \cdots, v_{n-1}, 1)}.$$

Therefore $w_{n+1}|H|m/w = m'/(v,1,1) = 0$, in $U_{n+1}^{-n-1}M$. Hence, by [**10**, 2.1], $0 = |H|m/w = m/u$. Now, the claim of this part follows from 2.5 (iii).

To prove the second part, let $n \in \mathbf{N}_0$, $m \div u \in \Phi_{n+1}^{-n-1}(M)$ and $\psi_n(m \div u) = m'/v$ for some $m' \in M$ and $v \in U_{n+1}$. Then there exist $H, K \in D_{n+1}(R)$ and $w = (w_1, \cdots, w_{n+1}) \in U_{n+1}$ such that$Hu^T = w^T = Kv^T$.Now,by 2.5 (iii) and 2.7

$$\begin{aligned} w_{n+1}\psi_n(|H|m \div w) &= \psi_n(|H|m \div (w_1, \cdots, w_n, 1) \\ &= \psi_n \mathrm{o} d^n(|H|m \div (w_1, \cdots, w_n)) \\ &= e^n \mathrm{o} \theta^{n-1}(|H|m \div (w_1, \cdots, w_n)) \\ &= |H|m/(w_1, \cdots, w_n, 1) = w_{n+1}|H|m/w. \end{aligned}$$

On the other hand, we have

$$w_{n+1}\psi_n(|H|m \div w) = w_{n+1}m'/v = w_{n+1}|K|m'/w.$$

Hence $w_{n+1}|H|m/w = w_{n+1}|K|m'/w$. It therefore follows, in view of [**10**, 2.1], that $|K|m'/w = |H|m/w$, in $U_{n+1}^{-n-1}M$. Hence, $\psi_n(m \div u) = \theta^n(m \div u)$; and consequently the proof of the second part is complete.

Now, it is immidiate from 2.7 that Θ is a unique morphism of complexes.

In the following, we prove one of the main theorems of this paper.

THEOREM 3.3. *Assume that, for all $i \in \mathbf{N}$ and $\mathfrak{b} \in \Phi_i$, there exists $(v_1, \cdots, v_i) \in U_i$ such that $\sum_{j=1}^{i} Rv_j \subseteq \mathfrak{b}$. Then there is a unique morphism of complexes $\Theta = (\theta^i)_{i \geq -2} : \mathcal{H}(\mathcal{S}, M) \longrightarrow C(\mathcal{U}, M)$ over Id_M. Furthermore, if $i \in \mathbf{N}_0$, then the following statements are equivalent.*

(i) $\theta^{-1}, \theta^0, \cdots, \theta^i$ are epimorphisms.

(ii) $\theta^{-1}, \theta^0, \cdots, \theta^i$ are isomorphisms.

(iii) For $n = 0, 1, \cdots, i$, each element of U_{n+1} divides each element of M.

Proof. The existance and uniqueness of the morphism Θ can be deduced from 3.1 and 3.2. So we need to show that the above conditions are equivalent.

(i) $\Longrightarrow$ (ii) we prove, inductively, that $\theta^{-1}, \theta^0, \cdots, \theta^i$ are isomorphisms. Let $0 \leq n \leq i$ and suppose that $\theta^{-1}, \theta^0, \cdots, \theta^{n-1}$ are isomorphisms. Since $D_{\Phi_{n+1}}(\text{ Coker } h^{n-2}) \cong D_{\Phi(U_{n+1})}(\text{ Coker } h^{n-2})$, by the same arguments in the proof of [**7**, 2.5], we may define an R-monomorphism $\psi_n : K^n \to U_{n+1}^{-n-1}M$ such that the diagram

$$\begin{array}{ccc} K^{n-1} & \xrightarrow{h^{n-1}} & K^n \\ \theta^{n-1}\big\downarrow & & \big\downarrow\psi_n \\ U_n^{-n}M & \xrightarrow{e^n} & U_{n+1}^{-n-1}M \end{array}$$

commutes. It therefore follows from 3.1 and 3.2 that $\psi_n = \theta^n$; hence θ^n is an isomorphism.

(ii) $\Longrightarrow$ (iii) Let $0 \leq n \leq i$ and suppose, inductively, that each element of U_n divides each element of M. Let $x \in M$ and $u = (u_1, \cdots, u_{n+1}) \in U_{n+1}$. Since $x/u \in U_{n+1}^{-n-1}M$ and θ^n is an epimorphism, by 3.1, there exist $m \in M$ and $v = (v_1, \cdots, v_{n+1}) \in U_{n+1}$ such that v divides m and $x/u = m/v$, in $U_{n+1}^{-n-1}M$. By 2.4, there are $P \in D_{n+1}(R), w = (w_1, \cdots, w_{n+1}) \in U_{n+1}$ and $f_n \in \text{Hom}_R(\sum_{j=1}^{n+1} Rw_j, \text{ Coker } h^{n-2})$ such that $Pv^T = w^T$ and

$$f_n(\sum_{j=1}^{n+1} a_j w_j) = a_{n+1}p_{n+1}\pi_{n-1}(m \div (v_1, \cdots, v_n)) \text{ for } a_1, \cdots, a_{n+1} \in R.$$

Next, since $m/v = |P|m/w$, we have $x/u = |P|m/w$. Hence, there are $H, Q \in D_{n+1}(R)$ and $z = (z_1, \cdots, z_{n+1}) \in U_{n+1}$ such that $Hu^T = z^T = Qw^T$ and $|H|x - |QP|m \in \sum_{j=1}^{n} z_j M$. Therefor, by inductive hypothesis and 2.5,

$$\pi_{n-1}(|H|x \div (z_1, \cdots, z_n)) = \pi_{n-1}(|QP|m \div (z_1, \cdots, z_n));$$

consequently

$$\begin{aligned} f_n(\sum_{j=1}^{n+1} b_j z_j) &= b_{n+1}q_{n+1}p_{n+1}\pi_{n-1}(m \div (v_1, \cdots, v_n)) \\ &= b_{n+1}\pi_{n-1}(|QP|m \div (z_1, \cdots, z_n)) \text{ (by 2.5 (i) and (iii))} \\ &= b_{n+1}\pi_{n-1}(|H|x \div (z_1, \cdots, z_n)) \\ &= b_{n+1}h_{n+1}\pi_{n-1}(x \div (u_1, \cdots, u_n)) \end{aligned}$$

for all $b_1, \cdots, b_{n+1} \in R$. Therefore, since $Hu^T = z^T$, we have, by inductive hypothesis and 2.4, that u divides x, as required.

(iii) $\Longrightarrow$ (i) This is clear.

COROLLARY 3.4. *Let the situation be as in 3.3. Then the unique morphism of complexes* $\mathcal{H}(\mathcal{S}, R) \longrightarrow C(\mathcal{U}, R)$ *over* Id_R *is an isomorphism if and only if each element of* $U_{n+1}(n \in \mathrm{N}_0)$ *divides the identity element of* R.

4. Isomorphism of complexes

In this section, we shall use the results of §3 to obtain isomorphism between the complexes $\mathcal{H}(\mathcal{S}, M)$ and $C(\mathcal{U}, M)$ in certain cases. Also, we shall provide a necessary and sufficient condition for the exactness of the certain generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$.

In the following, we shall use the concept of a d-sequence. The theory of d-sequences was introduced by Huneke in [**4**]. Let $n \in \mathrm{N}$ and $x_1 \cdots, x_n \in R$. Then the sequence $x_1, \cdots, x_n$ is called a d-sequence on M if, for all $i = 0, \cdots, n-1$ and all $k \geq i+1$,

$$(\sum_{j=1}^{i} Rx_j)M :_M x_{i+1}x_k = (\sum_{j=1}^{i} Rx_j)M :_M x_k.$$

(this is actually a slight weakening of Huneke's definition).

THEOREM 4.1. *Let* Θ *be the unique morphism of complexes of 3.3. Let* $i \in \mathrm{N}_0$ *and suppose that, for all* $j = 0, 1, \cdots, i$ *and all*

$(u_1, \cdots, u_{j+1}) \in U_{j+1}$, there exist an integer k with $0 \leq k \leq j+1$ such that $u_1, \cdots, u_k$ form a d-sequence on M and $u_{k+1} = \cdots = u_{j+1} = 1_R$. Then $\theta^{-1}, \theta^0, \cdots, \theta^i$ are isomorphisms: that is, if $x \in M$ and $u \in U_{j+1}(0 \leq j \leq i)$ then u divides x (see 3.3).

Proof. We prove the claim by induction on j. Suppose that $0 \leq j \leq i$ and the claim has been proved for smaller values of j. Let $x \in M$ and $u = (u_1, \cdots, u_{j+1}) \in U_{j+1}$. Then there exists $0 \leq k \leq j+1$ such that $u_1, \cdots, u_k$ is a d-sequence on M and $u_{k+1} = \cdots = u_{j+1} = 1_R$. If $k < j+1$, then, in view of the inductive hypothesis and 2.7, the claim is clear. So we assume that $u_1, \cdots, u_{j+1}$ form a d-sequence on M. Let $D = diag\ (1, \cdots, 1, u_{j+1}) \in D_{j+1}(R)$. Then $Du^T = (u_1, \cdots, u_j, u_{j+1}^2)^T$ and so, in view of 2.4, the inductive step will be complete if we show that

$$(\sum_{l=1}^{j} Ru_l :_R u_{j+1}^2) \subseteq ann_R(\pi_{j-1}(u_{j+1}m \div (u_1, \cdots, u_j))).$$

To achieve this, let $\alpha \in (\sum_{l=1}^{j} Ru_l :_R u_{j+1}^2)$. Then

$$\alpha m \in ((\sum_{l=1}^{j} Ru_l)M :_M u_{j+1}^2) = ((\sum_{l=1}^{j} Ru_l)M :_M u_{j+1}).$$

It therefore follows from inductive hypothesis and 2.5 (ii) that

$$\pi_{j-1}(u_{j+1}\alpha m \div (u_1, \cdots, u_j)) = 0.$$

Hence $\alpha \in ann_R(\pi_{j-1}(u_{j+1}m \div (u_1, \cdots, u_j)))$ and the proof is complete.

THEOREM 4.2. *Let the situation and notation be as in 3.3. If one of the complexes $C(\mathcal{U}, M)$ or $\mathcal{H}(\mathcal{S}, M)$ is exact, then the unique morphism of complexes Θ is an isomorphism.*

In particular, $\mathcal{H}(\mathcal{S}, M)$ is exact if and only if each element of U_i is a poor M-sequence.

Proof. The claim follows from [**5**, 3.1] and 4.1 whenever $C(\mathcal{U}, M)$ is exact. Therefore we may assume that $\mathcal{H}(\mathcal{S}, M)$ is exact. In view of 4.1 and the fact that each poor M-sequence is a d-sequence, it is enough to show that each of element of U_{n+1} is a poor M-sequence for all $n \in \mathbb{N}_0$. We prove this by induction on n. Suppose, inductively, that U_i consistes of poor M-sequences for all $1 \leq i \leq n$. Let $u = (u_1, \cdots, u_{n+1}) \in U_{n+1}$ and let $m \in M$ be such that $u_{n+1}m \in (\sum_{j=1}^{n} Ru_j)M$. Then $u_{n+1}m = \sum_{j=1}^{n} u_j m_j$ for some $m_1, \cdots, m_n \in M$. It follows from inductive hypothesis and 4.1 that $(u_1, \cdots, u_n)$ divides the elements $m, m_1, \cdots, m_n$. Hence, by 2.7 and 2.5, we have

$$d^n(m \div (u_1, \cdots, u_n)) = u_{n+1}m \div (u_1, \cdots, u_{n+1})$$

and

$$\pi_{n-1}(u_{n+1}m \div (u_1, \cdots, u_n)) = \sum_{j=1}^{n} \pi_{n-1}(u_j m_j \div (u_1, \cdots, u_n)) = 0.$$

Consequently, in view of difinition of divisibility, $d^n(m \div (u_1, \cdots, u_n)) = 0$, and so $m \div (u_1, \cdots, u_n) \in \operatorname{Ker} d^n = \operatorname{Im} d^{n-1}$. Therefore there exist $m' \in M$ and $(v_1, \cdots, v_{n-1}) \in U_{n-1}$ such that $m \div (u_1, \cdots, u_n) = m' \div (v_1, \cdots, v_{n-1}, 1)$. Hence, by 3.2,

$$\frac{m}{(u_1, \cdots, u_n)} = \frac{m'}{(v_1, \cdots, v_{n-1}, 1)},$$

in $U_n^{-n}M$. Now, by arguments entirely similar to those used in the proof of [**5**, 3.1], we can show that $m \in (\sum_{j=1}^{n} Ru_j)M$, and so the proof is complete.

In the following, we obtain a stronger form of [**8**, 3.5] and [**7**, 2.5] by using 3.3 and 2.6.

THEOREM 4.3. *Let the situation and notation be as in 3.3. If R is an N-ring (and so in particular, if R is Noetherian), then the unique morphism of complexes Θ is an isomorphism.*

REMARK 4.4. An example of a commutative ring R and a chain of triangular subsets on R, are given in [**7**,§ 3] which shows that the morphism of complexes Θ of [**7**, 2.5] is not always an isomorphism. But it is not shown that these complexes are not isomorphic. Now, by using the uniqueness property of 3.3, it is clear that the two complexes which are considered in that example are not isomorphic over Id_M.

Note that one may use 3.4 and 2.4 to provide a shorter proof of [**7**, § 3].

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