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MINIMAL INJECTIVE RESOLUTIONS OF MODULES OVER COHEN–MACAULAY RINGS

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Let R be a commutative Noetherian ring, M be an R-module. Using the minimal injective resolution of M, H. Bass defined the ith invariant $\mu_i(P, M)$ for any $P \in \text{Spec}R$ in [**1**]. In general, it is noted that the most nice properties of $\mu_i(P, M)$ depend on M being finitely generated. In [**11**] Xu studied the minimal injective resolution of modules, over a Gorenstein ring, of finite flat dimension and did not assume them to be finitely generated. He showed that, R is Gorenstien ring if and only if for any R-module M with finite flat dimension s, $\mu_i(P, M) \neq 0$ only if $i \leq \text{ht}(P) \leq i + s$. The aim of the present paper is to obtain information about the minimal injective resolutions of arbitrary modules of finite flat dimension over Cohen-Macaulay ring. For instance, Theorem 4 shows that R is Cohen-Macaulay if and only if for any R-module M with finite flat dimension s, $\mu_i(P, M) \neq 0$ only if $\text{ht}(P) \leq i+s$. Also, in this note, we give another version of Xu's theorem [**11**, 2.2] which provides a quick proof for [**11**, 2.2 ((1) $\Longrightarrow$ (2))]. The proof of this result is concerned with a complex $C(\mathcal{U}, M)$ of R-modules which involves modules of generalized fractions derived from M and poor M-sequences.

Throughout this paper, R is a commutative Noetherian ring with the identity and M is an R-module. For any R-module X, $\text{f.dim}_R X$ stands for the flat dimension of X, $\text{inj.dim}_R X$ stands for the injective dimension of X, and $E(X)$ stands for its injective envelope. All other notation is standard. For instance, $\text{ht}(P)$ means the height of P, and X_P means the localization of R-module X at a prime P. We use N to denote the set of positive integers.

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Let us recall the definition of the M-grade of an ideal $\mathfrak{a}$ of R (Note that M is not assumed to be finitely generated).

DEFINITION 1. Let $\mathfrak{a}$ be an ideal of R. The *M-grade* of $\mathfrak{a}$, written $\operatorname{grade}(\mathfrak{a}, M)$, is the least integer r such that $\operatorname{Ext}^r_R(\frac{R}{\mathfrak{a}}, M) \neq 0$ if this exists, and ∞ otherwise.

REMARKS 2. Recall that elements $a_1, \dots, a_n$ of R are said to form a *poor M-sequence* (of length n) if, for all $i = 1, \dots, n$, the element a_i is not a zerodivisor on $M / \sum_{j=1}^{i-1} a_j M$.

(i) If $\mathfrak{a}$ contains a poor M-sequence of length r, then $\operatorname{grade}(\mathfrak{a}, M) \geq r$.

(ii) If M is finitely generated and $M \neq \mathfrak{a}M$, then $\operatorname{grade}(\mathfrak{a}, M)$ is equal to the common length of all maximal M-sequences contained in $\mathfrak{a}$.

PROPOSITION 3. *Let $\mathfrak{a}$ be an ideal of R. Then*

$$\operatorname{grade}(\mathfrak{a}, R) \leq \operatorname{grade}(\mathfrak{a}, M) + f.\dim_R M.$$

Proof. We need to prove our assertion only in the case that $f.\dim_R M = s$ and grade $(\mathfrak{a}, M) = t$ are finite. We use induction on s. To begin, note that in the case when $s = 0$ the claim immediately follows; because every R-sequence is a poor M-sequence. Suppose that $f.\dim_R M = s > 0$ and that the result has been proved for all modules with flat dimension less than s. Consider an exact sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$$

with F flat. Since $\operatorname{grade}(\mathfrak{a}, R) \leq \operatorname{grade}(\mathfrak{a}, F)$, it is enough to prove the claim under assumption $\operatorname{grade}(\mathfrak{a}, F) > t$. Using the long exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(\frac{R}{\mathfrak{a}}, N) \longrightarrow \operatorname{Hom}_R(\frac{R}{\mathfrak{a}}, F) \longrightarrow \operatorname{Hom}_R(\frac{R}{\mathfrak{a}}, M) \longrightarrow \dots$$
$$\longrightarrow \operatorname{Ext}^i_R(\frac{R}{\mathfrak{a}}, N) \longrightarrow \operatorname{Ext}^i_R(\frac{R}{\mathfrak{a}}, F) \longrightarrow \operatorname{Ext}^i_R(\frac{R}{\mathfrak{a}}, M) \longrightarrow \dots$$

we may deduce that grade $(\mathfrak{a}, N) = t+1$. Now it follows from the inductive hypothesis that $\text{grade}(\mathfrak{a}, R) \leq \text{grade}(\mathfrak{a}, N) + \text{f.}\dim_R N = t+s$. The inductive step is therefore complete.

A minimal injective resolution of M is an exact sequence

$$0 \longrightarrow M \longrightarrow E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \dots \longrightarrow E_i \xrightarrow{d_i} \dots$$

such that for each $i \geq 0$, E_i is an injective envelope of $\ker(d_i)$. It is well known that each E_i has a unique decomposition $E_i = \bigoplus E(\frac{R}{P})$, $P \in \operatorname{Spec} R$ [**5**]. If $\mu_i(P, M)$ denotes the ith Bass number, it can be written in the form

$$E_i = \bigoplus_{P \in \operatorname{Spec} R} \mu_i(P, M) E(R/P).$$

Also, by [**1**, 2.7], $\mu_i(P, M)$ can be described as the dimension of a vector space: if $k(P)$ denote the residue field of the local ring R_P, then

$$\mu_i(P, M) = \dim_{k(P)} \operatorname{Ext}^i_{R_P}(k(P), M_P) = \dim_{k(P)}(\operatorname{Ext}^i_R(R/P, M))_P.$$

THEOREM 4. *The following statements are equivalent:*

(1) *R is Cohen-Macauley,*

(2) *Any R-module M with* $\text{f.}\dim_R M = s < \infty$ *admits a minimal injective resolution as*

$$0 \longrightarrow M \longrightarrow \bigoplus_{\text{ht}(P) \leq s} \mu_0(P, M) E(R/P) \longrightarrow \dots$$
$$\longrightarrow \bigoplus_{\text{ht}(P) \leq k+s} \mu_k(P, M) E(R/P) \longrightarrow \dots$$

Proof. (1) $\Longrightarrow$ (2) Suppose that $E(R/P)$ is contained in $E_i(M)$. Then $\operatorname{Ext}^i_R(R/P, M) \neq 0$ and hence grade $(P, M) \leq i$. Therefore, by Proposition 3, grade $(P, R) \leq i + s$. Hence $\text{ht}(P) \leq i + s$; because R is Cohen-Macaulay.

(2) $\Longrightarrow$ (1) Suppose that R admits a minimal injective resolution

$$0 \longrightarrow R \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \dots \longrightarrow E_i \longrightarrow \dots$$

such that $\mathrm{ht}(P) \leq i$ whenever $E(R/P) \subsetneq E_i$. Let $\mathfrak{m}$ be a maximal ideal of R. If $\mathrm{ht}(P) < \mathrm{ht}(\mathfrak{m})$ then, by [**10**, 2.31], $H^0_{\mathfrak{m}}(E(R/P)) = 0$; so $H^0_{\mathfrak{m}}(E_i(R)) = 0$ for all $i < \mathrm{ht}(\mathfrak{m})$ (because local cohomology functor 'Commutes' with arbitrary direct sums). This implies that $H^i_{\mathfrak{m}}(R) = 0$ for all $i < \mathrm{ht}(\mathfrak{m})$. Therefore, by [**3**, 3.10] or [**4**, 2.1], grade $(\mathfrak{m}, R) \geq \mathrm{ht}(\mathfrak{m})$. It therefore follows that R is Cohen-Macaulay.

From the above argument, we may establish the following.

COROLLARY 5. *The following statements are equivalent:*

(1) *R is Cohen-Macaulay,*
(2) *For any R-module M with $\mathrm{f.dim}_R M = s < \infty$,*

$$E_0(M) = \bigoplus_{\mathrm{ht}(P) \leq s} \mu_0(P, M) E(R/P)$$

(3) *For any finitely generated R-module M with $\mathrm{f.dim}_R M = s < \infty$,*

$$E_0(M) = \bigoplus_{\mathrm{ht}(P) \leq s} \mu_0(P, M) E(R/P).$$

Proof. In view of Theorem 4, the only non-obvious point is to show that (3) $\Longrightarrow$ (1). For any maximal ideal $\mathfrak{m}$ of R, let $x_1, \dots, x_t$ be a maximal R-sequence in $\mathfrak{m}$. The $\mathrm{f.dim}_R \frac{R}{(x_1,\dots,x_t)} = t$. Set $L = \frac{R}{(x_1,\dots,x_t)}$. Since $R/\mathfrak{m} \subseteq L$, we have that $E(\frac{R}{\mathfrak{m}}) \subseteq E_0(L)$. Hence by assumption $\mathrm{ht}(\mathfrak{m}) \leq t$ and the result follows.

Reminder 6 : Complexes of Modules of Generalized Fractions. The concept of a chain of triangular subsets on R is explained in [**7**, p. 420]. Such a chain $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ determines a complex

$$0 \xrightarrow{e^{-1}} M \xrightarrow{e^0} U_1^{-1}M \longrightarrow \dots \longrightarrow U_i^{-i}M \xrightarrow{e^i} U_{i+1}^{-i-1}M \longrightarrow \dots$$

of R-modules and R-homomorphisms which we denote by $C(\mathcal{U}, M)$. Here $U_i^{-i}M$ is the module of generalized fractions of M with respect to the triangular subset U_i of R^i, and the homomorphisms e^i (for $i \geq 0$) are given by the following formula: $e^0(m) = \frac{m}{(1)}$ for all $m \in M$ and $e^i\left(\frac{m}{(u_1,\ldots,u_i)}\right) = \frac{m}{(u_1,\ldots,u_i,1)}$ for all $m \in M$ and $(u_1, \ldots, u_i) \in U_i$. For each $i \in \mathbf{N}$, we set

$$U_i = \left\{(u_1, \ldots, u_i) \in R^i;\ u_1, \ldots, u_i \text{ form a poor } R\text{-sequence}\right\}.$$

It is easy to check that the family $\mathcal{U} = (U_i)_{i \in \mathbf{N}}$ is a chain of triangular subsets on R; and so we may form the complex $C(\mathcal{U}, M)$ as above.

The next proposition provides an explicit description of the minimal injective resolution of a flat module over Gorenstein ring.

PROPOSITION 7. *The following statemants are equivalent:*

(1) *R is Gorenstein,*
(2) *For any flat R-module F, $C(\mathcal{U}, F)$ provides the minimal injective resolution for F.*

Proof. (1) $\Longrightarrow$ (2). By [**9**, 5.8] $C(\mathcal{U}, R)$ provides the minimal injective resolution for R. Now, in view of [**9**, 3.17] and [**8**, 3.83], $C(\mathcal{U}, F)$ is an injective resolution for F. Also, it is immediate consequence of [**9**, 5.1] that, if $U_i^{-i}F \neq 0$, then it is an essential extension of $\operatorname{im} e^{i-1}$.

(2) $\Longrightarrow$ (1) By assumption, the complex

$$0 \longrightarrow R \longrightarrow U_1^{-1}R \longrightarrow U_2^{-2}R \longrightarrow \ldots \longrightarrow U_i^{-i}R \longrightarrow \ldots$$

is a minimal injective resolution for R. For any maximal ideal $\mathfrak{m}$ of R we have, by [**2**, 3.1], that $(U_i^{-i}R)_{\mathfrak{m}} = 0$ for all $i > \operatorname{ht}(\mathfrak{m}) + 1$. Then passing to localization, we see that $\operatorname{inj.dim}_{R_{\mathfrak{m}}} R_{\mathfrak{m}}$ is finite. It follows that R is Gorenstein.

We are now in a position to establish and prove another version of [**11**, 2.2] that was promised earlier at the beginning of the paper.

THEOREM 8. *The following statements are equivalent:*

(1) *R is Gorenstein,*

(2) *Any R-module M with $\operatorname{f.dim}_R M = s < \infty$ admits a minimal injective resolution as*

$$0 \longrightarrow M \longrightarrow \bigoplus_{0 \leq \operatorname{ht}(P) \leq s} \mu_0(P, M) E(R/P) \longrightarrow \dots$$
$$\longrightarrow \bigoplus_{i \leq \operatorname{ht}(P) \leq i+s} \mu_i(P, M) E(R/P) \longrightarrow \dots$$

Proof. By [**6**, 18.8], we only need to show that (1) $\Longrightarrow$ (2). We prove this by induction on s. Consider the case $s = 0$. If $E(R/P) \subseteq E_i(M)$, then, by Proposition 7, $P \in \operatorname{Supp}(U_{i+1}^{-i-1} M)$. Hence, by [**2**, 3.1], $\operatorname{ht}(P) \geq i$ and so, by Theorem 4, the result follows. Now, suppose inductively that $s > 1$ and the result has been proved for smaller values of s. As usual, we consider an exact sequence $0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$ with F flat. Let $E(R/P) \subseteq E_i(M)$. Then $\left(\operatorname{Ext}_R^i(R/P, M)\right)_P \neq 0$. If $\left(\operatorname{Ext}_R^i(R/P, F)\right)_P \neq 0$ then by the case $s = 0$, $\operatorname{ht}(P) = i$. If $\left(\operatorname{Ext}_R^i(R/P, F)\right)_P = 0$, then by applying the above short exact sequence we deduce that $\left(\operatorname{Ext}_R^{i+1}(R/P, N)\right)_P \neq 0$. Hence by inductive hypothesis $i + 1 \leq \operatorname{ht}(P) \leq (i+1) + (s-1)$. Thus if $E(R/P) \subseteq E_i(M)$, then $i \leq \operatorname{ht}(P) \leq i + s$. Now the result follows, by induction.

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