Honam Mathematical J. **19**(1997), No. 1,pp. 87–95

THE INDEX FOR THE TYPE III FACTORS $\mathcal{W}^*(\mathcal{R}_G)$

DONG-SEO KIM AND CHANG-HO BYUN

Dept. of Mathematics,
Chonnam National University, Kwangju 500-757, Korea.

1. Introduction

From the Jones'index theory[3] for II_1-factors, many researchers developed index theory for general factors in [3,4,5]. In [2], H. Kosaki extended the notion of an index to any normal faithful conditional expectation from a factor onto a subfactor. Specially, Kosaki's definition of the index of an expectation relies on the notion of spatial derivatives due to A. Connes[6] as well as the theory of operator-valued weights due to U. Haagerup[7].

In § 2, we define the index for a pair of factors and we have an example. Also we show the relations of each index and the some properties of the index in the case of type II_1 factor pairs.

In § 3, we construct the group von Neumann algebras $\mathcal{W}^*(\mathcal{R}_G)$ on the Hilbert space $L^2(\mathcal{R}_G)$ and also we study the relation between $L^2(X) \otimes l^2(G)$ and $\mathcal{W}^*(\mathcal{R}_G)$.

Finially, we calculate the index for the pair of type III factors $\mathcal{W}^*(\mathcal{R}_G)$.

2. The Index for Factors $\mathcal{W}^*(\mathcal{R}_G)$

If M is a finite factor acting on a Hilbert space $\mathcal{H}$ with finite commutant M', the coupling constant $dim_M(\mathcal{H})$ of M is defined as $tr_M(E_\omega^{M'})/tr_{M'}(E_\omega^M)$ where ω is a non-zero vector in $\mathcal{H}$, tr_A denotes the normalized trace and E_ω^A is the projection onto the closure of the subspace $A\omega$.

Received May 23, 1997.

DEFINITION 2.1. If N is a subfactor of M, the number $dim_N(\mathcal{H})/dim_M(\mathcal{H})$ is called the (global) *Index* of N in M and is written $[M : N]$. Note that $[M : N] = \infty$ means that N' is infinite for any normal representation of M.

Since $[M : N] = dim_M(L^2(M, tr)), [M : N]$ is a conjugacy invariant for N as a subfactor of M.

Let M be a finite von Neumann algebra with faithful normal normalized trace tr and let N be a von Neumann subalgebra. By [8] there is a conditional expectation $E_N : M \to N$ defined by the relation $tr(E_N(x)y) = tr(xy)$ for $x \in M, y \in N$. The map E_N is normal and has the following properties:

$$E_N(axb) = aE_N(x)b \quad \text{for} \quad x \in M, \ a, b \in N \ \text{(the bimodule property)}$$

$$E_N(x^*) = E_N(x)^* \quad for \quad all \quad x \in M$$

$$E_N(x^*)E_N(x) \leq E_N(x^*x) \quad and \quad E_N(x^*x) = 0 \quad implies \quad x = 0.$$

Let ω be the canonical cyclic trace vector in $L^2(M, tr)$. Identify M with the algebra of the left multiplication operators on $L^2(M, tr)$. The conditional expectation E_N extends to a orthogonal projection e_N on $\mathcal{H}$ via $e_N(x\omega) = E_N(x)\omega$. We denote by $< M, e_N >$ the von Neumann algebra on $L^2(M, tr)$ generated by M and e_N. Let J be the conjugate linear isometry of $L^2(M, tr)$ extending the map $x \to x^*$ on M.

PROPOSITION 2.2 [1,3].
(i) $N' = \{M' \cup \{e_N\}\}''$.
(ii) $< M, e_N >= JN'J$.

DEFINITION 2.3. If L is a subalgebra of $< M, e_N >$, a trace Tr on $< M, e_N >$ is called a (τ, L)trace if Tr extends tr and $Tr(e_N x) = \tau tr(x)$ for $x \in L$.

PROPOSITION 2.4 [3]. *If* M *and* N *are factors then* $[M : N] < \infty$ *iff* $< M, e_N >$ *is finite and in this case the canonical trace* Tr *on* $< M, e_N >$ *is a* (τ, M) *trace where* $\tau = [M : N]^{-1}$*. In particular* $Tr(e_N) = [M : N]^{-1}$*. Also* $[< M, e_N >: M] = [M : N]$.

Let N be a proper von Neumann subalgebra of the finite von Neumann algebra M with faithful normal normalized trace tr.

Suppose there is a faithful normal (τ, M) trace Tr on $< M, e_N >$. Then we may form the extension $<< M, e_N >, e_M >$.

THEOREM 2.5 [1,3]. *Let M be a von Neumann algebra with faithful normal normalized trace tr. Let $\{e_i | i = 1, 2, \cdots\}$ be projections in M satisfying*

a) $e_i e_{i\pm 1} e_i = \tau e_i$ for some $\tau \leq 1$

b) $e_i e_j = e_j e_i$ for $|i - j| \geq 2$

c) $tr(we_i) = \tau tr(w)$ if w is a word on $1, e_1, e_2, \cdots, e_{i-1}$

Then if P denotes the von Neumann algebra generated by the $e_i's$,

(i) $P \cong R$(the hyperfinite II_1 factor),

(ii) $P_\tau = \{e_2, e_3, \cdots\}$" is a subfactor of P with $[P : P_\tau] = \tau^{-1}$,

(iii) $\tau \leq \frac{1}{4}$ or $\tau = \frac{1}{4}\sec^2\frac{\pi}{n}, n = 3, 4, \cdots$.

LEMMA 2.6. *If N is a subfactor of the II_1 factor M then either $[M : N] \geq 4$ or $[M : N] = 4\cos^2\frac{\pi}{n}$ for some $n \geq 3$.*

Proof. If $[M : N] < \infty$, define the increasing sequence $M_i, i = 0, 1, 2, \cdots$ of II_1 factors by the relations $M_0 = M, M_1 =< M, e_N >, M_{i+1} =< M_i, e_{M_{i-1}} >$ for $i \geq 1$. The inductive limit becomes a II_1 factor with faithful normal normalized unique trace tr. If $\tau = [M : N]^{-1}$ and $e_i = e_{M_i}$ then the $e_i's$ satisfy the conditions of theorem 2.5 by proposition 2.4. Thus, by theorem 2.5, either $[M : N] \geq 4$ or $[M : N] = 4\cos^2\pi/n, n = 3, 4 \cdots$.

EXAMPLE 2.7. If M is a II_1 factor and G is a finite group of outer automorphisms of M with fixed point algebra M^G, $[M : M^G] = |G|$.

Proof. Let M act on $L^2(M, tr)$ and let u_g be the unitaries extending the action of G on M. Then the $u_g's$ act on M and it is that $(M^G)'$ is isomorphic in the obvious way to $M' \rtimes G$. The projection onto $\overline{M^G}$ is $|G|^{-1}\sum_{g\in G} u_g$ by the isomorphism with the cross product its trace is $|G|^{-1}$. Hence $[M : M^G] = |G|$.

The H.Kosaki's index based on Haagerup's theory on the operator-valued weights [7] and Connes' spatial theory [6]. Let M be an arbitrary factor with a subfactor N. We assume the existence of a normal conditional expectation $E : M \to N$. By E, $IndexE$ will be defined. If M and N are II_1 factors, the index of the canonical conditional expectation determined by the unique normalized trace on M is exactly Jones' index $[M : N]$ [3] based on the coupling constant.

Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and ψ be a normal faithful semifinite (n.f.s.) weight on the commutant M'.

Let ϕ be a n.f.s. weight on M and let N be a von Neumann subalgebra in M. An operator-valued weight $E : M \to N$ is a map $M^+ \to \hat{N}_+$ such that

(a) E is additive,

(b) $E(axa^*) = aE(x)a^*, x \in M_+, a \in N$.

The set of all (n.f.s.) operator-valued weights from M to N is denoted by $P(M, N)$. Haagerup showed that

$$P(M, N) \neq \emptyset \Leftrightarrow P(N', M') \neq \emptyset$$

and constructed an order-reversing bijection between $P(M, N)$ and $P(N', M')$. By the spatial theory, it is possible to construct the canonical order-reversing bijection from $P(M, N)$ onto $P(N', M')$ (denoted by $E \to E^{-1}$). For a given $E \in P(M, N)$, the canonical $E^{-1} \in P(N', M')$ is characterized by

$$d(\phi \circ E)/d\psi = d\phi/d(\psi \circ E^{-1}).$$

Let M be a (σ-finite) factor on a Hilbert space $\mathcal{H}$ with a subfactor N. Since E is an operator-valued weight, we get $E^{-1} \in P(N', M')$. For any unitary $u \in M'$, we have

$$uE^{-1}(1)u^* = E^{-1}(u1u^*) = E^{-1}(1).$$

Since M is a factor, this means that $E^{-1}(1)$ is a scalar.

DEFINITION 2.8. *IndexE* is the scalar $E^{-1}(1)$.

When the *Index* $E < +\infty$, the operator-valued weight $E^{-1} \in P(N', M')$ is a scalar multiple of a conditional expectation.

3. The von Neumann algebra $\mathcal{W}^*(\mathcal{R}_G)$

Let (X,μ) be a Lebesgue space and G be a countable group acting on X as automorphisms. An equivalence relation $\mathcal{R}_G$ is a subset $\{(x,y)\in X\times X|\exists g\in G \text{ s.t } y=gx\}$ of $X\times X$. Then $(\mathcal{R}_G,\mu_l)$ becomes a Lebesgue space, where μ_l is a left counting measure. Hence we define a Hilbert space $L^2(\mathcal{R}_G,\mu_l)$ on $(\mathcal{R}_G,\mu_l)$. If f is integrable on $(\mathcal{R}_G,\mu_l)$,

$$\int_{\mathcal{R}_G} f(x,y)d\mu_l(x,y)=\int_X\sum_{x\sim y}f(x,y)d\mu(y)$$

For $f,g\in L^2(\mathcal{R}_G,\mu_l)$, the operation of f and g on $L^2(\mathcal{R}_G,\mu_l)$ is the convolution product of f and g as

$$(f*g)(x,y)=\sum_{z\sim x}f(x,z)g(z,y).$$

We shall construct the von Neumann algebra $\mathcal{W}^*(\mathcal{R}_G)$ on $L^2(\mathcal{R}_G,\mu_l)$. Let $\mathcal{R}_G$ be a countable relation on $(X,\mathfrak{B},\mu)$. For $f\in L^2(\mathcal{R}_G,\mu_l)$ with *supp f small* (i.e, $f(x,gx)=0$ a.e. except for finitely many g), $L_f:L^2(\mathcal{R}_G,\mu_l)\to L^2(\mathcal{R}_G,\mu_l)$ is defined by $L_fg\equiv f*g$ for $g\in L^2(\mathcal{R}_G,\mu_l)$. Then we have $L_fL_g=L_{f*g}$, and $L_f{}^*=L_{f^*}$ with $f^*(x,y)=\bar{f}(y,x)$.

DEFINITION 3.1. The operators L_f form a *-algebra of operators with unit; we denote its weak closure by $\mathcal{W}^*(\mathcal{R}_G)$.

For $f\in L^2(\mathcal{R}_G,\mu_l)$ with *supp* $f\subseteq D=\{(x,x)|x\in X\}$ the diagonal, we define $F\in L^\infty(X,\mu)$ by

$$f(x,y)=\begin{cases}F(x) & x=y\\ 0 & x\neq y.\end{cases}$$

Hence we can correspond $\{L_f|suppf\subseteq D\}$ to $L^\infty(X,\mu)$. For $f,f'\in L^2(\mathcal{R}_G)$ with *supp* f in D, if $f(x,y)=\delta_{xy}F(x)$ and $f'(x,y)=\delta_{xy}F'(x)$, then

$$(f*f')(x,y)=\sum_{z\sim x}f(x,z)f'(z,y)=\delta_{xy}F(x)F'(x).$$

Therefore $\{L_f|suppf\subseteq D\}$ already forms an abelian von Neumann subalgebra $\mathcal{A}$ of $\mathcal{W}^*(\mathcal{R}_G)$, isomorphic to $L^\infty(X)$.

PROPOSITION 3.2 [9]. *The abelian algebra $\mathcal{A}$ is a MASA(maximal abelian subalgebra) in $\mathcal{W}^*(\mathcal{R}_G)$. i.e, $\mathcal{W}^*(\mathcal{R}_G) \cap \mathcal{A}' = \mathcal{A}$.*

We suppose that X is $\mathbb{R}$. Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space and G_0 be a countable group of all mappings $g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ of $x \mapsto ax + b$, where $a, b \in \mathbb{Q}$, $a > 0$. That is, $G_0 = \{g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : X \to X | a, b \in \mathbb{Q}, a > 0\}$. Let $G_0' = \{g = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} | g \in G_0\}$ be a subgroup of G_0. Then G_0' acts ergodically on X. Hence G_0 acts ergodically.
Let $G = \{g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : X \to X | a, b \in \mathbb{Q}, a \neq 0\}$.

PROPOSITION 3.3 [9]. $L^2(X) \otimes l^2(G) \cong L^2(\mathcal{R}_G)$.

LEMMA 3.4 [9]. *For $\mathcal{A} = L^\infty(X)$ and a unitary V as in proposition 3.3,*

$$V(\mathcal{A} \rtimes_{\alpha_g} G)V^{-1} = \mathcal{W}^*(\mathcal{R}_G).$$

4. Main Results

Let $\Gamma(g) = \{(x, gx) | x \in X\}$ be the graph of g in $\mathcal{R}_G$. If $L_f \in \mathcal{W}^*(\mathcal{R}_G) \cap \mathcal{W}^*(\mathcal{R}_G)' \subseteq \mathcal{W}^*(\mathcal{R}_G) \cap \mathcal{A}' = \mathcal{A}$, then L_f commutes with $L_{\chi_{\Gamma(g)}}$ where $\mathcal{A}$ is as in proposition 3.2 and $\chi_{\Gamma(g)}$ is the characteristic function on $\Gamma(g)$. For $L_f \in \mathcal{A}$ and $f(x,y) = \delta_{xy}F(x)$, $F(gx) = F(x)$, since

$$L_{\chi_{\Gamma(g)}} L_f h(x, gx) = L_f L_{\chi_{\Gamma(g)}} h(x, gx) \quad \text{for all} \quad h \in L^2(\mathcal{R}_G, \mu_l).$$

Thus we have the following theorem.

THEOREM 4.1. *$\mathcal{W}^*(\mathcal{R}_G)$ is a factor of type III.*

Let G be a countable group of all mappings $g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ of $x \mapsto ax + b$, where $a, b \in \mathbb{Q}$, $a \neq 0$. We denote by $A =$

$\{(x,y) \in \mathcal{R}_G | \exists g \in G_0 \quad s.t \quad y = gx\}, B = \{(x,y) \in \mathcal{R}_G | \exists g \in G \setminus G_0 \quad s.t \quad y = gx\}$. Also we define the multiplication operator M_{χ_A} by $M_{\chi_A} k(x,y) = \chi_A(x,y)k(x,y)$. Now we define $E : \mathcal{W}^*(\mathcal{R}_G) \to \mathcal{W}^*(\mathcal{R}_{G_0})$ by $E(L_f)k(x,y) = L_{M_{\chi_A} f} k(x,y)$ for $k \in L^2(\mathcal{R}_{G_0})$.

LEMMA 4.2. *E is a (n.f.s.)conditional expectation.*

Proof. For $L_h \in \mathcal{W}^*(\mathcal{R}_{G_0}), L_f \in \mathcal{W}^*(\mathcal{R}_G)$, and $k \in L^2(\mathcal{R}_{G_0})$,

$$
\begin{aligned}
(E(L_h L_f L_{h^*})k)(x,y) &= (E(L_{h^* * f * h^*})k)(x,y) \\
&= L_{M_{\chi_A} h * f * h^*} k(x,y) \\
&= \sum_{g \in G_0} h * f * h^*(x, gx) k(gx, y) \\
&= \sum_{g \in G_0} \sum_{w,v \sim x} h(x,v) f(v,w) h^*(w, gx) k(gx, y) \\
&= \sum_{v,w \sim x} h(x,v) f(v,w) L_{h^*} k(w,y) \\
&= \sum_{v \sim x} h(x,v) L_f (L_{h^*} k)(v,y) \\
&= (L_h (L_{M_{\chi_A} f} (L_{h^*} k)))(x,y) \\
&= (L_h E(L_f) L_{h^*} k)(x,y).
\end{aligned}
$$

Also we have $(E(L_{\chi_{\Gamma(g_1)}})k)(x,y) = L_{M_{\chi_A} \chi_{\Gamma(g_1)}} k(x,y) = L_{\chi_{\Gamma(g_1)}} k(x,y)$ because of $M_{\chi_A} \chi_{\Gamma(g_1)} k(x,y) = \chi_{\Gamma(g_1)} k(x,y)$ for $g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in G. Since $L_{\chi_{\Gamma(g_1)}}$ is identity in $\mathcal{W}^*(\mathcal{R}_G)$, E is a conditional expectation. Let $L_{f_i} \in \mathcal{W}^*(\mathcal{R}_G) \nearrow a \in \mathcal{W}^*(\mathcal{R}_{G_0})$. Since $E(a) - E(L_{f_i}) = (a - L_{f_i})|_{L^2(\mathcal{R}_{G_0})}$, $E(L_{f_i}) \nearrow E(a)$. So E is normal. Let $E(L_{f^*} L_f) = 0$ for $L_f \in \mathcal{W}^*(\mathcal{R}_G)$. For $k \in L^2(\mathcal{R}_{G_0})$ and

$l \in L^2(\mathcal{R}_{G_0})^{\perp}$,

$$< E(L_{f^*}L_f)k, k >= \int_X \sum_{y\sim x}(E(L_{f^**f})k)(x,y)\overline{k(x,y)}d\mu(y)$$
$$= \int_X \sum_{y\sim x}(L_{M_{\chi_A}f^**f})k(x,y)\overline{k(x,y)}d\mu(y)$$
$$= \int_X \sum_{y\sim x}\sum_{g\in G_0} f^* * f(x,gx)k(gx,y)\overline{k(x,y)}d\mu(y)$$
$$= \int_X \sum_{y\sim x}\sum_{g\in G_0}\sum_{h\in G} f^*(x,hx)f(hx,gx)k(gx,y)\overline{k(x,y)}d\mu(y)$$
$$= \int_X \sum_{y\sim x}\sum_{g\in G}\sum_{h\in G} f^*(x,hx)f(hx,gx)M_{\chi_A}k(gx,y)\overline{M_{\chi_A}k(x,y)}d\mu(y)$$
$$=< L_{f^*}L_f M_{\chi_A}k, M_{\chi_A}k >, \text{and}$$

$$< L_{f^*}L_f M_{\chi_B}l, M_{\chi_B}l >$$
$$= \int_X \sum_{y\sim x}\sum_{g\in G}\sum_{h\in G} f^*(x,hx)f(hx,gx)M_{\chi_B}l(gx,y)\overline{M_{\chi_B}l(x,y)}d\mu(y)$$
$$= \int_X \sum_{y\sim x}\sum_{g\in G_0}\sum_{h\in G} f^*(x,hx)f(hx,gx)l(gx,y)\overline{l(x,y)}d\mu(y)$$
$$= \int_X \sum_{y\sim x}(L_{M_{\chi_A}f^**f})l(x,y)\overline{l(x,y)}d\mu(y) =< E(L_{f^*}L_f)l, l > .$$

Thus E is faithful since $L_f = 0$ by above. We put

$$n_E = \{L_f \in \mathcal{W}^*(\mathcal{R}_G) | \|E(L_{f^*}L_f)\| < \infty\}$$

$$m_E = n_E^* n_E = span\{L_{f^*}L_h | L_f, L_h \in n_E\}$$

$$D_E = \{L_f \in \mathcal{W}^*(\mathcal{R}_G)_+ | \|E(L_f)\| < \infty\}$$

Let $L_f \in m_E$. Since D_E is dense in m_E, there exist net $\{L_{f_i}\}$ of D_E with $L_{f_i} \nearrow L_f$ and $E(L_{f_i}) \nearrow E(L_f)$ by the normality of E. Hence E is semifinite.

Let $\phi \in \mathcal{W}^*(\mathcal{R}_{G_0})_*^+$ be a fixed normal faithful state. By representing $\mathcal{W}^*(\mathcal{R}_G)$, we assume $\phi \circ E = \omega_{\xi_0}$ with a cyclic and separating vector ξ_0. Let e be the projection defined by $e(L_f\xi_0)(x,y) = E(L_f)\xi_0(x,y)$, for $L_f \in \mathcal{W}^*(\mathcal{R}_G)$.

THEOREM 4.3. *Let G_0 be a countable subgroup of G as above and let $E : \mathcal{W}^*(\mathcal{R}_G) \to \mathcal{W}^*(\mathcal{R}_{G_0})$ be a (n.f.s.) conditional expectation. Then $IndexE = 2$.*

Proof. Let $g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $g_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then G is the disjoint union of g_1G_0 and g_2G_0. Let J_G be the modular conjugation of $\mathcal{W}^*(\mathcal{R}_G)$. By the basic construction, $<\mathcal{W}^*(\mathcal{R}_G), e>= J_G\mathcal{W}^*(\mathcal{R}_G)'J_G$ is the basic extension of $\mathcal{W}^*(\mathcal{R}_G) \supseteq \mathcal{W}^*(\mathcal{R}_{G_0})$. Also $\{L_{\chi_{\Gamma(g_i)}} e L^*_{\chi_{\Gamma(g_i)}}\}_{i\in 1,2}$ is a partition of the unit of $<\mathcal{W}^*(\mathcal{R}_G), e>$. By the U. Haagerup [12], there exist the (n.f.s.) operator valued weight $E^{-1} : \mathcal{W}^*(\mathcal{R}_{G_0})' \to \mathcal{W}^*(\mathcal{R}_G)'(J_{G_0}E(L_f)J_{G_0} \mapsto E^{-1}(J_{G_0}E(L_f)J_{G_0}))$. Define $E_1 :<\mathcal{W}^*(\mathcal{R}_G), e>\to \mathcal{W}^*(\mathcal{R}_G)$ by $E_1 = J_GE^{-1}(J_{G_0} \cdot J_{G_0})J_G$. Then E_1 is the (n.f.s.) operator valued weight $<\mathcal{W}^*(\mathcal{R}_G), e>$ into $\mathcal{W}^*(\mathcal{R}_G)$. Hence

$$IndexE = E^{-1}(1) = E_1(1) = E_1(\sum_{i=1}^{2} L_{\chi_{\Gamma(g_i)}} e L^*_{\chi_{\Gamma(g_i)}}) = 2.$$

References

1. F.M.Goodman, P.de la Harpe, V.F.R.Jones, *Coxeter Graphs and Towers of Algebras*, Springer-Verlag, 1989.
2. H.Kosaki, *Extension of Jones' Theory on Index to Arbitrary Factors*, J. Funct. Anal. **66** (1986).
3. V.F.R.Jones, *Index for subfactors*, INvent math. **72** (1983).
4. H.Choda, *Full factors with non integer index*, preprint.
5. Longo, *Index for subfactors and statistics of quantum fields I*, Comm.Math. Phys. **126** (1989).
6. connes, *Spatial theory of von Neumann algebras*, J. Funct. anal. **35** (1980).
7. U.Haagerup, *Operator-valued weights in von Neumann algebras I,II*, J.Funct. Anal. **32,33** (1979).
8. Umegaki, *Conditional expectation in an operator algebra I*, Tohoku math.J **6** (1954).
9. Dongseo Kim, Changho Byun, *Notes on the Factor $\mathcal{W}^*(\mathcal{R}_G)$ over the Hilbert space*, to appear.