Honam Mathematical J. 19(1997), No. 1,pp. 117–123

ON THE LOCALIZATION OF THE LOCALLY NILPOTENT SPACE AND CONDITIONS (T^*) AND (T^{**})

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1. Introduction

There are many results on the nilpotent space with relation to the localization, completion [1,2,6,9,11]. There is an effort extending the concept of the nilpotent space with respect to localization [4,10].

In this paper, we make some results of the locally nilpotent spaces with relation to the conditions (T^*) and (T^{**}).

Furthermore, we refer the product and hereditary properties of the locally nilpotent spaces and spaces satisfying condition (T^*) or (T^{**}). The problems of the R-good property of the spaces satisfying condition (T^*) or (T^{**}) and the existence of the idempotent completion functor will be studied.

We work in the category of the connected CW-complexes with base point and denoted as the T. We denote "$\approx$" as the same homotopy type.

Received May 24, 1997.

This work was partially supported by NON DIRECTED RESEARCH FUND, Korea research Foundation, 1996 and BSRI program, Ministry of education under the project number BSRI-96-1425.

2. Some properties of the locally nilpotent space and conditions (T^*) and (T^{**})

In this section, we define the locally nilpotent space which extends the concept of the nilpotent space. The conditions (T^*), (T^{**}), and their properties will be studied.

We recall that a locally nilpotent group is the group whose all finitely generated subgroups are nilpotent groups [13].

And we denote the category of nilpotent spaces and continuous maps by T_N.

Now we extend the concept of the nilpotent space as follows; we recall that a space $X(\in T)$ is said to be a locally nilpotent space if

(1) $\pi_1(X)$ is a locally nilpotent group,
(2) the action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all $n \geq 2$ [8,14].

And we denote the category of locally nilpotent spaces and continuous maps by T_{LN}.

We know that the category T_N is a full subcategory of T_{LN}.

Generally, for a group G and a fixed $g \in G$, we denote by $[g, G]$ the subgroup of G generated by all commutators $[g, a]$, where $a \in G$. Since $[g, a]^b = [g, b]^{-1}[g, ab]$ for each $a, b \in G$ (where $a^b = b^{-1}ab$), we know that [g,G] is a normal subgroup of G.

We recall that a space $X(\in T)$ satisfies the *condition* (T^*) if for all $g, t \in \pi_1(X)$

(1) $g[g, \pi_1(X)] = t[t, \pi_1(X)]$ or
(2) $g[g, \pi_1(X)] \bigcap t[t, \pi_1(X)] = \phi$ [8].

Let G be an arbitrary group. If $b \in a[a, G](a, b \in G)$ then $b[b, G] \subset a[a, G]$ [5].

LEMMA 2.1 [14]. *For $X(\in T)$, the following conditions are equivalent.*

(1) *X satisfies the condition (T^*).*
(2) *For each $a, b \in \pi_1(X)$, $a[a, \pi_1(X)] \subset b[b, \pi_1(X)]$ $\Rightarrow a[a, \pi_1(X)] = b[b, \pi_1(X)]$.*
(3) *For each $a \in \pi_1(X)$, $h \in [a, \pi_1(X)] \Rightarrow [ah, \pi_1(X)] = [a, \pi_1(X)]$.*

Thus we have the following theorem by use of the Lemma 2.1.

THEOREM 2.2. *For* $X \in T_{LN}$, *if* $b \in [a, \pi_1(X)]$ *then* $a[a, \pi_1(X)] = b[b, \pi_1(X)]$, *for* $a,b \in \pi_1(X)$.

Proof. For $X \in T_{LN}, X$ satisfies the condition (T^*) [14]. If $b \in [a, \pi_1(X)]$ $(a, b \in \pi_1(X))$ then $b[b, \pi_1(X)] \subset a[a, \pi_1(X)]$. Thus our proof is completed by Lemma 2.1.

Now we define an effective concept with respect to the locally nilpotent space.

We recall the following [8]; for $X(\in T)$, we say that X satisfies the *condition* (T^{**})

if for all $g(\neq 1) \in \pi_1(X)$, then $g \notin [g, \pi_1(X)]$.

Since the $[g, \pi_1(X)]$ is a normal subgroup of $\pi_1(X)$,the condition (T^{**}) is homotopy invariant property. Furthermore, the condition (T^{**}) is more powerful than the condition (T^*). And the condition (T^{**}) has the interesting direct limit property [7].

THEOREM 2.3 [8]. *For set* $\{X_\alpha | \alpha \in M : finite\}$, X_α *satisfies the condition* (T^{**}) *for each* $\alpha \in M$ *if and only if* $\prod_{\alpha \in M} X_\alpha$ *satisfies the condition* (T^{**}).

THEOREM 2.4 [14]. *For set* $\{X_\alpha | \alpha \in M : finite\}$, $X_\alpha \in T_{LN}$ *for each* α

if and only if $\prod_{\alpha \in M} X_\alpha \in T_{LN}$.

We recall that a space X is said to be a virtually nilpotent space if

(1) $\pi_1(X)$ has a nilpotent normal subgroup N of finite index,
(2) $\pi_1(X)$ has a subgroup of finite index which acts nilpotently on $\pi_n(X)$ for $n \geq 2$.

We denote the category of the virtually nilpotent spaces and continuous maps as T_{VN}. And T_N is the full subcategory of T_{VN}. We know that the real projective space RP^n and supersoluble spaces are all virtually nilpotent spaces [6].

LEMMA 2.5 [12, PROPOSITION 2.1]. *A space* X *is nilpotent if and only if* $\pi_1(X)$ *is nilpotent and for all* $i \geq 0$ *the* $\pi_1(X)$*-modules* $H_i(\tilde{X}; \mathbb{Z})$ *are nilpotent.*

We denote a fixed set of prime by P, and its complement in the set of all primes by P'. Moreover, we write $n \in P'$ if n belongs to the multiplicative closure of the primes in P'.

A group G is said to be *P-local* if the mapping $*: G \longrightarrow G$ defined by $*(x) = x^n$ is bijective for all $n \in P'$.

DEFINITION 2.6. Let G be a group and A a (left) $\mathbb{Z}G$-module. Denote by $\omega: \mathbb{Z}G \longrightarrow \mathrm{End}(A)$ the associated ring homomorphism. We say that A is *a P-local* $\mathbb{Z}G$*-module* if

$$\omega(1 + x + x^2 + \cdots + x^{n-1}) \in \mathrm{Aut}(A)$$

for all $x \in G$, $n \in P'$.

We use the notation

$$\rho_{n,x} = 1 + x + x^2 + \cdots + x^{n-1}.$$

Given a group G and $\mathbb{Z}G$-module A, and the identity

$$(a, x)^n = (\rho_{n,x} a, x^n), \quad n \geq 1,$$

in the group $A \rtimes G$, $A \rtimes G$ is P-local if and only if G is a P-local group and A is a P-local $\mathbb{Z}G$-module [4]. Here "$\rtimes$" means the semi-direct product.

LEMMA 2.7 [4]. *For X, the followings are equivalent:*

(1) *On the loop space ΩX, the n-th power map*

$$\omega \longmapsto \omega^n$$

is a self homotopy equivalence for every $n \in P'$.

(2) *The group $[W, \Omega X]$ is P-local for every space W. Here $[\ ,\ ]$ means the base point preserving homotopy class.*

(3) *The groups $\pi_1(X)$ and $\pi_k(X) \rtimes \pi_1(X)$, $k \geq 2$, are P-local, where $\rtimes$ denotes the semi-direct product with respect to the standard action.*

(4) *The group $\pi_1(X)$ is a P-local group and each $\pi_k(X)$, $k \geq 2$, is a P-local $\mathbb{Z}[\pi_1(X)]$-module.*

DEFINITION 2.8. We call *P-local spaces* those X satisfying the equivalent conditions of Lemma 2.7.

Note that, if X is nilpotent, then the groups $\pi_k(X) \rtimes \pi_1(X)$ are nilpotent. Therefore a nilpotent X is P-local in our sense if and only if the groups $\pi_k(X)$ are P-local for $k \geq 1$, because P-localization is exact in the category of nilpotent groups [4].

3. Main Results

Let R be a subring of the rational number Q i.e., $\mathbb{Z}[J]$ where J is the set of given prime numbers or $R = \mathbb{Z}_p$; the integers modulo a prime p. R-localization and R-completion are well known for nilpotent space [3]. But the case of the non-nilpotent space is obscure. We put the R-completion of X as $R_\infty X$. In fact, the functor R_∞ is covariant and put $X_{(p)}$ as the $\mathbb{Z}_p$-localization of X [3].

We know that the space X is R-good if $\tilde{H}_n(X; R) \to \tilde{H}_n(R_\infty X; R)$ is an isomorphism for $n \geq 0$. We recall that the space X with $\pi_1(X)$ finite is $\mathbb{Z}_p$ -good for all prime p and for the space X with $\pi_i(X)$ finite for all i then X is a $\mathbb{Z}$-good space. However the space with $\pi_1(X)$ finite need not be $\mathbb{Z}$-good [3]. Thus we have a question; under what condition is the space with $\pi_1(X)$ finite $\mathbb{Z}$-good? Furthermore, under what condition is there an idempotent completion functor? For a given category B, we make the orthogonal pair [ ] in B by the idempotent monad [4]. The localization of B is discribed by idempotent monad and every idempotent monad determines an orthogonal pair [4]. Thus we make the following idempotent functor R_∞ on the given orthogonal pair i.e., $R_\infty \circ R_\infty = R_\infty$.

We recall the well known facts; the Bousfield - Kan's completion functor R_∞ is not idempotent in general.

THEOREM 3.1. *For X satisfying condition (T^{**}) with*

(1) *$\pi_1(X)$ is finite,*
(2) *the action $\pi_1(X) \times H_n(\tilde{X}) \to H_n(\tilde{X})$ is nilpotent where $n \geq 0$,*

then there exists an idempotent functor R-completion; R_∞, where $R = \mathbb{Z}[J]$; subring of the rational numbers or $R = \mathbb{Z}_p$.

Proof. By use of the upper central series of $\pi_1(X)$ by the virtue of center of $\pi_1(X)$, we check the nilpotent structure of the $\pi_1(X)$ [8]. Thus $X \in T_N$ by Lemma 2.5. Furthermore, the space X is R -good space, $R_\infty X$ and $R_\infty(R_\infty X)$ are weak homotopy equivalence. Hence our proof is completed.

COROLLARY 3.2. *Under the same conditions with Theorem 3.1, there exists an idempotent completion functor $(\mathbb{Z}_p)_\infty$ such that $(\mathbb{Z}_p)_\infty X \approx X_{(p)}$.*

COROLLARY 3.3. *For finite X satisfying condition (T^*), if*

(1) *$\pi_1(X)(\neq 1)$ is finite,*
(2) *the action $\pi_1(X) \times H_n(\tilde{X}) \to H_n(\tilde{X})$ is nilpotent for $n \geq 0$,*

then there exists an idempotent completion functor R_∞.

Proof. We know the following; if X satisfies the condition (T^*) then X also satisfies the condition (T^{**}) [14].

We recall that a group G satisfies the maximal condition if it has no infinite strictly increasing chain of subgroups [13].

Let F be a free group on an infinitely countable number of generators, BF is not an R-good space.

THEOREM 3.4. *For $X(\in T_{LN})$, if*

(1) *$\pi_1(X)(\neq 1)$ is finite,*
   or
(2) *$\pi_1(X)$ is infinite with the maximal condition on normal subgroups of $\pi_1(X)$,*

then there exist a localization $X_{(p)}$ of X such that $X_{(p)} \approx (\mathbb{Z}_p)_\infty X$. Furthermore, $l_* : \tilde{H}_n(X) \to \tilde{H}_n(X_p)$ is a localization for all $n \geq 1$.

Proof. (Case 1) when $\pi_1(X)$ is finite, so it is finitely generated. Thus $X \in T_N$. Thus we get the localization and completion of X such that $(\mathbb{Z}_p)_\infty X \approx X_{(p)}$.

(Case 2)when $\pi_1(X)$ is infinite and $\pi_1(X)$ satisfies the maximal condition on normal subgroups then $\pi_1(X)$ is a finitely generated nilpotent group. Hence $X \in T_N$. Thus our proof is completed.

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