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STRONG HOMOLOGY GROUPS W.R.T. $\sum_{p\in\mathbb{Z}}\bar{C}^p(\mathcal{X};R)$

HONG-JAE LEE AND DAE-WOONG LEE

Dept. of Mathematics,
Chonbuk National University, Chonju 561-756, Korea.

1. Introduction

Strong homology group, a new type of homology, of inverse systems (of pairs) was introduced by J. T. Lisica and S. Mardešić ([2],[3]). This group also defined in the category *invTop* of inverse systems of topological spaces and the related category *proTop* and the coherent pro-homotopy category *cphTop*. Some desirable results are appeared in the previous papers([4],[5]).

In this paper, we define a strong evaluation map and strong cup product of strong cochain groups. As a consequence of these definitions and well known facts, we construct a strong cochain ring $\sum_{p\in\mathbb{Z}}\bar{C}^p(\mathcal{X};R)$ and a ring homomorphism.

2. Some Properties of (Relative) Strong Homology Groups

Let $\mathcal{X} = (X_\alpha, p_{\alpha\alpha'}, D)$ be an inverse system of topological spaces X_α and continuous maps $p_{\alpha\alpha'} : X_{\alpha'} \to X_\alpha$, $\alpha \leq \alpha'$, over a directed set D. Let $D^n, n \geq 0$, be the set of all $a = (\alpha_0, \alpha_1, \cdots, \alpha_n), \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n, \alpha_i \in D$ and G be an abelian group. If $n \geq 1, 0 \leq j \leq n$ and $a = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in D^n$, let $a_j \in D^{n-1}$ be obtained from $a = (\alpha_0, \alpha_1, \cdots, \alpha_n)$ by deleting the j-th factor α_j. For an integer p, a *strong p-chain* [2] of $\mathcal{X}$ with coefficients in G is defined by a function x which assigns

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to every $n \geq 0$ and every $a = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in D^n$, a singular $(p+n)$-chain $x_a \in C_{p+n}(X_a; G)$, where X_a is defined by X_{α_0}. The sum $x + y$ of two strong p-chains of $\mathcal{X}$ is given by $(x+y)_a = x_a + y_a, a \in D^n$. Strong p-chains form an abelian group $\bar{C}_p(\mathcal{X}; G)$ which is called *strong p- chain group* of $\mathcal{X}$ with coefficients in G. That is to say,

$$\bar{C}_p(\mathcal{X}; G) = \prod_{n=0}^{\infty} \prod_{a \in D^n} C_{p+n}(X_a; G),$$

where $C_{p+n}(X_a; G)$ is a $(p+n)$-chain group of $X_a = X_{\alpha_0}$. A boundary operator $d_p : \bar{C}_{p+1}(\mathcal{X}; G) \to \bar{C}_p(\mathcal{X}; G)$ was defined by

$$\begin{cases} (d_p(x))_{(\alpha_0)} = \partial(x_{(\alpha_0)}) \text{ for } n = 0 \\ (-1)^n (d_p x)_a = \partial(x_a) - p_{\alpha_0 \alpha_1}(x_{a_0}) - \sum_{j=1}^{n} (-1)^j x_{a_j} \text{ for } n \geq 1, \end{cases}$$

where $x \in \bar{C}_{p+1}(\mathcal{X}; G)$. Note that $d_p \circ d_{p+1} = 0$, $p \geq -n$ so that $(\bar{C}, d)$ is a chain complex. The *p-dimensional strong homology group* $\bar{H}_p(\mathcal{X}; G)$ of the inverse system $\mathcal{X} = (X_\alpha, p_{\alpha\alpha'}, D)$ with coefficients in G is defined by the homology group of this chain complex $(\bar{C}, d)$,*i.e.*,

$$\bar{H}_p(\mathcal{X}; G) = \ker(d_{p-1} : \bar{C}_p(\mathcal{X};G) \to \bar{C}_{p-1}(\mathcal{X}; G))/ \\ \operatorname{im}(d_p : \bar{C}_{p+1}(\mathcal{X}; G) \to \bar{C}_p(\mathcal{X}; G)).$$

DEFINITION 2.1. The *p-dimensional strong cochain group* $\bar{C}^p(\mathcal{X}; G)$ of the inverse system $\mathcal{X}$ with coefficients in G is defined by

$$\bar{C}^p(\mathcal{X}; G) = Hom(\bar{C}_p(\mathcal{X}), G).$$

The coboundary operator $\delta : \bar{C}^p(\mathcal{X}; G) \to \bar{C}^{p+1}(\mathcal{X}; G)$ is defined to be the dual homomorphism of the boundary operator $d : \bar{C}_{p+1}(\mathcal{X}) \to \bar{C}_p(\mathcal{X})$.

We consider inverse system $(\mathcal{X}, \mathcal{A}) = ((X, A)_\alpha, p_{\alpha\alpha'}, D)$ of pair of spaces and maps of pairs, where $(X, A)_\alpha = (X_\alpha, A_\alpha), A_\alpha \subset$

X_α. We also consider the inverse system $\mathcal{A} = (A_\alpha, p_{\alpha\alpha'}|_{A_{\alpha'}}, D)$ of topological subspace A_α of $X_\alpha$for each $\alpha \in D$ and continuous maps $p_{\alpha\alpha'}|_{A_{\alpha'}}$ over the same directed set D. We can view the strong chain group $\bar{C}^p(\mathcal{A};G)$ as a subgroup of $\bar{C}^p(\mathcal{X};G)$ and the boundary operator $d_\mathcal{A} : \bar{C}_p(\mathcal{A};G) \to \bar{C}_{p-1}(\mathcal{A};G)$ as a restriction map $d_\mathcal{A} = d|_{\bar{C}_p(\mathcal{A};G)}$. The *relative strong chain group* $\bar{C}^p(\mathcal{X},\mathcal{A};G)$ of $(\mathcal{X},\mathcal{A})$ with coefficents in G is defined by

$$\bar{C}_p(\mathcal{X},\mathcal{A};G) = \bar{C}_p(\mathcal{X};G)/\bar{C}_p(\mathcal{A};G).$$

Similarly, the *relative strong homology group* $\bar{H}_p(\mathcal{X},\mathcal{A};G)$ of $(\mathcal{X},\mathcal{A})$ with coefficients in G is defined by

$$\bar{H}_p(\mathcal{X},\mathcal{A};G) = \ker(d_{p-1} : \bar{C}_p(\mathcal{X},\mathcal{A};G) \to \bar{C}_{p-1}(\mathcal{X},\mathcal{A};G))/ \\ \mathrm{im}(d_p : \bar{C}_{p+1}(\mathcal{X},\mathcal{A};G) \to \bar{C}_p(\mathcal{X},\mathcal{A};G)).$$

Let $\mathcal{Y} = (Y_\beta, q_{\beta\beta'}, E)$ be an inverse system of topological spaces Y_β and continuous maps $q_{\beta\beta'} : Y_{\beta'} \to Y_\beta, \beta \le \beta'$ over a directed set E and let $f : \mathcal{X} \to \mathcal{Y}$ be a map of inverse systems given by an increasing function $\varphi : E \to D$ and continuous maps $f_\beta : X_{\varphi(\beta)} \to Y_\beta, \beta \in E$, and the following diagram

$$\begin{array}{ccc} X_{\varphi(\beta')} & \xrightarrow{p_{\varphi(\beta)\varphi(\beta')}} & X_{\varphi(\beta)} \\ f_{\beta'}\downarrow & & f_\beta\downarrow \\ Y_{\beta'} & \xrightarrow{q_{\beta\beta'}} & Y_\beta \end{array}$$

commutes for $\beta \le \beta'$. f induces a chain map $f_\sharp : \bar{C}_\sharp(\mathcal{X};G) \to \bar{C}_\sharp(\mathcal{Y};G)$, given by the homomorphisms

$$(f_\sharp x)_{(\beta_0,\beta_1,\cdots,\beta_n)} = f_{\beta_0\sharp}(x_{\varphi(\beta_0)\varphi(\beta_1),\cdots,\varphi(\beta_n)}),\ (\beta_0,\beta_1,\cdots,\beta_n) \in E^n$$

Let $(\mathcal{X},\mathcal{A},\mathcal{B}) = ((X,A,B)_\alpha, p_{\alpha\alpha'}, D)$ be an inverse system of traids, where $(X,A,B)_\alpha = (X_\alpha, A_\alpha, B_\alpha), A_\alpha, B_\alpha \subset X_\alpha$. Let $\bar{C}_p(\mathcal{A};G) + \bar{C}_p(\mathcal{B};G)$ denote the p-dimensional strong subchain group of $\bar{C}_p(\mathcal{X};G)$ generated by $\bar{C}_p(\mathcal{A};G) \cup \bar{C}_p(\mathcal{B};G)$.

Let $\mathcal{A} = (A_\alpha, p_{\alpha\alpha'}|_{A_{\alpha'}}, D)$ and $\mathcal{B} = (B_\alpha, p_{\alpha\alpha'}|_{B_{\alpha'}}, D)$ be two inverse systems

DEFINITION 2.2. The pair $(\mathcal{A}, \mathcal{B})$ of inverse systems is said to be *strong excisive couple* if the inclusion chain map

$$\bar{C}_\sharp(\mathcal{A};G) + \bar{C}_\sharp(\mathcal{B};G) \hookrightarrow \bar{C}_\sharp(\mathcal{A}\cup\mathcal{B};G)$$

induces an isomorphism of strong homology group.

PROPOSITION 2.3. *$(\mathcal{A}, \mathcal{B})$ is a strong excisive couple if and only if the inclusion $j : (\mathcal{A}, \mathcal{A}\cap\mathcal{B}) \hookrightarrow (\mathcal{A}\cup\mathcal{B}, \mathcal{B})$ induces an isomorphism of strong homology groups.*

proof. Consider the commutative diagram of chain maps

$$\begin{array}{ccc}
\bar{C}_\sharp(\mathcal{A},\mathcal{A}\cap\mathcal{B};G) & \xrightarrow{j_\sharp} & \bar{C}_\sharp(\mathcal{A}\cup\mathcal{B},\mathcal{B};G) \\
{\scriptstyle i_\sharp}\downarrow & & \| \\
[\bar{C}_\sharp(\mathcal{A};G)+\bar{C}_\sharp(\mathcal{B};G)]/\bar{C}_\sharp(\mathcal{B};G) & \xrightarrow{j'_\sharp} & \bar{C}_\sharp(\mathcal{A}\cup\mathcal{B},\mathcal{B};G)
\end{array}$$

induced by inclusions. By the Noether isomorphism theorem [6], $i_\sharp$ is an isomorphism. Therefore $j_* = j'_* \circ i_* : \bar{H}_*(\mathcal{A},\mathcal{A}\cap\mathcal{B};G) \to \bar{H}_*(\mathcal{A}\cup\mathcal{B},\mathcal{B};G)$ is an isomorphism if and only if j'_* is an isomorphism. If we use the following strong homology exact sequence of pair [3] and five lemma;

$$\begin{array}{ccccccccc}
\cdots\to & \bar{H}_p(\mathcal{B};G) & \to & \bar{H}_p(\mathcal{A}+\mathcal{B};G) & \to & \bar{H}_p(\mathcal{A}+\mathcal{B},\mathcal{B};G) & \to & \bar{H}_{p-1}(\mathcal{B};G) & \to\cdots \\
& \| & & \downarrow & & \downarrow & & \| & \\
\cdots\to & \bar{H}_p(\mathcal{B};G) & \to & \bar{H}_p(\mathcal{A}\cup\mathcal{B};G) & \to & \bar{H}_p(\mathcal{A}\cup\mathcal{B},\mathcal{B};G) & \to & \bar{H}_{p-1}(\mathcal{B};G) & \to\cdots
\end{array}$$

then we see that $j'_* : \bar{H}_p(\mathcal{A}+\mathcal{B},\mathcal{B};G) \to \bar{H}_p(\mathcal{A}\cup\mathcal{B},\mathcal{B};G)$ is an isomorphism if and only if the inclusion $j'_{1\sharp} : \bar{C}_p(\mathcal{A};G)+\bar{C}_p(\mathcal{B};G) \hookrightarrow \bar{C}_p(\mathcal{A}\cup\mathcal{B};G)$ induces an isomorpism $j'_{1*} : \bar{H}_p(\mathcal{A}+\mathcal{B};G) \to \bar{H}_p(\mathcal{A}\cup\mathcal{B};G)$ of strong homology groups. The proof follows from this fact and the definition of strong excisive couple.

DEFINITION 2.4. An inverse system $\mathcal{A} = (A_\alpha, p_{\alpha\alpha'}|_{A_{\alpha'}}, D)$ is called a *strong retract* of the inverse system $\mathcal{X} = (X_\alpha, p_{\alpha\alpha'}, D)$ if A_α is a subspace of X_α and the inclusion map $s_\alpha : A_\alpha \hookrightarrow X_\alpha$ has a left inverse $r_\alpha : X_\alpha \hookrightarrow A_\alpha$ for each $\alpha \in D$.

PROPOSITION 2.5. *If $\mathcal{A} = (A_\alpha, p_{\alpha\alpha'}|_{A_{\alpha'}}, D)$ is a strong retract of $\mathcal{X}$, then we have the following;*

$$\bar{H}_*(\mathcal{X}; G) \cong \bar{H}_*(\mathcal{A}; G) \oplus \bar{H}_*(\mathcal{X}, \mathcal{A}; G).$$

Proof. We can consider the following strong homology long exact sequence ([1],[3],[6])

$$\cdots \to \bar{H}_p(\mathcal{A}; G) \xrightarrow{s_*} \bar{H}_p(\mathcal{X}; G) \xrightarrow{t_*} \bar{H}_p(\mathcal{X}, \mathcal{A}; G) \xrightarrow{\partial} \bar{H}_{p-1}(\mathcal{A}; G) \to \cdots$$

induced by inclusions $s : \mathcal{A} \hookrightarrow \mathcal{X}$ and $t : (\mathcal{X}, \phi) \hookrightarrow (\mathcal{X}, \mathcal{A})$. Since $r_\alpha \circ s_\alpha = I_{A_\alpha}$ for each $\alpha \in D$, that is to say $r \circ s = (\cdots, r_\alpha, \cdots) \circ (\cdots, s_\alpha, \cdots) = (\cdots, I_{A_\alpha}, \cdots) = I_{\mathcal{A}}$, we have $r_* \circ s_* = I_{\bar{H}_p(\mathcal{A};G)}$. This shows that s_* is a monomorphism and the boundary operator ∂ is trivial. Thus we easily obtain the following split exact sequence

$$0 \to \bar{H}_p(\mathcal{A}; G) \xrightarrow{s_*} \bar{H}_p(\mathcal{X}; G) \xrightarrow{t_*} \bar{H}_p(\mathcal{X}, \mathcal{A}; G) \to 0$$

Therefore, we complete the proof.

3. A Ring $\sum_{p\in\mathbb{Z}} C^p(\mathcal{X}, R)$ with Unity $e_{\mathcal{X}}$

DEFINITION 3.1. Define a *strong evaluation map*

$$<,>: Hom(\bar{C}_p(\mathcal{X}), G) \times \bar{C}_p(\mathcal{X}) \to G$$

by $< c^p, c_p >_{(a,n)} = < c_a^{p+n}, c_{a,p+n} >$ for each $a = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in D^n, n = 0, 1, 2, \cdots$, where c_a^{p+n}(resp.$c_{a,p+n}$) is a $(p+n)$-cochain (resp. $(p+n)$-chain) on X_a.

It is easy to see that the strong evaluation map $<,>$ is bilinear. If $c^p \in \bar{C}^p(\mathcal{X}; G))$ and $c_{p+1} \in \bar{C}_{p+1}(\mathcal{X})$, then

$$< \delta c^p, c_{p+1} > = < c^p, dc_{p+1} >$$

and if $c_p \in \bar{C}_p(\mathcal{X})$ and $f : \mathcal{X} \to \mathcal{Y} = (Y_\alpha, q_{\alpha\alpha'}, D)$ is a map of inverse systems, then we have

$$< f^\sharp(c^p), c_p > = < c^p, f_\sharp(c_p) > .$$

Let R be a commutative ring with unity 1.

DEFINITION 3.2. Define a map

$$\bar{\cup} : \bar{C}^p(\mathcal{X}; R) \times \bar{C}^q(\mathcal{X}; R) \to \bar{C}^{p+q}(\mathcal{X}; R)$$

by

$$< c^p \bar{\cup} c^q, T >_{(a,n)} = < c^p, T_p >_{(a,i)} \cdot < c^q, T_q >_{(a,j)}$$

for each $a = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in D^n, i, j, n = 0, 1, 2, \cdots, i + j = n$, where T is a strong $(p+q)$-simplex and T_p(resp.T_q) denotes the strong p(resp.q)-simplex of the inverse system $\mathcal{X}$. The strong cochain $c^p \bar{\cup} c^q$ is called the *strong cup product* of the strong cochains c^p and c^q.

LEMMA 3.3. *The map* $\bar{\cup}$ *is bilinear.*

Proof. Let $c_1^p, c_2^p \in \bar{C}^p(\mathcal{X}; R)$ and $c_1^q, c_2^q \in \bar{C}^q(\mathcal{X}; R)$. If T is a strong $(p+q)$-simplex, then we obtain

$$\begin{aligned}
&< c_1^p \bar{\cup} (c_1^q + c_2^q), T >_{(a,n)} \\
&=< c_1^p, T_p >_{(a,i)} \cdot < (c_1^q + c_2^q), T_q >_{(a,j)} \\
&=< c_1^p, T_p >_{(a,i)} \cdot [< c_1^q, T_q >_{(a,j)} + < c_2^q, T_q >_{(a,j)}] \\
&= [< c_1^p, T_p >_{(a,i)} \cdot < c_1^q, T_q >_{(a,j)}] + [< c_1^p, T_p >_{(a,i)} \cdot < c_2^q, T_q >_{(a,j)}] \\
&= [< c_1^p \bar{\cup} c_1^q, T >_{(a,n)}] + [< c_1^p \bar{\cup} c_2^q, T >_{(a,n)}]
\end{aligned}$$

and

$$\begin{aligned}
&< (c_1^p + c_2^p) \bar{\cup} c_1^q, T >_{(a,n)} \\
&\quad =< (c_1^p + c_2^p), T_p >_{(a,i)} \cdot < c_1^q, T_q >_{(a,j)} \\
&\quad = [< c_1^p, T_p >_{(a,i)} + < c_2^p, T_p >_{(a,i)}] \cdot < c_1^q, T_q >_{(a,j)} \\
&\quad = [< c_1^p, T_p >_{(a,i)} \cdot < c_1^q, T_q >_{(a,j)}] + [< c_2^p, T_p >_{(a,i)} \cdot < c_1^q, T_q >_{(a,j)}] \\
&\quad =< c_1^p \bar{\cup} c_1^q, T >_{(a,n)} + < c_2^p \bar{\cup} c_1^q, T >_{(a,n)}
\end{aligned}$$

for each $a \in D^n, i + j = n$ and $i, j, n = 0, 1, 2, \cdots$. That is to say,

$$c_1^p \bar{\cup} (c_1^q + c_2^q) = (c_1^p \bar{\cup} c_1^q) + (c_1^p \bar{\cup} c_2^q)$$

and

$$(c_1^p + c_2^p) \bar{\cup} c_1^q = (c_1^p \bar{\cup} c_1^q) + (c_2^p \bar{\cup} c_1^q).$$

These complete the proof of the lemma.

THEOREM 3.4. *$\sum_{p\in\mathbb{Z}}\bar{C}^p(\mathcal{X};R)$ is a ring with unity $e_{\mathcal{X}}$ under the strong cup product.*

Proof. we now consider the associative law and unity. If $c^p \in \bar{C}^p(\mathcal{X};R)$, $c^q \in \bar{C}^q(\mathcal{X};R)$, $c^k \in \bar{C}^k(\mathcal{X};R)$ and if σ is a strong $(p+q+k)$-simplex, then we see that

$$< c^p\bar{\cup}(c^q\bar{\cup}c^k),\sigma >_{(a,n)}=< c^p,\sigma_p >_{(a,i)} \cdot < (c^q\bar{\cup}c^k),\sigma_{(q+k)} >_{(a,j)}$$

and

$$< (c^q\bar{\cup}c^k),\sigma_{(q+k)} >_{(a,j)}=< c^q,\sigma_q >_{(a,l)} \cdot < c^k,\sigma_k >_{(a,m)},$$

where σ_q(resp.σ_k) denotes the strong q(resp.k)- simplex of $\mathcal{X}, l+m=j$ and $l,m=0,1,2,\cdots$. Therefore we obtain

$$\begin{aligned} &< c^p\bar{\cup}(c^q\bar{\cup}c^k),\sigma >_{(a,n)} \\ &\quad =< c^p,\sigma_p >_{(a,i)} \cdot[< c^q,\sigma_q >_{(a,l)} \cdot < c^k,\sigma_k >_{(a,m)}] \\ &\quad = [< c^p,\sigma_p >_{(a,i)} \cdot < c^q,\sigma_q >_{(a,l)}]\cdot < c^k,\sigma_k >_{(a,m)} \\ &\quad =< c^p\bar{\cup}c^q,\sigma_{p+q} >_{(a,i+l)} \cdot < c^k,\sigma_k >_{(a,m)} \\ &\quad =< (c^p\bar{\cup}c^q)\cup c^k,\sigma >_{(a,i+l+m)} \\ &\quad =< (c^p\bar{\cup}c^q)\cup c^k,\sigma >_{(a,n)} . \end{aligned}$$

Thus, the associative law is completed. Define $e_{\mathcal{X}} \in \bar{C}^0(\mathcal{X};R)$ by

$$< e_{\mathcal{X}},T >_{(a,n)}= 1$$

for all strong 0-simplex T. Then we easily see that $e_{\mathcal{X}}$ is a unity on $\sum_{p\in\mathbb{Z}}\bar{C}^p(\mathcal{X};R)$. The lemma 3.3 shows that the condition of the distributive law is satisfied.

THEOREM 3.5. *The map $f:\mathcal{X}\to\mathcal{Y}$ of inverse systems induces a ring homomorphism $f^\sharp:\sum_{p\in\mathbb{Z}}\bar{C}^p(\mathcal{Y};R)\to\sum_{p\in\mathbb{Z}}\bar{C}^p(\mathcal{X};R)$.*

Proof. Let $e^p \in \bar{C}^p(\mathcal{Y};R), e^q \in \bar{C}^q(\mathcal{Y};R)$ and T be a strong $(p+q)$-simplex of the inverse system $\mathcal{X}$. Then we obtain

$$\begin{aligned} < f^\sharp(e^p\bar{\cup}e^q),T >_{(a,n)} &=< e^p\bar{\cup}e^q, f_\sharp(T) >_{(a,n)} \\ &=< e^p, f_\sharp T_p >_{(a,i)} \cdot < e^q, f_\sharp T_q >_{(a,j)} \\ &=< f^\sharp(e^p),T_p >_{(a,i)} \cdot < f^\sharp(e^q),T_q >_{(a,j)} \\ &=< f^\sharp(e^p)\bar{\cup}f^\sharp(e^q),T >_{(a,n)} \end{aligned}$$

and

$$
\begin{aligned}
< f^{\sharp}(e_1^p + e_2^p), T >_{(a,n)} &=< e_1^p + e_2^p, f_{\sharp}(T) >_{(a,n)} \\
&=< e_1^p, f_{\sharp}T >_{(a,n)} + < e_2^p, f_{\sharp}T >_{(a,n)} \\
&=< f^{\sharp}(e_1^p), T >_{(a,n)} + < f^{\sharp}(e_2^p), T >_{(a,n)} \\
&=< f^{\sharp}(e_1^p) + f^{\sharp}(e_2^p), T >_{(a,n)}
\end{aligned}
$$

for each $a = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in D^n, i, j, n = 0, 1, 2, \cdots, i + j = n$. If $e_{\mathcal{X}}$ and $e_{\mathcal{Y}}$ are unities of $\sum_{p\in\mathbb{Z}} \bar{C}^p(\mathcal{X}; R)$ and $\sum_{p\in\mathbb{Z}} \bar{C}^p(\mathcal{Y}; R)$ respectively, then we see that

$$
\begin{aligned}
< f^{\sharp}(e_{\mathcal{Y}}), T >_{(a,n)} &=< e_{\mathcal{Y}}, f_{\sharp}(T) >_{(a,n)} \\
&= 1.
\end{aligned}
$$

for all strong 0-simplex T of the inverse system $\mathcal{X}$. Thus $f^{\sharp}(e_{\mathcal{Y}}) = e_{\mathcal{X}}$. Therefore, the map $f : \mathcal{X} \to \mathcal{Y}$ induces a ring homomorphism.

References

1. K. A. Lee, *Foundation of Topology (Vol. 2)*, Hakmoonsa, Seoul, 1984.
2. J. T. Lisica and S. Mardešić, *Strong Homology of Inverse System of Spaces I,II*, Topology and its Appl. **19** (1985), 29-64.
3. J. T. Lisica and S. Mardešić, *Strong Homology of Inverse System of Spaces III*, Topology and its Appl. **20** (1985), 29-37.
4. S. Mardešić and A. V. Parasolov, *Strong Homology is not Additive*, Trans. of the A.M.S. **307(2)** (1988), 725-744.
5. S. Mardešić and T. Watanabe, *Strong Homology and Dimension*, Topology and its Appl. **29** (1988), 185-205.
6. E. H. Spanier, *Algebraic Topology*, Mcgraw-Hill Book Co., New York, 1966.