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ON THE DIRECT LIMIT OF THE LOCALLY NILPOTENT DIRECT SYSTEM

SANG-EON HAN

Dept. of Mathematics, Honam University, Kwangju 506-090, Korea.

Abstract In this paper, we make some results on the direct limit of the locally nilpotent direct system. We study the (T^{**}) - properties of the above direct limit and homotopy equivalence of the direct limits.

1. Introduction

In this paper, the properties of the direct limit of the locally nilpotent direct system will be studied. We make some results on the above direct limit with relations to the conditions (T^*) and (T^{**}) and homotopy equivalence.

Furthermore, we study the homotopy equivalent conditions of the direct limits from the locally nilpotent direct systems.

We work in the category of the topological spaces having the same homotopy type of connected CW-complexes with base point and denote it as T.

2. Some properties of the condition (T^{**}) and locally nilpotent direct system

In this section, we recall the locally nilpotent space and condition (T^*) and condition (T^{**}) [5,11]. Furthermore, nilpotent or locally nilpotent direct system will be introduced and we study their properties respectively.

In the direct system $\{(X_\alpha, f_\alpha)|\alpha \in \Lambda\}$, the direct limit $\varinjlim X_\alpha$ has the weak topology with respect to X_α for each $\alpha \in \Lambda$.

We recall that locally nilpotent group is the group whose finitely generated subgroups are nilpotent groups [9].

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And we denote the category of nilpotent spaces and continuous maps as T_N [6,7].

Now we extend the concept of the nilpotent space like following; we recall that a space $X(\in T)$ is said to be a locally nilpotent space if

(1) $\pi_1(X)$ is a locally nilpotent group,
(2) the action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all $n \geq 2$ [1].

And we denote the category of locally nilpotent spaces and continuous maps as T_{LN}. Trivially, the category T_N is a full subcategory of T_{LN}.

Generally, for a group G and a fixed $g \in G$, we denote by $[g, G]$ the subgroup of G generated by all commutators $[g, a]$ where $a \in G$. And $[g, G]$ is a normal subgroup of G [3].

We recall that a space $X(\in T)$ satisfies condition (T^*) if for all $g, t \in \pi_1(X)$ either $g[g, \pi_1(X)] = t[t, \pi_1(X)]$ or $g[g, \pi_1(X)] \bigcap t[t, \pi_1(X)] = \phi$ [6].

Now we define the effective concept with respect to extending the concept of nilpotent space.

DEFINITION 2.1. For $X(\in T)$, we say that X satisfies the condition (T^{**})
if for all $g(\neq 1) \in \pi_1(X)$, then $g \notin [g, \pi_1(X)]$.

Since the $[g, \pi_1(X)]$ is a normal subgroup of $\pi_1(X)$, *the* condition (T^{**}) has the homotopy invariant useful property.

DEFINITION 2.2. In the direct system $\{(X_\alpha, f_\alpha) | \alpha \in \Lambda\} \cdots (*)$,
we call the above $(*)$ nilpotent direct system if $X_\alpha \in T_N$ for each α.

Similarly, we call the above $(*)$ locally nilpotent direct system if $X_\alpha \in T_{LN}$ for each α.

LEMMA 2.3 [2]. *Let G be an arbitrary group. If $b \in a[a, G]$ $(a, b \in G)$ then $b[b, G] \subset a[a, G]$.*

LEMMA 2.4 [5,10]. *For $X \in T_{LN}$, then X satisfies the condition (T^*).*

LEMMA 2.5 [10]. *For $X(\in T)$, the following conditions are equivalent.*

(1) *X satisfies the condition (T^*).*
(2) *For each $a, b \in \pi_1(X)$, $a[a, \pi_1(X)] \subset b[b, \pi_1(X)]$ $\Rightarrow a[a, \pi_1(X)] = b[b, \pi_1(X)]$.*
(3) *For each $a \in \pi_1(X)$, $h \in [a, \pi_1(X)] \Rightarrow [ah, \pi_1(X)] = [a, \pi_1(X)]$.*

THEOREM 2.6. *The conditions (T^*) and (T^{**}) are equivalent up to nilpotency of the $\pi_1(X)$ if $\pi_1(X)$ is finite.*

Proof. First, if $\pi_1(X)$ is finite,by the topological reformation of the Douchaev's result [3], we know that X satisfies the condition (T^*) if and only if $\pi_1(X)$ is nilpotent group.

Next, if X satisfies the condition (T^{**}) we know that $\pi_1(X)$ is a nilpotent group if $\pi_1(X)$ is finite [5]. Thus the conditions (T^*) and (T^{**}) are equivalent up to nilpotency of the $\pi_1(X)$ if $\pi_1(X)$ is finite. Conversely, if $\pi_1(X)$ is a nilpotent group then for each $g(\neq 1) \in \pi_1(X)$, then $g \notin [g, \pi_1(X)]$. Furthermore, for all $g, t \in \pi_1(X)$ whether $g[g, \pi_1(X)] = t[t, \pi_1(X)]$, or $g[g, \pi_1(X)] \bigcap t[t, \pi_1(X)] = \phi$. Thus our proof is completed.

In fibration $F \to E \to B$, any path $\alpha : I \to B$ and singular q-complex $g : \Delta^q \to p^{-1}(\alpha(0))$ determine a map $G : \Delta^q \times I \to E$ over $\alpha \circ pr_2 : \Delta^q \times I \to I \to B$ and extend $G_0 = g : \Delta^q \times \{ 0 \} \to E$. If α is a loop, then
$G_1 : \Delta^q \times \{ 1 \} \to E$ is a q-simplex in $p^{-1}(\alpha(1)) = p^{-1}(\alpha(0))$. Now do elements of $\pi_1(B)$ operate on $H_*(F)$ [8].

DEFINITION 2.7. A fibration $F \to E \to B$ is said to be quasi-nilpotent if the action of $\pi_1(B)$ on $H_*(F)$ is nilpotent, $* \geq 0$.

3. Main Results

In the direct systems $\{(X_\alpha, f_\alpha) | \alpha \in \Lambda\}$ with $X_\alpha \subset X_{\alpha+1}$ then for any compact space Y, we know that
$[Y, \varinjlim X_\alpha] = \varinjlim [Y, X_\alpha]$, where [ ] means the homotopy class since X_α is a T_1-space for each α [4]. Apparently,
$\pi_n \varinjlim X_\alpha = \varinjlim \pi_n X_\alpha$.

LEMMA 3.1. *For $X \in T_{LN}$, if $b \in [a, \pi_1(X)]$ then $a[a, \pi_1(X)] = b[b, \pi_1(X)]$, for $a, b \in \pi_1(X)$.*

Proof. If $b \in a[a, \pi_1(X)]$ $(for \quad a, b \in \pi_1(X))$ then $b[b, \pi_1(X)] \subset a[a, \pi_1(X)]$ by Lemma 2.3. Since X satisfies the condition (T^*) by Lemma 2.4, thus our proof is completed by Lemma 2.5.

THEOREM 3.2. *In the direct system*
$\{(X_\alpha, i_\alpha) | \alpha \in \Lambda, i_\alpha : X_\alpha \subset X_{\alpha+1}, X_\alpha \in T_N\}$
then the $\varinjlim X_\alpha$ satisfies the condition (T^*).

Proof. Since $\pi_1 \varinjlim X_\alpha = \varinjlim \pi_1 X_\alpha$. and furthermore $\varinjlim \pi_1 X_\alpha$ is a locally nilpotent group for each α. Put $\varinjlim \pi_1 X_\alpha$ as G. Now, suppose $c \in a[a, G] \cap b[b, G]$ for some $a, b, c \in G$. We know that $a[a, G] = b[b, G]$ by Lemma 3.1. Our proof is completed.

COROLLARY 3.3. *In the direct system*
$\{(X_\alpha, i_\alpha) | \alpha \in \Lambda, i_\alpha : X_\alpha \subset X_{\alpha+1}, X_\alpha \in T_{LN}\}$
then the $\varinjlim X_\alpha$ satisfies the condition (T^*).

Proof. Since the direct limit of the locally nilpotent group is a locally nilpotent group. Thus $\varinjlim X_\alpha$ satisfies the condition (T^*) by Lemma 2.4.

THEOREM 3.4. *Under the same hypothesis with Theorem 3.2, $\varinjlim X_\alpha$ satisfies condition (T^{**}).*

Proof. Since $\varinjlim X_\alpha$ satisfies the condition (T^*) by the Theorem 3.2, assume that for $g(\neq 1) \in \varinjlim \pi_1 X_\alpha, g \in [g, \varinjlim \pi_1 X_\alpha]$.

Since $[g, \varinjlim \pi_1 X_\alpha]$ is a normal subgroup of $\varinjlim \pi_1 X_\alpha$, there exists an element $g^{-1} \in [g, \varinjlim \pi_1 X_\alpha]$. Hence $1 \in g[g, \varinjlim \pi_1 X_\alpha]$. Therefore

$g[g, \varinjlim \pi_1 X_\alpha] \bigcap 1[1, \varinjlim \pi_1 X_\alpha] \neq \phi$.

By the condition (T^*), $g[g, \varinjlim \pi_1 X_\alpha] = 1[1, \varinjlim \pi_1 X_\alpha] = 1$. But $g \in g[g, \varinjlim \pi_1(X)]$. Thus we have a contradiction.

THEOREM 3.5. *In the direct system* $\{(X_\alpha, i_\alpha)|\alpha \in \Lambda, i_\alpha; X_\alpha \subset X_{\alpha+1}\}$, *if* X_α *satisfies the condition* (T^{**}), *the direct limit space* $\varinjlim X_\alpha$ *satisfies the condition* (T^{**}).

Proof. For all $g_\alpha(\neq 1) \in \pi_1(X_\alpha)$, then $g_\alpha \notin [g_\alpha, \pi_1(X_\alpha)]$. Since for any $g(\neq 1) \in \pi_1(\varinjlim X_\alpha)$, we know that $[g, \pi_1(\varinjlim X_\alpha)] = [g, \varinjlim \pi_1(X_\alpha)]$. Thus we get $g \notin [g, \pi_1(\varinjlim X_\alpha)]$.

THEOREM 3.6. *In the direct systems* $\{(X_\alpha, i_\alpha)|\alpha \in \Lambda, i_\alpha : X_\alpha \subset X_{\alpha+1}, X_\alpha \in T_{LN}\}$, *and*
$\{(Y_\alpha, i_\alpha)|\alpha \in \Lambda, i_\alpha : Y_\alpha \subset Y_{\alpha+1}, Y_\alpha \in T_{LN}\}$ *if there exists a map*
$h_\alpha : X_\alpha \to Y_\alpha$ *satisfying at most one of the following;*

(1) h_α *is a quasi-nilpotent homology equivalence and maximal perfect subgroup* $P\pi_1(X_\alpha) = 0$,
(2) h_α *is a nilpotent homology equivalence,*
(3) h_α *is an acylic map such that* $\pi_1(h_\alpha)$ *is an isomorphism.*

then $h : X \to Y$ *is a homotopy equivalence where* $\varinjlim X_\alpha = X$, $\varinjlim Y_\alpha = Y$, *and* $\varinjlim h_\alpha = h$.

Proof. Since the map $h = \varinjlim h_\alpha$ is a nilpotent map, our proof is completed.

THEOREM 3.7. *In the direct systems* $\{(X_\alpha, i_\alpha)|\alpha \in \Lambda, i_\alpha : X_\alpha \subset X_{\alpha+1}, X_\alpha$ *satisfies the condition* (T^*) $\}$ *and* $\{(Y_\alpha, i_\alpha)|\alpha \in \Lambda, i_\alpha : Y_\alpha \subset Y_{\alpha+1}, Y_\alpha$ *satisfies the condition* (T^*) $\}$, *if there exists a map* $h_\alpha : X_\alpha \to Y_\alpha$ *satisfying at most one of the following;*

(1) h_α *is a quasi-nilpotent homology equivalence,*
(2) h_α *is a nilpotent homology equivalence,*
(3) h_α *is an acylic map*

and if $\pi_1(X)$ *is finite, then* $h : X \to Y$ *is a homotopy equivalence where* $\varinjlim X_\alpha = X$, $\varinjlim Y_\alpha = Y$, *and* $\varinjlim h_\alpha = h$,

Proof. Under the above conditions, since the $\varinjlim X_\alpha$ and $\varinjlim Y_\alpha$ satisfies the condition (T^*). In cases (1) and (2), $\pi(X_\alpha)/P\pi_1(X_\alpha) \cong \pi_1(Y_\alpha)$. And $X \in T_N$ since $\pi_1(X)$ is finite, thus $P\pi_1(X) = 0$. $\pi_1(h)$ is a homotopy equivalence and h is an acyclic, hence h is a homotopy equivalence. In case (3), our proof is trivial since h_α is a homotopy equivalence.

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