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TIGHT CLOSURE AND INTEGRAL CLOSURE OF AN IDEAL

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The theory of tight closure was created by Mel Hochster and Craig Hunuke [2,3,4 and 6]. They have continuously developed the theory, and 'tight closure' can now be regarded as a synonym for 'characteristic p methods in commutative algebra'.

In this paper we study about the tight closure and the integral closure of an ideal. In fact, there is a very strong connection between tight closure and integral closure. With the properties of the integral closure of an ideal we will prove the following theorem related to the tight closure: Let R be a Cohen-Macaulay local ring of dimension one. If there exists a system of parameters which generates a tightly closed ideal, then R is weakly F-regular.

And we will also prove following, which is the well-known theorem in the commutative ring theory, by using the above theorem: Let R be a Noetherian local ring of dimension one. Then R is regular if and only if R is normal.

1. Definitions and Basic Theorems

All rings are commutative, Noetherian with identity of prime characteristic p, unless otherwise specified.

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DEFINITION 1.1.[2]. Let $I \subseteq R$ be an ideal and R^o denote the complement of the union of the minimal primes of R and let $I^{[q]}$ denote
the ideal $(i^q : i \in I)$. We say that $x \in I^*$, the *tight closure* of I, if there exists $c \in R^o$ such that $cx^q \in I^{[q]}$ for all $q \gg 0$, i.e. for all sufficiently large q of the form p^e. If $I = I^*$, we say that I is *tightly closed* . And a Noetherian ring is called *weakly F-regular* if every ideal is tightly closed.

REMARK 1.2. Note that if R is a domain, then the condition that $c \in R^o$ means that c is not zero. Note also that if R is reduced, then $cx^q \in I^{[q]}$ if and only if $c^{1/q}x \in IR^{1/q}$. Thus, if $x \in I^*$, we have $c^{1/q}x \in IR^\infty$ for some $c \in R^o$, where $R^\infty = \bigcup_q R^{1/q}$.

Now we state the general properties about the tight closure in the following proposition.

PROPOSITION 1.3. *Let R be a Noetherian ring and let I, J be ideals of R. Then,*

(1) *I^* is an ideal of R containing I.*
(2) *If $I \subseteq J$, then $I^* \subseteq J^*$. The intersection of an arbitrary family of tightly closed ideals is tightly closed.*
(3) *If I has a positive height or if R is reduced, then $x \in I^*$ if and only if there exists $c \in R^o$ such that $cx^q \in I^{[q]}$ for all $q = p^e$.*
(4) *$I^* = I^{**}$.*
(5) *$(0)^* = Rad(0)$. In particular, I^* contains the nilradical of R for all $I \subseteq R$.*
(6) *If I is tightly closed, then $I : J$ is tightly closed for any ideal J of R.*

Proof. See [5, Proposition 4.1].

To study the relationship between tight closure and integral closure, we need the definition of the integral closure of an ideal.

DEFINITION 1.4. Let R be an arbitrary commutative ring with identity. We call that an element $x \in R$ is in $\overline{I}$, the *integral closure* of I if there exists a positive integer k and an equation

$$x^k + i_1 x^{k-1} + \cdots + i_j x^{k-j} + \cdots + i_{k-1} x + i_k = 0,$$

where $i_j \in I^j$ for $1 \leq j \leq k$.

REMARK 1.5. We note that $\overline{I}$ is an ideal. It is easy to see that $x \in \overline{I}$ if and only if there is an integer $k \geq 1$ such that $x^k \in I(I+Rx)^{k-1}$ and this holds if and only if $(I+Rx)^k = I(I+Rx)^{k-1}$. From this equation it is trivial to prove by induction on l that

($\sharp$) $(I + Rx)^{k+l} = I^{l+1}(I + Rx)^{k-1}$ for every integer $l \in \mathbf{N}$.

Thus $x \in \overline{I}$ if and only if there exists a positive integer $k > 0$ such that ($\sharp$) holds for all $l \in \mathbf{N}$.
Hence it is easy to see that if x is in $\overline{I}$, then there exists an integer k such that for all $n \geq 0$, (*) $x^{n+k} \in I^n$. The converse is also true for Noetherian rings.

Another useful characterization of the integral closure is that $x \in \overline{I}$ if and only if for all valuation v (resp. discrete valuations in the Noetherian case) nonnegative on R and infinite only on a minimal prime ideal of R, $v(x) \geq v(I) = \min \{v(i) : i \in I\}$.

In charateristic p this criterion yields that $I^* \subset \overline{I}$. For if $ca^q \in I^{[q]}$ for all $q = p^e \gg 0$, then we may apply such a discrete valuation v and let q be sufficiently large ; we obtain that $v(c)/q + v(a) \geq v(I)$ and so $v(a) \geq v(I)$ unless $v(c) = \infty$. Since $c \in R^o$, $v(c)$ is not ∞, however.

Of course, in general I^* is much smaller than $\overline{I}$. Nevertheless, the analogy with integral closure has proved very useful for tight closure theory.

Note that any prime ideal is integrally closed, and thus tightly closed. For, let P be any prime ideal and $x \in \overline{P}$. Then for all n, $x^{n+k} \in P^n$ for some integer k. Thus $x^{n+k} \in P$. Since P is prime, we have $x \in P$.

DEFINITION 1.6 [7]. If I and J are ideals in a local ring R, then J is a *reduction* of I if $J \subseteq I$ and $JI^n = I^{n+1}$ for some positive integer n.

It is easy to see that J is a reduction of I if and only if $J \subseteq I$ and I is integral over J, that is, $\overline{I} = \overline{J}$.

Note that any ideal is integral over a local ring R if its reduction is integral over R.

PROPOSITION 1.7. *If R is a local ring with an infinite residue field, then every ideal is integral over an ideal generated by* dim$R$ *elements.*

Proof. See [7, Theorem 1].

The elegant application of tight closure is a proof of the Briançon-Skoda theorem in characteristic p.

PROPOSITION 1.8. *(Generalized Briançon-Skoda Theorem) Let R be a Noetherian ring of characteristic p, and let I be an ideal of positive height generated by n elements, say $u_1, \cdots, u_n$. Then for every $l \in \mathbf{N}$, $\overline{(I^{n+l})} \subseteq (I^{l+1})^*$ In paticular, $\overline{(I^n)} \subseteq I^*$ Hence, if R is weakly F-regular and, in particular, if R is regular, then $\overline{(I^{n+l})} \subseteq I^{l+1}$ and $\overline{(I^n)} \subseteq I$.*

Proof. See [5, Theorem 5.4].

COROLLARY 1.9. *Let R be a Noetherian ring and let x be an element of R^o. Then $(x)^* = \overline{(x)}$.*

2. Main Theorem

THEOREM 2.1. *Let R be a Cohen-Macaulay local ring of dimension one. If there exists a system of parameters(briefly s.o.p.) which generates a tightly closed ideal, then R is weakly F-regular.*

Proof. [Cf. 1, Theorem 2.8] Let I be any $\underline{m}$-primary ideal of R, where $\underline{m}$ is the maximal ideal of R. It suffices to show that $I = I^*$.[5,Proposition 4.16].

By the flat change of local rings $R \longrightarrow R[t]_{\underline{m}R[t]}$, where t is an indeterminate, we may assume $R/\underline{m}$ is infinite. Then by Proposition 1.7 , there exists a reduction J of I with $\dim R = \mu(J)$, the number of generators of J. Thus J is an s.o.p. ideal and $J = J^*$ by the F-rationality of a Cohen-Macaulay local ring R. Since J is principal, we have $\overline{J} = J^*$ by Corollary 1.9. Thus we have $J = \overline{J}$ and $\overline{I} = \overline{J}$. And these imply that I is integrally closed, and hence I is tightly closed.

The proof of this theorem shows the following.

COROLLARY 2.2. *Let R be the same as in Theorem 2.1. If there exists an s.o.p. which generates a tightly closed ideal, then every $\underline{m}$-primary ideal is a complete intersection and integrally closed.* (We say that an ideal I is a *complete intersection* if $\mathrm{ht}I = \mu(I)$, the number of generators of I). *Thus R is a regular local ring.*

COROLLARY 2.3. *Let R be a Cohen-Macaulay local ring of dimension one, and let $x \notin Z(R)$. If R/xR is weakly F-regular, then so is R.*

Proof. Let I be any $\underline{m}$-primary ideal of R, where $\underline{m}$ is the maximal ideal of R. Now we have the result $I = I^*$ by the proof of Theorem 2.1. Thus R is weakly F-regular.

At the end of this section, we will prove that the regularity is equivalent to the normality in a one dimensional Noetherian local ring by using our previous results. This theorem is well known in the commutative ring theory. To see this, the following lemma is needed.

LEMMA 2.4. *Let R be a Noetherian ring and let $\overline{R}$ be the integral closure of R in its field of fractions. If $x \notin Z(R)$, then $x\overline{R} \cap R = \overline{xR}$, the integral closure of xR. Thus, in a normal ring any principal ideal of height one is integrally closed.*

Proof. Let $y \in x\overline{R} \cap R$. Then $y/x \in \overline{R}$ and hence we have the equation $(y/x)^n + r_1(y/x)^{n-1} + \cdots + r_n = 0$, where $r_i \in R$. Multiplying this equation by x^n, we get $y^n + r_1xy^{n-1} + \cdots + r_nx^n = 0$. Since $r_ix^i \in (x^i)$, we have $y \in \overline{xR}$. Therefore $x\overline{R} \cap R \subset \overline{xR}$.

Conversely, let $y \in \overline{xR}$. Then we get the following equation

$$y^n + a_1 y^{n-1} + \cdots + a_n = 0, \text{ where } a_i \in x^i R.$$

Dividing this equation by x^n shows that $y/x \in \overline{R}$, for

$$(y/x)^n + (a_1/x)(y/x)^{n-1} + \cdots + (a_n/x^n) = 0$$

and $a_i/x^i \in R$. Thus $y \in x\overline{R}$ and $y \in R$.

THEOREM 2.5. *Let R be a Noetherian local ring of dimension one. Then R is regular if and only if R is normal.*

Proof. In a regular ring R, any principal ideal is tightly closed, hence integrally closed. Thus R is normal.
Now assume R is normal. Then $\overline{R} = R$ and $x\overline{R} \cap R = \overline{xR} = xR$ for any $x \notin Z(R)$. However $\overline{xR} = (x)^*$ by Corollary 1.9. Hence xR is tightly closed. Since any normal ring is reduced, R is Cohen-Macaulay. By Theorem 2.1, R is weakly F-regular. And by Corollary 2.2, R is regular, which is our desired result.

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