Honam Mathematical J. 19(1997), No. 1,pp. 81-86

THE BOUNDARY BEHAVIOR BETWEEN THE KOBAYASHI-ROYDEN AND CARATHÉODORY METRICS ON STRONGLY PSEUDOCONVEX DOMAIN IN $\mathbb{C}^n$

KIM JONG JIN AND PARK SUNG HEE

Dept. of Mathematics,
Chonbuk National University, Chonju 561-756, Korea.

Abstract The aim of this paper is to prove the boundary behavior between the Carathéodory and Kobayashi-Royden metrics in a strongly pseudoconvex bounded domain with C^2-boundary in $\mathbb{C}^n$ and to show that the converse does not holds. S. Venturini([Ven]) proved the corresponding result with distances in place of the infinitesimal metrics.

1. Definitions and preliminaries

We recall at first the definition of the Kobayashi-Royden metric and the Carathéodory metric on a complex manifold. On all complex manifold we are under consideration we assume the connectedness, Hausdroff and the countably compactness.

By Δ and ds_Δ, we mean the open unit disc in $\mathbb{C}$ and the Poincaré metric on Δ, respectively. Also by $N(M)$ we denote the function space of all holomorphic mappings of M into N.

Let $T(M)$ be the tangent bundle for a complex manifold M. Then we define the *Kobayashi-Royden metric* $F_K^M : T(M) \to \mathbb{R}^+ \cup \{0\}$ on M by

$$F_K^M(z,\xi) := \inf_{v \in T_0(\Delta)} \{ ds_\Delta(0,v) : \exists f \in M(\Delta) \text{ s.t. } f(0) = z,\ df(0)v = \xi \}$$

Received May 1, 1997.
1991 Mathematics Subject Classification : 32F05, 32F15
Key words and phrases : Kobayashi-Royden metrics and Carathéodory metrics

for all $(z,\xi) \in T(M)$.

The *Carathéodory metric* $F_C^M : T(M) \to \mathbb{R}^+ \cup \{0\}$ on M is defined by

$$F_C^M(z,\xi) := \sup\{\ ds_\Delta(0, df(z)\xi) : f \in \Delta(M) \text{ with } f(z) = 0\}$$

for all $(z,\xi) \in T(M)$.

In F_K^M and F_C^M, we mean that the upperscript M depends on M. F without a subscript C or K refers to either metric unless specified otherwise.

The proof of the following proposition is an ease result from definitions and also see [Gra],[Roy], etc.

PROPOSITION 1. *Let M and N be complex manifolds. Then*

(1) *For a holomorphic map* $f : M \to N$, $F^M(z,\xi) \geq F^N(f(z), f_*(\xi))$.
(2) $F_C^M(z,\xi) \leq F_K^M(z,\xi)$.
(3) $F_C^\Delta(z,\xi) = F_K^\Delta(z,\xi) = ds_\Delta(z,\xi)$.

THEOREM 2. *([Lem]) Let $\Omega \subset \mathbb{C}^n$ be a domain biholomorphic to a bounded convex domain. Then,*

$$F_K^\Omega(z,\xi) = F_C^\Omega(z,\xi)$$

for all $(z,\xi) \in T(\Omega)$.

A bounded domain $\Omega \subset \mathbb{R}^N$ [resp. $\mathbb{C}^n$] is said to have C^k-*boundary* ($k \geq 1$) if there is a real-valued C^k function φ defined on a neighborhood U of the closure $\overline{\Omega}$ of Ω such that

(1) $\Omega = \{x \in U \mid \varphi(x) < 0\}$
(2) $\nabla\varphi := (\frac{\partial\varphi}{\partial z_1}, \cdots, \frac{\partial\varphi}{\partial z_n}) \neq 0$ on $\partial\Omega$ (the boundary of Ω)

We call the function φ a C^k *defining function* for Ω.

REMARK 1. It follows from the implicit function theorem that Ω has a C^k defining function if and only if $\partial\Omega$ is a C^k manifold.

Let Ω be a bounded domain with C^2-boundary and let $p \in \partial\Omega$.

Ⓡ In case of $\Omega \subset \mathbb{R}^N$, we say that $\partial\Omega$ is *strongly convex* at p if

$$\sum_{j,k=1}^{N} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(p) w_j w_k > 0$$

for all $w(\neq 0) \in \mathbb{R}^N$ satisfying $\sum_{j=1}^{N} \frac{\partial \varphi}{\partial x_j}(p) w_j = 0$.

We say that Ω is *strongly convex* if $\partial\Omega$ is strongly convex at each boundary point of Ω.

Ⓢ In case of $\Omega \subset \mathbb{C}^n$, we say that $\partial\Omega$ is *strongly pseudoconvex* at p if the Levi form

$$\mathcal{L}_{\varphi,p}(\xi) := \sum_{j,k=1}^{N} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}(p) \xi_j \overline{\xi}_k > 0$$

for all $\xi(\neq 0) \in \mathbb{C}^n$ satisfying $\sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j}(p) \xi_j = 0$.

We say that Ω is *strongly pseudoconvex* if $\partial\Omega$ is strongly pseudoconvex at each boundary point of Ω.

REMARK 2.

(1) Any strongly convex boundary point of Ω is extreme point of $\overline{\Omega}$.
(2) The Levi form does transform canonically under any biholomorphic mapping.

THEOREM 3. *([Gra]) Let $\Omega \subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2-boundary and let $p \in \partial\Omega$. Then for any neighborhood U of p and for all vector $\xi \in \mathbb{C}^n$,*

$$\lim_{\Omega \cap U \ni z \to p} \frac{F_K^{\Omega \cap U}(z,\xi)}{F_K^{\Omega}(z,\xi)} = 1.$$

2. Main theorem

THEOREM. *Let $\Omega \subset \mathbb{C}^n$ be a strongly pseudoconvex bounded domain with C^2-boundary. Then given $\epsilon > 0$, there is a compact subset $K(\epsilon) \subset \Omega$ depending on ϵ such that*

$$F_K^{\Omega}(z,\xi) \leq (1+\epsilon)\, F_C^{\Omega}(z,\xi)$$

for all $z \in \Omega \setminus K(\epsilon)$ *and* $\xi \in \mathbb{C}^n$.

Proof. By Fornaess imbedding theorem ([Kr 2]), there exists $n' (> n) \in \mathbb{N}$, a strongly convex domain $\Omega' \subset \mathbb{C}^{n'}$, a neighborhood $\hat{\Omega}$ of $\overline{\Omega}$ and a proper holomorphic embedding $\Psi : \hat{\Omega} \to \mathbb{C}^{n'}$ such that

(1) $\Psi(\Omega) \subset \Omega'$
(2) $\Psi(\partial\Omega) \subset \partial\Omega'$
(3) $\Psi(\hat{\Omega} \setminus \overline{\Omega}) \subset \mathbb{C}^{n'} \setminus \overline{\Omega'}$
(4) $\Psi(\hat{\Omega})$ is transversal to $\partial\Omega'$.

For any given $\epsilon > 0$, we claim that there exist a compact subset $K(\epsilon) \subset \Omega$ such that

$$F_K^{\Omega}(z,\xi) \leq (1+\epsilon)\, F_K^{\Omega'}(\Psi(z), \Psi_*(\xi))$$

for all $z \in \Omega \setminus K(\epsilon)$ and $\xi \in \mathbb{C}^n$.
Then we get the required result. In fact, since Ω' is a convex domain,

$$F_K^{\Omega'}(\Psi(z), \Psi_*(\xi)) = F_C^{\Omega'}(\Psi(z), \Psi_*(\xi))$$

by Theorem 2. And also by the monotonicity of the Carathéodory metric (Proposition 1),

$$F_C^{\Omega'}(\Psi(z), \Psi_*(\xi)) \leq F_C^{\Omega}(z,\xi).$$

To prove our claim, suppose that the claim is not hold. Then there are $\epsilon > 0$ and sequences $\{z_\nu\} \subset \Omega$, $\{\xi_\nu\} \subset \mathbb{C}^n$ such that $z_\nu \to z \in \partial\Omega$ and

$$F_K^{\Omega}(z_\nu, \xi_\nu) \geq (1+\epsilon)\, F_K^{\Omega'}(\Psi(z_\nu), \Psi_*(\xi_\nu)). \qquad (\dagger)$$

By the transversality assumption and since the domains are strongly pseudoconvex, there are open neighborhoods U of z and V of $\Psi(z) \in \partial\Omega'$ for which $\Omega \cap U$ and $\Omega' \cap V$ are connected, $\Psi(\Omega \cap U) \subset \Omega' \cap V$ and a holomorphic retraction $\Phi : \Omega' \cap V \to \Omega \cap U$ for Ψ. Then

$$\begin{aligned} F_K^{\Omega\cap U}(z,\xi) &\geq F_K^{\Omega'\cap V}(\Psi(z), \Psi_*(\xi)) \\ &\geq F_K^{\Omega\cap U}(\Phi(\Psi(z)), \Phi_*(\Psi_*(\xi))) \\ &= F_K^{\Omega\cap U}(z,\xi). \end{aligned}$$

Hence $F_K^{\Omega\cap U}(z,\xi) = F_K^{\Omega'\cap V}(\Psi(z),\Psi_*(\xi))$ and so we have the following equality by Theorem 3

$$\lim_{\nu\to\infty} \frac{F_K^{\Omega}(z_\nu,\xi_\nu)}{F_K^{\Omega'}(\Psi(z_\nu),\Psi_*(\xi_\nu))} = \lim_{\nu\to\infty} \frac{F_K^{\Omega}(z_\nu,\xi_\nu)}{F_K^{\Omega\cap U}(z_\nu,\xi_\nu)} \frac{F_K^{\Omega'\cap V}(\Psi(z_\nu),\Psi_*(\xi_\nu))}{F_K^{\Omega'}(\Psi(z_\nu),\Psi_*(\xi_\nu))} = 1,$$

which is a contradiction to (†). Thus we complete the proof.

The next example shows that the converse of Theorem does not holds.

EXAMPLE. For $p > 0$, put $\Omega(p) = \left\{(z_1,z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^{\frac{2}{p}} < 1\right\}$. If $0 < p \leq 1$, then $\Omega(p)$ is a convex bounded domain with C^2-boundary. Hence by Theorem 2,

$$F_K^{\Omega(p)}(z,\xi) = F_C^{\Omega(p)}(z,\xi)$$

for all $(z,\xi) \in \Omega(p) \times \mathbb{C}^2$. But since the Levi form of the defining function $\varphi(z_1,z_2) = |z_1|^2 + |z_2|^{\frac{2}{p}} - 1$ for $\Omega(p)$ degenerates along the curve defined by the equations $|z_1| = 1$ and $z_2 = 0$, $\Omega(p)$ is not strongly pseudoconvex.

References

[Aba] M. Abate, *A Characterization of Hyperbolic Manifolds*, Proc. AMS **117** (1993), 789 - 793.

[Azu] K. Azukawa, *Hyperbolicity of circular domains*, Tohoku Math. J. **35** (1983), 403 - 413.

[Gra] I. Graham, *Boundary behavior of the Carathéodory and Kobayashi metrics on Strongly Pseudoconvex Domains in $\mathbb{C}^n$ with Smooth boundary*, Trans. AMS **207** (1975), 219 - 240.

[Ko1] S. Kobayashi, *Hyperbolic manifolds and Holomorphic mappings*, Dekker New York, 1970.

[Kr1] S.K. Krantz, *The boundary behavior of The Kobayashi metric*, Rocky Mountain J. Math. **22** (1992), 227 - 233.

[Kr2] ______, *Function theory of several complex variables, 2nd ed.*, Wadsworth, Belmount, 1992.

[Lem] L. Lempert, *Holomorphic retracts and intrinsic metrics in convex domains*, Analysis Mathematica **8** (1982), 257 - 261.

[Roy] H.L. Royden, *Remarks on the Kobayashi metric,* Several Complex Variables II, *Lecture Note in Math. vol 185,* Springer-verlag New York Berlin 1971.

[Ven] S. Venturini, *Comparison between the Kobayashi and Carathéodory distances on strongly pseudoconvex bounded domains in* $\mathbb{C}^n$, Proc. AMS **107** (1989), 725 - 730.