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SOME PROPERTIES OF CERTAIN CLASSES OF FUNCTIONS WITH BOUNDED RADIUS ROTATIONS

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Abstract Let $R_k(\alpha), 0 \le \alpha < 1, k \ge 2$ denote certain subclasses of analytic functions in the unit disc E with bounded radius rotation. A function f, analytic in E and given by $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$, is said to be in the family $R_k(n, \alpha) n \in N_o = \{0, 1, 2, ...\}$ and $\star$ denotes the Hadamard product. The classes $R_k(n, \alpha)$ are investigated and some properties are given. It is shown that $R_k(n+1, \alpha) \subset R_k(n, \alpha)$ for each n. Some integral operators defined on $R_k(n, \alpha)$ are also studied.

1. Introduction

A number of important classes of univalent functions (e.g. convex, starlike) are related through their derivatives to functions with positive real part. Convex and starlike functions of order α are defined by requiring the related functions to have real part greater than α. We replace functions with real part greater than α by certain weighted difference of such functions and obtain some new classes of functions.

For $0 \le \alpha < 1$, let $P(\alpha)$ be the class of functions p, analytic in the unit disc $E = \{z : |z| < 1\}$ with $p(0) = 1$, such that Re $p(z) > \alpha$ for $z \in E$. Also $C(\alpha)$ and $S^*(\alpha), 0 \le \alpha < 1$, denote the classes of convex and starlike functions of order α respectively. A

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function f, analytic in E and given by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1.1}$$

is starlike of order α if $\frac{zf'}{f} \in P(\alpha)$ and is convex of order α if $\frac{(zf')'}{f'} \in P(\alpha)$ for $z \in E$.

Let $P_k(\alpha), k \geq 2, 0 \leq \alpha < 1$, be the class of functions h, analytic in E, such that

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \tag{1.2}$$

where $p_1, p_2 \in P(\alpha)$.

The class $P_k(0) \equiv P_k$ was introduced in [7] by Pinchuk. We note that $h \in P_k(\alpha)$ if and only if there exists $p \in P_k$ such that

$$h(z) = (1 - \alpha)p(z) + \alpha. \tag{1.3}$$

DEFINITION 1.1. A function f, analytic in E and given by (1.1), is said to belong to the class $R_k(\alpha)$; $k \geq 2, 0 \leq \alpha < 1$, if and only if $\frac{zf'(z)}{f(z)} \in P_k(\alpha)$.

Clearly $R_2(\alpha) \equiv S^*(\alpha)$ and $R_k(0) \equiv U_k$, the class of functions with bounded radius rotation, see [2].

Similarly an analytic function f, given by (1.1), belongs to $V_k(\alpha)$ for $z \in E$ if and only if $\frac{(zf'(z))'}{f'(z)} \in P_k(\alpha)$. It is obvious that

$$f \in V_k(\alpha) \quad \text{if and only if} \quad zf' \in R_k(\alpha). \tag{1.4}$$

It may be noted that $V_2(\alpha) \equiv C(\alpha)$ and $V_k(0) \equiv V_k$, the class of functions with bounded boundary rotation first discussed by Paatero, see [2].

Let f and g be analytic in E with $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$. Then the convolution (or Hadamard product) of f and g is defined by

$$(f \star g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m. \tag{1.5}$$

For $n \in N_o = \{0, 1, 2, 3, ...\}$, let

$$D^n f = \frac{z}{(1-z)^{n+1}} \star f \quad \text{so that} \quad D^n f = \frac{z(z^{n-1}f)^{(n)}}{n!}.$$

We now define the following.

DEFINITION 1.2. For $n \in N_o, k \geq 2, 0 \leq \alpha < 1$, a function f analytic in E is said to belong to the class $R_k(n, \alpha)$ if and only if $\frac{z(D^n f(z))'}{D^n f(z)} \in P_k(\alpha)$ for $z \in E$.

We note that $R_2(0, \alpha) \equiv S^*(\alpha), R_2(1, \alpha) \equiv C(\alpha)$. Also $R_k(0, \alpha) \equiv R_K(\alpha)$ and $R_k(1, \alpha) \equiv V_k(\alpha)$. For $k = 2$ and $\alpha = 0$, we have the classes $R(n)$ which have been studied in [9] and it is known that the functions in $R(n)$ are starlike. It can easily be seen that

$$f \in R_k(n, \alpha) \quad \text{if and only if} \quad D^n f \in R_k(\alpha). \tag{1.6}$$

In order to develop some results for $R_k(n, \alpha)$, we shall need the following:

LEMMA 1.1. *[8]. Let ϕ be convex and g be starlike in E. Then, for F analytic in E with $F(0) = 1$, $\frac{\phi \star Fg}{\phi \star g}$ is contained in the convex hull of $F(E)$.*

LEMMA 1.2. *[8]. Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and $\psi(u, v)$ be a complex-valued function satisfying the conditions:*
(i) *$\psi(u, v)$ is continuous in a domain $D \subset C^2$,*
(ii) *$(1, 0) \in D$ and $\psi(1, 0) > 0$.*
(iii) *Re $\psi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.*
If $h(z) = 1 + \sum_{m=2}^{\infty} c_m z^m$ is a function, analytic in E such that $(h(z), zh'(z))$ $\in D$ and Re $\{\psi(h(z)), zh'(z))\} > 0$ for $z \in E$, then Re $h(z) > 0$ in E.

2. Main Results

We first prove that all functions in $R_k(n, \alpha)$ are of bounded radius rotation.

THEOREM 2.1. *$R_k(n+1,\alpha) \subset R_k(n,\alpha)$ for each $n \in N_o$.*

Proof. Let $f \in R_k(n+1,\alpha)$. Then, for $z \in E$, $\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} \in P_k(\alpha)$.

Set

$$\frac{z\,(D^n f(z))'}{D^n f(z)} = H(z). \tag{2.1}$$

$H(z)$ is analytic and $H(0) = 1$.

From (2.1) and the indentity

$$z\,(D^n f(z))' = (n+1)D^{n+1}f(z) - nD^n f(z), \tag{2.2}$$

we obtain

$$\frac{z\left(D^{n+1}f(z)\right)'}{D^{n+1}f(z)} = \left\{H(z) + \frac{zH'(z)}{H(z)+n}\right\} \in P_k(\alpha) \tag{2.3}$$

We want to show that $H \in P_k(\alpha)$.

Let

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z), \;\; h_i(0) = 1, i = 1,2.$$

Now, with $h_i = \frac{zs_i'}{s_i}, i = 1,2$, we have

$$H(z) = \frac{z\,(D^n f(z))'}{D^n f(z)} = \Big/ \left(\frac{k}{4}+\frac{1}{2}\right)\frac{zs_1'(z)}{s_1(z)} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{zs_2'(z)}{s_2(z)}.$$

This gives us

$$\begin{aligned} H + \frac{zH'}{H+n} &= \frac{z(D^n f)' \star \phi_n}{D^n f \star \phi_n} = \left(\frac{k}{4}+\frac{1}{2}\right)\frac{zs_1' \star \phi_n}{s_1 \star \phi_n} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{zs_2' \star \phi_n}{s_2 \star \phi_n} \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left(h_1 + \frac{zh_1'}{h_1}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(h_2 + \frac{zh_2'}{h_2}\right), \end{aligned}$$

where $\phi_n(z) = \frac{n}{n+1}\left(\frac{z}{1-z}\right) + \frac{1}{n+1}\frac{z}{(1-z)^2}$.

From (2.3) it follows that $\left\{h_i + \frac{zh_i'}{h_i+n}\right\} \in P(\alpha)$ for $z \in E, i = 1,2$, and using Lemma 1.2 it can easily be verified that $h_i \in P(\alpha)$ for $i = 1,2$. Hence $H \in P_k(\alpha)$ and consequently $f \in R_k(n,\alpha)$.

For $\alpha = 0, k = 2$, this result is proved in [9]. Also, for $n = 0$, it follows from Theorem 2.1 that

$$V_k(\alpha) \subset R_k(\alpha).$$

THEOREM 2.2. *Let $f \in R_k(n,\alpha)$ and be given by (1.1). Then, for $m > 3$, $k \geq 2$,*

$$a_m = 0(1).n^{\{(1-\alpha)(\frac{k}{2}+1)-(n+1)\}},$$

where 0(1) is a constant depending only on k, α and n. The exponent $\left\{(1-\alpha)\left(\frac{k}{2}+1\right)-(n+1)\right\}$ is best possible.

Proof.

$$\begin{aligned} D^n f(z) &= \frac{z}{(1-z)^{n+1}} \star f(z) \\ &= \left[z + \sum_{m=2}^{\infty} \frac{(m+n-1)!}{n!(m-1)!} z^m\right] \star \left[z + \sum_{m=2}^{\infty} a_m z^m\right] \\ &= z + \sum_{m=2}^{\infty} \frac{(m+n-1)!}{n!(m-1)!} a_m z^m. \end{aligned}$$

Now, since $D^n f \in R_k(\alpha)$, we use relation (1.4) together with a coefficient result for the class $V_k(\alpha)$ proved in [5] to have, for $m > 3, k \geq 2$

$$\frac{(m+n-1)!}{n!(m-1)!} |a_m| < \left\{k^2(1-\alpha)^2 + k(1-\alpha)\right\} 2^{-2\alpha} \left(\frac{2}{3} m\right)^{(1-\alpha)(\frac{k}{2})-1},$$

and this gives us the required result.

The function $F_o \in R_k(n,\alpha)$ defined by

$$D^n F_o(z) = \frac{z(1+\delta_1 z)^{(\frac{k}{2}-1)(1-\alpha)}}{(1-\delta_2 z)^{(\frac{k}{2}+1)(1-\alpha)}}, \quad |\delta_1| = |\delta_2| = 1$$

shows that the exponent $\{(1-\alpha)(\frac{k}{2}+1) - 1 - n\}$ is best possible.

THEOREM 2.3. *$\bigcap_{n=0}^{\infty} R_k(n,\alpha) = \{id\}$, were id is the identity function.*

Proof. Let $f(z) = z$. Then it follows trivially that $z \in R_k(n,\alpha)$ for $n \in N_o$.

On the contrary, assume that $f \in \bigcap_{n=0}^{\infty} R_k(n,\alpha)$ with $f(z)$ given by (1.1). Then, from Theorem 2.2, we deduce that $f(z) = z$.

For a function f analytic in E, we define the integral operator I_β by

$$I_\beta(f) = \frac{(\beta+1)}{z^\beta}\int_0^z t^{\beta-1} f(t)dt, \qquad (\beta > -1). \tag{2.4}$$

The operator I_β, when $\beta \in N = \{1,2,3,...\}$ was introduced by Bernardi [1]. In particular, the operator I_1 was studied earlier by Libera [3] and Livingston [4].

We now prove the following.

THEOREM 2.4. *Let $f \in R_k(n,0)$ and let $I_\beta(f)$ be defined by (2.4). Then, for $z \in E$, $I_\beta(f) \in R_k(n,\alpha)$, where $0 < \alpha < 1$ and*

$$\alpha = \frac{1}{4}\left\{-(2\beta+1) + \sqrt{4\beta^2 + 4\beta + 9}\right\}. \tag{2.5}$$

Proof. Let

$$\frac{z(D^n I_\beta(f))'}{D^n(I_\beta(f))} = H,$$

where H is analytic and $H(0) = 1$.

Simple computations show that, for $z \in E$,

$$\frac{z\,(D^n(z))'}{D^n f(z)} = \left\{H(z) + \frac{zH'(z)}{H(z)+\beta}\right\} \in P_k.$$

Now following the same procedure of Theorem 2.1, we have

$$\begin{aligned} H(z) + \frac{zH'(z)}{H(z)+\beta} &= \left(\frac{k}{4}+\frac{1}{2}\right)\left(h_1(z) + \frac{zh_1'(z)}{h_1(z)+\beta}\right) \\ &\quad - \left(\frac{k}{4}-\frac{1}{2}\right)\left(h_2(z) + \frac{zh_z'(z)}{h_2(z)+\beta}\right), \end{aligned}$$

where h_i are analytic in E, with $h_i(0) = 1$ and $\left\{h_i(z) + \frac{zh_i'(z)}{h_i(z)+\beta}\right\} \in P, i = 1, 2$ for $z \in E$. We want to show that $h_i \in P(\alpha)$ where α is defined by (2.5). Let $h_i(z) = (1-\alpha)p_i + \alpha$. Then

$$\left\{(1-\alpha)p_i + \alpha + \frac{(1-\alpha)zp_i'(z)}{(1-\alpha)p_i(z) + \alpha + \beta}\right\} \in P \quad \text{for} \quad z \in E.$$

We form the functional $\psi(u, v)$ by choosing $u = p(z), v = zp'(z)$. Thus

$$\psi(u, v) = (1-\alpha)u + \alpha + \frac{(1-\alpha)v}{(1-\alpha)u + (\alpha+\beta)}.$$

The first two conditions of Lemma 1.2 are clearly satisfied. We verify the condition (iii) as follows.

$$\text{Re } \psi(iu_2, v_1) = \alpha + \frac{(1-\alpha)(\alpha+\beta)v_1}{(\alpha+\beta)^2 + (1-\alpha)^2u_2^2}.$$

By putting $v_1 \leq -\frac{1}{2}(1+u_2^2)$, we obtain

$$\begin{aligned}\text{Re } \psi(iu_2, v_1) &\leq \alpha - \frac{1}{2}\frac{(1-\alpha)(\alpha+\beta)(1+u_2^2)}{(\alpha+\beta)^2 + (1-\alpha)^2u_2^2} \\ &= \frac{2\alpha(\alpha+\beta)^2 + 2\alpha(1-\alpha)^2u_2^2 - (1-\alpha)(\alpha+\beta) - (1-\alpha)(\alpha+\beta)u_2^2}{2[(\alpha+\beta)^2 + (1-\alpha)^2u_2^2]} \\ &= \frac{A + Bu_2^2}{2C},\end{aligned}$$

where

$$A = 2\alpha(\alpha+\beta)^2 - (1-\alpha)(\alpha+\beta) = (\alpha+\beta)[2\alpha(\alpha+\beta) - (1-\alpha)],$$
$$B = 2\alpha(1-\alpha)^2 - (1-\alpha)(\alpha+\beta) = (1-\alpha)[2\alpha(1-\alpha) - (\alpha+\beta)],$$
$$C = (\alpha+\beta)^2 + (1-\alpha)^2u_2^2 > 0.$$

We notice that Re $\psi(iu_2, v_1) \leq 0$ if and only if $A \leq 0$ and $B \leq 0$. From $A \leq 0$, we obtain α as defined by (2.5) and $B \leq 0$ gives us $0 < \alpha < 1$. This proves that $h_i \in P(\alpha), i = 1, 2$ and hence $H \in P_k(\alpha)$.

As a special case, we note that, for $f \in R_k(n, 0), I_1(f) \in R_k\left(n, \frac{-3+\sqrt{17}}{4}\right)$.

If we put $\beta = n$ in (2.4), we have the following.

THEOREM 2.5. *Let $f \in R_k(n,\alpha)$. Then $I_n(f) \in R_k(n+1,\alpha)$ where $I_n(f)$ is defined by (2.4) with $\beta = n$.*

The proof is straightforward when we note from (2.2) and (2.4) that

$$D^n f(z) = D^{n+1} I_n(f).$$

Next we consider the converse of the problem involving the operator (2.4).

THEOREM 2.6. *Let $I_\beta(f)$, defined by (2.4), belong to $R_k(n,\alpha)$. Then $f \in R_k(n,\alpha)$ for $|z| < r_\beta$, where*

$$r_\beta = (1+\beta)/\left[2+\sqrt{\beta^2+3}\right]. \tag{2.6}$$

This result is best possible.

Proof. Proceeding as in Theorem 2.2, we have

$$\frac{z(D^n f(z))'}{D^n f(z)} = H(z) + \frac{zH'(z)}{H(z)+\beta},$$

where

$$H(z) = \frac{z(D^n I_\beta(f))'}{D^n I_\beta(f)} \in P_k(\alpha) \quad \text{for} \quad z \in E.$$

Define

$$\phi_\beta(z) = \sum_{j=1}^{\infty} \frac{\beta+j}{\beta+1} z^j = \frac{\beta}{\beta+1}\frac{z}{(1-z)} + \frac{1}{1+\beta}\frac{z}{(1-z)^2}.$$

Then it can easily be verified that ϕ_β is convex for $|z| < r_\beta$ where the exact value of r_β is given by (2.6).

Since $H \in P_k(\alpha)$, we can write

$$\begin{aligned} H(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z), \quad p_1,p_2 \in P(\alpha) \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{zs_1'(z)}{s_1(z)} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{zs_2'(z)}{s_2(z)}, \quad s_1,s_2 \in S^*(\alpha) \end{aligned}$$

Now

$$\frac{z(D^n f)'}{D^n f} = H + \frac{zH'}{H+\beta} = \left(\frac{k}{4}+\frac{1}{2}\right)\frac{zs_1'\star\phi_\beta}{s_1\star\phi_\beta} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{zs_2'\star\phi_\beta}{s_2\star\phi_\beta}$$
$$= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{\phi_\beta\star F_1 s_1}{\phi_\beta\star s_1} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{\phi_\beta\star F_2 s_2}{\phi_\beta\star s_2}.$$

Since $s_i \in S^*(\alpha) \subset S^*, F_i = \frac{zs_i'}{s_i} \in P(\alpha), i = 1,2$ and for the exact radius r_β, ϕ_β is convex in $|z| < r_\beta$, we conclude, by using Lemma 1.1, that $\frac{\phi_\beta \star F_i s_i}{\phi_\beta \star s_i} \in P(\alpha)$ for $|z| < r_\beta$. Hence $f \in R_k(n,\alpha)$ for $|z| < r_\beta$ where r_β is given by (2.6).

The case $\beta = 1$ gives us the Livingston's operator [4] for the class $R_k(n,\alpha)$.

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