Honam Mathematical J. **19**(1997), No. 1,pp. 107–116

GEOMETRIC PROPERTIES OF CURVES IN THE MINKOWSKI PLANE

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Abstract In this paper, we get some properties of curves of constant relative breadth in the Minkowski plane.

1. Introduction

Spherical balls certainly have the property that they can be arbitrarily rotated between two fixed parallel planes without losing contact with either plane. It has been known that there are other convex bodies with the property. They are called convex bodies of constant breadth. Convex bodies of constant breadth are of general interest. Many mathematicians including Gray [8] and Chakerian [6] have studied convex bodies of constant breadth. Good references in this line are Gruber and Wills [9] and Chakerian and Groemer [7].

In this paper, we study corresponding properties of curves of constant breadth in the Minkowski plane.

2. Preliminaries

For a centrally symmetric closed convex curve U *enclosing area* π with center at the origin O of the Euclidean plane E^2, a usual metric d on E^2 defines a Minkowski metric, m, using the formula

$$m(x,y) = \frac{d(x,y)}{r(x,y)}, \tag{1}$$

Received January 6, 1997.

1991 Mathematics Subject Classification : 53C65, 52A10.

Key words and phrases : Minkowski plane, relative breadth, isoperimetrix, indicatrix .

This paper was partially supported by BSRI-96-1419.

where $d(x, y)$ is the Euclidean distance from x to y, and $r(x, y)$ is the radius of U in the direction of a vector $x - y$ (See [2]). Trivially $m(x, y)$ satisfies the properties of a metric such that

$$\begin{aligned}
&(M_1) \quad m(x, y) \geq 0 \\
&(M_2) \quad m(x, y) = 0 \Leftrightarrow x = y \\
&(M_3) \quad m(x, y) = m(y, x) \\
&(M_4) \quad m(x, y) + m(y, z) \geq m(x, z).
\end{aligned}$$

Since $d(x, y)$ is a metric on the Euclidean plane the properties (M_1) and (M_2) are trivial. And since U is centrally symmetric we have the property (M_3). And if we define the function F by

$$F(x) = \frac{|x|}{|r(x)|}, \tag{2}$$

where $r(x)$ is the radius of U in the direction of a vector x and $|\cdot|$ is the Euclidean norm, then the inequality (M_4) above is equivalent to the inequality

$$F(x + y) \leq F(x) + F(y). \tag{3}$$

For any $x, y \in E^2$ we have $\frac{x}{F(x)}, \frac{y}{F(y)} \in U$. Since U is convex we have

$$\frac{F(x)}{F(x) + F(y)} \frac{x}{F(x)} + \frac{F(y)}{F(x) + F(y)} \frac{y}{F(y)} \in U.$$

Thus $\frac{x+y}{F(x)+F(y)} \in U$, i.e., $F\left(\frac{x+y}{F(x)+F(y)}\right) \leq 1$. Since the function F is homogeneous, we have the property (3). We shall assume throughout that U is smooth and has positive finite curvature everywhere. The set of points of E^2 together with metric m is the Minkowski plane denoted by M^2. Certainly U is the unit circle in M^2 and it shall be referred to as the indicatrix. Now following Chakerian [5], we parametrize U by twice its sectorial area, θ, and write the equation of U as

$$t = t(\theta), \quad 0 \leq \theta \leq 2\pi, \quad m(O, t) = 1 \tag{4}$$

and define $n(\theta)$ by

$$n(\theta) = \frac{dt(\theta)}{d\theta}, \quad 0 \leq \theta \leq 2\pi.$$

The trace of $n(\theta)$ is a convex curve I, called the isoperimetrix. For more informations see [3]. It is easy to verify that I is the polar reciprocal of U, with respect to the Euclidean unit circle, rotated through deg 90 [5]. *We shall always denote by T the area enclosed by I.*

Let C be an analytic closed convex curve in the Minkowski plane with the indicatrix U. For the point $x = (x_1, x_2) \in E^2$ the equation $F(x_1, x_2) = 1$ given by the function F in (2) determines the indicatrix U of the Minkowski plane M^2. And if $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, and $F(\cos\theta, \sin\theta) = \rho(\theta)$, then this equation means that

$$\begin{aligned} F(x_1, x_2) = 1 &\Leftrightarrow F(r\cos\theta, r\sin\theta) = 1 \\ &\Leftrightarrow rF(\cos\theta, \sin\theta) = 1 \\ &\Leftrightarrow r = \frac{1}{F(\cos\theta, \sin\theta)} \\ &\Leftrightarrow r = \frac{1}{\rho(\theta)}. \end{aligned}$$

Thus $r(\theta) = \rho^{-1}(\theta)$ is the equation of U in polar coordinate. If $h(\theta)$ is the supporting function of C, i.e., the distance of the origin from the tangent of C at the point p where the exterior normal of C has direction θ. Then the radius of curvature of C at p equals $(h(\theta) + h''(\theta))d\theta$. From the formula (1) the Minkowski length(see [3]) of C is given by the formula

$$L(C) = \int_0^{2\pi} \{h(\theta) + h''(\theta)\}\rho(\theta \pm \frac{\pi}{2})d\theta. \tag{5}$$

3. Constant relative breadth curves

In this section we consider some properties of curves of constant relative breadth in the Minkowski plane M^2.

DEFINITION 1. Let C be a closed convex plane curve in the Minkowski plane M^2. Then the Minkowski distance between two parallel lines of support to C which are perpendicular to the direction θ and which contain C between them is called the relative breadth of C in the direction θ and denoted by $Br(C, \theta)$.

From Definition 1 one can show that $Br(C, \theta) = \frac{h(\theta)+h(\theta+\pi)}{p(\theta)}$, where $h(\theta)$ and $p(\theta)$ are the support functions of C and U, respectively. Thus all the homotheties ωU of U are curves of constant relative breadth 2ω. Now we guarantee existence of nontrivial curve of constant relative breadth in the following example.

EXAMPLE 1. There exists a nontrivial curve of constant relative breadth.

Proof. Assume that U is the indicatrix in the Minkowski plane M^2. Let X_* be a point obtained by rotating a point X through deg 180. Let P be the point of the intersection of U and positive y-axis. Let $\overline{RQ}$ be a chord of U parallel to $\overline{OP}$ of length $1 = m(O, P)$ such that the ray $\overrightarrow{RQ}$ has the same direction as that of the ray $\overrightarrow{OP}$. And assume that $\overline{RQ}$ is in the right half plane. Let $\overset{\frown}{PO}$ be the arc obtained by translating a point Y on the arc $\overset{\frown}{R_*Q_*}$ of U to the point $Y + Q$ and let $\overset{\frown}{OQ}$ be the arc obtained by translating a point $\bar{Y}$ on the arc $\overset{\frown}{P_*R}$ of U to the point $\bar{Y}+Q$. Then we will show that the curve C determined by these two arcs and the arc $\overset{\frown}{PQ}$ is a curve such that $Br(C, \theta) = 1$ for all $\theta \in [0, 2\pi]$. As a matter of convenience we assume that the indicatrix U contains the origin O in its interior. Let S be a point on the interior of the arc $\overset{\frown}{OQ}$. Let m_1 be the supporting line to C at S. We want to show that the relative breadth of C determined by the line m_1 is always equal to 1. Let m_2 be the line parallel to m_1 through P. Then m_2 is a supporting line to C. Let l_1 and l_2 be the supporting lines of U parallel to m_1 at the points S_1 and S_2, respectively. Then the Minkowski distance between m_1 and m_2 is

$$m(m_1, m_2) = \frac{d(A, B)}{d(O, S_1)},$$

where $A = m_2 \cap \overline{S_1S_2}$ and $B = m_1 \cap \overline{S_1S_2}$. Because the arc $\overset{\frown}{OQ}$ is a translation of the arc $\overset{\frown}{P_*R}$ and the line m_1 is parallel to the line l_2, the segment $\overline{SS_2}$ is parallel to the segment $\overline{PO}$. Thus $d(P, S) = d(O, S_2) = d(O, S_1)$. And since m_1 is parallel to m_2 and $\overline{AB}$ is parallel to $\overline{PS}$, $d(A, B) = d(P, S)$. Thus $m(m_1, m_2) = 1$. But a point S is arbitrary. So the relative breadth of C determined by the supporting line of C at a point on the arc $\overset{\frown}{OQ}$ is always equal to 1. Similarly we have the same result for the arcs $\overset{\frown}{OP}$ and $\overset{\frown}{PQ}$.

Now we have to show that m_2 is a supporting line to C. Now assume that m_2 is not a supporting line of C. Then m_2 should separate the arc $\overset{\frown}{PQ}$ or the arc $\overset{\frown}{OP}$. If m_2 separates the arc $\overset{\frown}{PQ}$, there is a point Q' on the interior of the arc $\overset{\frown}{PQ}$ such that m_2 separates the points Q and Q'. Then m_2 separates the segment $\overline{S_1Q'}$ and Q. Then the segment $\overline{S_1Q'}$ can not be contained in the interior of U. This is a contradiction to the fact that U is convex. On the other hand, if m_2 separates the arc $\overset{\frown}{OP}$, then the line l_3 through R_* and parallel to l_1 separates the arc $\overset{\frown}{R_*Q_*}$. Thus there is a point Q'' on the interior of the arc $\overset{\frown}{R_*Q_*}$ such that l_3 separates the points Q_* and Q''. Then m_2 separates the segment $\overline{S_1Q''}$ and Q_*. Then the segment $\overline{S_1Q''}$ can not be contained in the interior of U. This is a contradiction to the fact that U is convex. This completes the proof.

Let C_1 and C_2 be two closed convex curves on the plane whose support functions with reference to O_1 and O_2 are, respectively, h_1 and h_2, assumed of class C^2. Consider the function $h(\theta) = h_1(\theta) + h_2(\theta)$. Since C_1 and C_2 are convex, $h + h'' > 0$. Therefore the function $h = h_1 + h_2$ is always the support function of a convex curve C_{12} called the mixed convex curve of C_1 and C_2. The area of the region enclosed by C_{12} has the form

$$A = \frac{1}{2}\int_0^{2\pi} (h^2 - {h'}^2) d\theta = A(C_1) + A(C_2) + 2A(C_1, C_2),$$

where $A(C_1)$, $A(C_2)$ are the areas of the regions enclosed by C_1

and C_2, respectively. The quantity

$$A(C_1, C_2) = \frac{1}{2}\int_0^{2\pi} (h_1 h_2 - h_1' h_2')d\theta$$

is the so-called mixed area of C_1 and C_2 (See [10]). *we shall always denote by c the mixed area of the indicatrix U and the isoperimetrix I.*

For all plane convex curves C of constant breadth ω in the Euclidean plane, Barbier's theorem [1] says that

$$L = \pi\omega, \tag{6}$$

where L is the perimeter of C.

Now we consider this problem in the Minkowski plane.

THEOREM 1. *Let C be a plane curve of constant relative breadth ω and $L(C)$ the Minkowski perimeter of C. Then*

$$L(C) = \omega c. \tag{7}$$

Proof. Let $h(\theta)$ be the support function of the curve C. By the definition (5) of the Minkowski length of C, we have

$$L(C) = \int_0^{2\pi} \{h(\theta) + h''(\theta)\}\rho(\theta \pm \frac{\pi}{2})d\theta.$$

Since the isoperimetrix I of M^2 is also centrally symmetric we have

$$\begin{aligned}
&\int_0^{2\pi} \{h(\theta) + h''(\theta)\}\rho(\theta \pm \frac{\pi}{2})d\theta \\
&= \int_0^{\pi} \{h(\theta) + h(\theta + \pi) + h''(\theta) + h''(\theta + \pi)\}\rho(\theta \pm \frac{\pi}{2})d\theta \\
&= \omega \int_0^{\pi} \{p(\theta) + p''(\theta)\}\rho(\theta \pm \frac{\pi}{2})d\theta,
\end{aligned} \tag{8}$$

where $p(\theta)$ is the support function of the indicatrix U. Because $\rho(\theta \pm \frac{\pi}{2})$ is equal to the support function of the isoperimetrix I , the quantity $\int_0^\pi \{p(\theta) + p''(\theta)\}\rho(\theta \pm \frac{\pi}{2})d\theta$ in (8) is the mixed area of U and I. This completes the proof.

Theorem 1 is a generalization of Barbier's theorem (6) to the Minkowski plane.

COROLLARY 1. *Let C be a Euclidean plane curve of constant breadth ω, then the perimeter $L(C)$ of C is equal to $\pi\omega$*

Proof. Because the value of c in the equation (7) is equal to π in the Euclidean plane we prove the corollary.

COROLLARY 2. *Let C be a plane curve of constant relative breadth ω. Then*

$$A(C, I) = \frac{\omega c}{2},$$

where $A(C, I)$ is the mixed area of C and I.

Proof. In the right side of the equation (5), $\rho(\theta \pm \frac{\pi}{2})$ is the support function of I. Thus the integral $\int_0^{2\pi} \{h(\theta) + h''(\theta)\}\rho(\theta \pm \frac{\pi}{2})d\theta$ is equal to two times the mixed area $A(C, I)$ of C and I. This completes the proof.

4. The dual Minkowski plane M^{2*}

Let G be a line parallel to the direction $t(\theta)$ in (4). Then the equation of G can be given by the formula:

$$|[t(\theta), X]| = P, \tag{9}$$

where $[X, Y] = x_1y_2 - x_2y_1$ for $X = (x_1, x_2), Y = (y_1, y_2)$ and $|\cdot|$ denotes the absolute value of the number. We shall denote the line G by $G = G(P, \theta)$. In fact, $[X, JY]$ is equivalent to $< X, Y >$ where J is rotation by $\frac{\pi}{2}$ and $<, >$ is the Euclidean dot product. The authors investigate geometry of the number P in [4].

THEOREM 2. *Let $G(P,\theta)$ be a line given by the equation (9). Then the point $Pn(\theta)$ lies on the line $G(P,\theta)$ and P is the Minkowski distance in the new Minkowski plane with the indicatrix I from the origin to the line $G(P,\theta)$.*

Proof. Let e_1 be the unit vector in the direction $t(\theta)$ and let e_2 be the unit vector obtained by rotating e_1 through $\frac{\pi}{2}$. Note that $\|[t(\theta), \frac{1}{r}e_2]\| =< re_1, \frac{1}{r}e_1 >= 1$. But component of $n(\theta)$ in the vector e_2 is $\frac{1}{r}$. Thus $\|[t(\theta), n(\theta)]\| = 1$. From the linearity $\|[t(\theta), Pn(\theta)]\| = P$. Thus the point $Pn(\theta)$ lies on $G(P,\theta)$.

Now we prove the second part of the theorem. Note the fact that I is the polar reciprocal of U, with respect to the Euclidean unit circle, rotated through deg 90. Thus the line parallel to the line $G(P,\theta)$ and passing through the point $n(\theta)$ is the support line of I at $n(\theta)$. So the shortest distance from the origin to the line $G(P,\theta)$ in the new Minkowski plane with the indicatrix I is $\frac{|Pn(\theta)|}{|n(\theta)|} = P$. This completes the proof.

In Theorem 2 we define the new Minkowski plane with the isoperimetrix I as its indicatrix. We call this new plane *the dual plane* of the original Minkowski plane and denote it by M^{2*}. Also we call the distance m^* on M^{2*} the *dual distance.* We shall denote the relative breadth in M^{2*} by $Br^*(C,\theta)$.

Now we consider some properties of curves of constant relative bread- th in the new Minkowski plane M^{2*}.

THEOREM 3. *Let C be a plane curve of constant relative breadth $Br^*(C,\theta) = \omega$ in M^{2*}. Then*

$$A(C) + A(C, -C) = \frac{\omega^2}{2}T,$$

where $-C$ is a curve obtained by rotating C through deg 180.

Proof. Let $h(\theta)$ be the support function of C. Then $A(C) = \frac{1}{2}\int_0^{2\pi}\{h^2(\theta) - h'^2(\theta)\}d\theta$. Since C is of constant relative breadth in M^{2*} $\frac{h(\theta)+h(\theta+\pi)}{p(\theta)} = \omega$, where $p(\theta)$ is the support function of the isoperimetrix I of M^2. Thus

$$2A(C) = \omega^2 \int_0^{\pi} \{p^2(\theta) - p'^2(\theta)\}d\theta - 2\int_0^{\pi} \{h(\theta)h(\theta+\pi) - h'(\theta)h'(\theta+\pi)\}d\theta$$
$$= \omega^2 T - 2A(C, -C).$$

So $A(C) + A(C, -C) = \frac{\omega^2}{2}T$.

COROLLARY 3. *If C is centrally symmetric in the above theorem, then*

$$A(C) = \frac{w^2}{4}T.$$

Proof. The proof follows from the fact that $A(C, -C) = A(C)$ and Theorem 3 immediately.

In Theorem 1 we show that the Minkowski length of constant relative breadth curve can be represented by the mixed area of the indicatrix and the isoperimetrix. Thus we guess that there is some relation between M^2 and M^{2^*}. Now we consider the relation between the Minkowski plane M^2 and the dual Minkowski plane M^{2^*} of M^2 in the following theorem.

THEOREM 4. *Let C be a plane curve of constant relative breadth $Br^*(C,\theta) = \omega$ in M^{2^*} and $L(C)$ the Minkowski perimeter of C. Then*

$$L(C) = \omega T.$$

Proof. In the proof of the Theorem 1 the quantity $\int_0^{\pi}\{p(\theta) + p''(\theta)\}\rho(\theta \pm \frac{\pi}{2})d\theta$ in (8) is equal to T because $p(\theta)$ is the support function of the isoperimetrix I. This completes the proof.

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