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A MINIMAL INTERPOLATION PROBLEM FOR SYMMETRIC RATIONAL MATRIX FUNCTIONS

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1. Introduction

In this paper, we solve the problem of finding an $m \times m$ rational matrix function F satisfying the following interpolation conditions

(1.1) $$x_i F(z_i) = y_i, \quad i = 1, \ldots, n$$

(1.2) $$F^T(z) = F(z)$$

(1.3)
F has the minimal possible $McMillan\, degree$,

where $\{z_1, \ldots, z_n\}$ be an n-tuple of distinct points in the complex plaine C , nonzero vectors $x_i \in C^{1\times m}$ and vectors $y_i \in C^{1\times m}$ for $i = 1, \ldots, n$. The *McMillan degree* of a rational matrix function $F(z)$, denoted by $\delta(F)$, measures the complexity of F (see [ABKW][BGR] for the definition). The problem of finding the solutions of (1.1) (1.3) in the scalar case is studied in [AA] and in the matrix case in [ABKW] with more general interpolation conditions. The problem together with the symmetric condition (1.2) and with the more general interpolation conditions in place of (1.1) is addressed elsewhere (see [K2]).

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Throughout this paper, σ denotes a subset of the complex plane C and V denotes a $2m \times 2m$ constant matrix given by

$$V = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

By a σ-admissibe Sylvester data set, it means a set of matrices

$$\tau = (C_\pi,\, A_\pi\, ;\, A_\zeta,\, B_\zeta\, ;\, \Gamma) \tag{1.4}$$

of sizes $m \times n_\pi$, $n_\pi \times n_\pi$, $n_\zeta \times n_\zeta$, $n_\zeta \times m$, $n_\zeta \times n_\pi$, respectively, where

$$\sigma(A_\pi) \cup \sigma(A_\zeta) \subset \sigma,$$

(C_π, A_π) is a null-kernel pair, i.e, $\displaystyle\bigcap_{j=0}^{n_\pi - 1} Ker\, C_\pi A_\pi^j = \{0\}$,

(A_ζ, B_ζ) is a full-range pair, i.e, $\displaystyle\sum_{j=0}^{n_\zeta - 1} Im\, A_\zeta^j B_\zeta = C^{n_\zeta}$,

Γ satisfies the following Sylvester matrix equation

$$\Gamma A_\pi - A_\zeta \Gamma = B_\zeta C_\pi. \tag{1.5}$$

For a given τ, we associate another set of matrices

$$\tau^T = (-V^{-1} B_\zeta^T,\, A_\zeta^T;\, A_\pi^T,\, C_\pi^T V;\, \Gamma^T).$$

It is easy to check that τ^T is a σ-admissible Sylvester data set if and only if τ is.

Let $\Theta(z)$ be an $M \times M$ rational matrix function. For a Sylvester data set τ given by (1.4) with M in place of m, $\Theta(z)$ is said to have τ as its C-null-pole triple if

$$\Theta \mathcal{P}_M = \{C_\pi (zI - A_\pi)^{-1} x + h(z) \,|\, x \in C^{n_\pi}, h \in \mathcal{P}_M \text{ such that } \sum_{z_0 \in C} Res_{z=z_0} (zI - A_\zeta)^{-1} B_\zeta h(z) = \Gamma x\},$$

where $\mathcal{P}_M$ denotes the set of polynomials with coefficients in C^M.

2. Main Results

Throughout this section, we set

$$A_\zeta = \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{bmatrix}, \quad B_\zeta = [B_+ \, B_-] \tag{2.1}$$

with

$$B_+ = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad B_- = - \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

where z_i, x_i, y_i are the same as in (1.1). If we set

$$A_\pi = A_\zeta, \qquad C_\pi := \begin{bmatrix} C_+ \\ C_- \end{bmatrix} = \begin{bmatrix} B_-^T \\ -B_+^T \end{bmatrix} \tag{2.2}$$

and

$$\Gamma = (\gamma_{ij}), \qquad \text{with } \gamma_{ij} = \begin{cases} \frac{x_i y_j^T - y_i x_j^T}{z_j - z_i}, & i \neq j \\ \text{arbitrarily given }, & i = j \end{cases} \tag{2.3}$$

so that the Sylvester equation (1.5) is satisfied. It is straight foward to see τ given as in (1.4) with all the matrices defined by (2.1)-(2.3) is a σ- admissible Sylvester data set which is similar to τ^T with $\sigma(A_\pi) \subset \sigma$. From the main results of [K2], we can see that there exists a $2m \times 2m$ rational matrix function $\Theta(z)$ for which

(2.4) Θ has τ as its C-null-pole triple,
(2.5) Θ is column reduced at infinity,
(2.6) $\Theta^T(z) V \Theta(z) = V, \qquad \forall z \in C.$

In this case the column indices of $\Theta(z)$ are given by

$$-\alpha_1, -\alpha_2, \cdots . -\alpha_t, \underbrace{0, \cdots, 0}_{(2m-2t) \text{ times}}, \alpha_t, \cdots, \alpha_1$$

where $\alpha_1 \geq \cdots \geq \alpha_t$ are the nonzero observability indices of $(\rho_\pi C_\pi|_{Ker\Gamma},$

$A_\pi|_{Ker\Gamma}$), where ρ_π is a projection onto $Ker\Gamma$.

The problem of finding a rational matrix function having the prescribed null-pole structure(as in (2.4) with $\sigma \in C$ in place of C, in general), known as the inverse spectral problem for rational matrix functions, is studied in literature. In particular, the problem of finding a rational matrix function Θ satisfying (2.4)(2.5) is studied in [ABKW] [BKGK] related to the problem of finding a minimal solution of given nonhomogeneous interpolation problem, which is a generalization of the interpolation condition (1.1). For the definition of the observability indices and more general interpolation conditons, see [BGR] or [ABKW].

Let $\mathcal{R}^{m\times m}(\sigma)$ represent the set of all $m \times m$ rational matrix functions which are analytic in σ. If $\sigma = C$, we use the notation $\mathcal{P}^{m\times m}$ for the set of polynomials having coefficients in $C^{m\times m}$. The following Theorem parametrizes all the solutions satisfying (1.1)(1.2).

THEOREM 2.1. *For σ an arbitrary countable set containing $\sigma(A_\pi)$, there exists $F \in \mathcal{R}^{m\times m}(\sigma)$ satisfying the conditons (1.1), (1.2). Let Θ be any $2m \times 2m$ matrix function represented as a 2×2 block matrix*

$$\Theta(z) = \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix} \tag{2.7}$$

which satisfies (2.4)-(2.6). Then the set of functions $F \in \mathcal{R}^{m\times m}(\sigma)$ satisfying (1.1)(1.2) coincides with the set of functions F having a representation in the form

$$F = (\Theta_{11}Q_1 + \Theta_{12}Q_2)(\Theta_{21}Q_1 + \Theta_{22}Q_2)^{-1} \tag{2.8}$$

where $Q_1, Q_2 \in \mathcal{R}^{m\times m}(\sigma)$ are such that

$$(\Theta_{21}Q_1 + \Theta_{22}Q_2)\mathcal{R}^m(\sigma) = [\Theta_{21}\,\Theta_{22}]\mathcal{R}^{2m}(\sigma) \tag{2.9}$$

(2.10)

$$[\,Q_1^T(z) \quad Q_2^T(z)\,]\,V \begin{bmatrix} Q_1(z) \\ Q_2(z) \end{bmatrix} = 0$$

at all points z of analyticity of Q_1 and Q_2.

Proof. First we note that the condition (1.1) is equivalent to the contour integral condition $\int_\gamma (zI - A_\zeta)^{-1} B_+ F(z)dz = -B_-$ where γ is a rectifiable closed contour having $\sigma(A_\zeta)$ inside. The condition (1.2) together with (1.1) forces F also to satisfy extra conditions that is

$$\int_\gamma F(z)C_-(zI - A_\pi)^{-1}dz = C_+$$

$$\int_\gamma (zI - A_\zeta)^{-1}B_+F(z)C_-(zI - A_\pi)^{-1}dz = \Gamma,$$

that is, F is a solution of so called *two-sided residue interpolatoin problem* for the given interpolation data $\omega = (C_-, C_+, A_\pi, A_\zeta, B_+, B_-, \Gamma)$. Observing that for our choice of V, it satisfies the relation $\omega = \omega^{T_V} := (-B_+^T, B_-^T, A_\zeta^T, A_\pi^T, C_-^T, C_+^T, \Gamma^T)$ with $V = -V^T$ and the condition (1.2) is equivalent to $[F^T I]V \begin{bmatrix} F \\ I \end{bmatrix} = 0$, this theorem is straight forward from Theorem 4.5 of [BK]. Here we also note that such parameters Q_1, Q_2 satisfying (2.9)(2.10) simultaneously exist by Lemma 4.6 of [BK] in the case σ is countable.

The next theorem gives a reparametrization of all the solutions of (1.1) (1.2) so that McMillan degree of the solutions can be read from the parametrization.

THEOREM 2.2. *Let $\Theta(z)$ as in Theorem 2.1 and $\sigma = \sigma(A_\pi)$. Then, $F \in \mathcal{R}^{m\times m}(\sigma)$ satisfies the conditons (1.1)(1.2) if and only if there exist Q_1, $Q_2 \in \mathcal{P}^{m\times m}$ for which*

(2.11)

$$F = (\Theta_{11}Q_1 + \Theta_{12}Q_2)(\Theta_{21}Q_1 + \Theta_{22}Q_2)^{-1}$$

(2.12) $(\Theta_{21}Q_1 + \Theta_{22}Q_2)$ *has no zeros in* σ

(2.13) Q_1, Q_2 *are right coprime*

(2.14) $\Theta \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ *is column reduced at infinity*

(2.15) $[Q_1^T Q_2^T]V \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = 0.$

If (2.11)-(2.15) are satisfied, then

$$\delta(F) = n + \sum_{j=1}^{m} l_j, \tag{2.16}$$

where l_j is the j^{th} column index of $\Theta \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$.

Proof. By Theorem 2.1, any $F \in \mathcal{R}^{n\times n}(\sigma)$ satisfying (1.1)(1.2) is given by (2.11) for $\widetilde{Q_1}, \widetilde{Q_2} \in \mathcal{R}^{n\times n}(\sigma)$ for which the conditions (2.12)(2.15) are satisfied with $\widetilde{Q_1}, \widetilde{Q_2}$ with in palces of Q_1, Q_2, respectively. On the other hand, the proof of Theorem 6.2 of [ABKW] shows how to derive $Q_1, Q_2 \in \mathcal{P}^{m\times m}$ from $\widetilde{Q_1}, \widetilde{Q_2}$ satisfying (2.12) so that Q_1, Q_2 fulfill the extra conditions (2.13)(2.14). There, we can observe that $Q_1, Q_2 \in \mathcal{P}^{m\times m}$ are obtained so that they satisfy the extra constraints (2.13)(2.14) by

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \widetilde{Q_1} \\ \widetilde{Q_2} \end{bmatrix} U(z) \tag{2.17,}$$

where $U(z)$ is a product of finite unimodular matrices and the respective inverses of finite diagonal matrix polynomials. Hence it is easy to see that Q_1, Q_2 refined from $\widetilde{Q_1}, \widetilde{Q_2}$ by the algorithm given in [ABKW], by (2.17) in short, also satisfy (2.15). It is a result of Theorem 6.2 of [ABKW] to compute the McMillan degree of F as in (2.16) in the case F is parametrized as (2.11) so that it satisfies (2.12)-(2.14). This completes the proof.

The next theorem and the following Corollary give a so called minimal solution which has the minimal possible McMillan degree among all possible solutions.

THEOREM 2.3. *Let $\Theta(z)$ and σ be as in Theorem2.2. Then there exist constant $m \times m$ matrices Q_{10}, Q_{20} satisfying (2.11)-(2.15) with Q_{10}, Q_{20} in places of Q_1, Q_2 and also the sum of the* *column indices of* $\Theta \begin{bmatrix} Q_{10} \\ Q_{20} \end{bmatrix}$ *is minimal possible among all the other* $\Theta \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ *with* $Q_1\, Q_2$ *satisfying (2.11)-(2.15).*

Proof. We will construct m columns $\tilde{\theta}_{i_1}(z), \ldots, \tilde{\theta}_{i_m}(z)$ of $\tilde{\Theta}_I(z) := \Theta(z)\Delta$ recursively with Δ, a $2m \times m$ constant matrix, so that $\Delta^T = [Q_{10}^T\, Q_{20}^T]$ satisfies the theorem. In this proof, by $\underline{k}$ for integer k, we mean the set $\{1, \ldots, k\}$, and $\Theta(z)$ is given as in Theorem 2.1. Let

$$[\Theta(z_k)]_l$$

denote the first l columns of $\Theta(z)$ evaluated at $z = z_k$ with z_k, $k = 1, \cdots, n$, given by (1.1), let

$$i_1 := \min\{l \mid rank([0\ I][\Theta(z_k)]_l) \geq 1,\ k \in \underline{n}\}$$

and let f_{oj} denote the j^{th} column of $\Theta(z)$.
(**step I**) First, we assert that there exists a set of complex numbers $\{\beta_j\}_{j=1}^{i_1-1}$ such that

$$\tilde{\theta}_{i_1}(z) := \left\{ f_{oi_1}(z) + \sum_{j=1}^{i_1-1} \beta_j f_{oj}(z_k) \right\}$$

satisfies

$$[0\ I]\tilde{\theta}_{i_1}(z_k) \neq 0 \quad \text{for } k \in \underline{n}.$$

For the details about how to construct such $\beta_j{}'s$, see [K1]. It is worthy to mention that $\tilde{\theta}_{i_1}(z)^T V \tilde{\theta}_{i_1}(z) = 0$ for $\forall z \in C$ for the $\tilde{\theta}_{i_1}(z)$ constructed exactly as in [K1] where the extra condition, $\tilde{\theta}_{i_1}(z)^T V \tilde{\theta}_{i_1}(z) = 0$, is not considered, since $y^T V y = 0$ for all $y \in C^{2m}$.
(**step II**) Assume a set of integers $\{i_1, \ldots, i_p\}$ and a set of $2m$-dimensional rational vector functions $\{\tilde{\theta}_{i_1}(z), \ldots, \tilde{\theta}_{i_p}(z)\}$ are chosen for $p < m$ so that

$$i_1 < i_2 < \cdots < i_p$$

$$\tilde{\theta}_{i_p}(z) := f_{p-1,i_p}(z) + \sum_{j=1}^{i_p-1} \gamma_j f_{p-1,j}(z) \tag{2.18}$$

for some constants $\gamma_1, \ldots, \gamma_{i_p-1}$

$$rank\ \left([0\ I][\tilde{\theta}_{i_1}(z_k)\ldots\tilde{\theta}_{i_p}(z_k)]\right) = p, \quad k \in \underline{n}, \tag{2.19}$$

$$[\tilde{\theta}_{i_1}(z)\ldots\tilde{\theta}_{i_p}(z)]^T V[\tilde{\theta}_{i_1}(z)\ldots\tilde{\theta}_{i_p}(z)] = 0, \quad \forall z \in C$$

where $f_{p-1,j}(z)$ is the j^{th} column of

$$\Theta_{p-1}(z) := \Theta(z)U_0\ldots U_{p-1}, \quad (U_0 = I),$$

where U_j is a $2m \times 2m$ constant upper triangular matrix with main diagonal entries equal to 1.
(**step III**) Define

$$\Theta_p(z) := \Theta_{p-1}(z)U_p,$$

where U_p is a $2m \times 2m$ constant upper triangular matrix satisfying (2.20)

$$\text{the } j^{\text{th}} \text{ column of } (I_{2m} - U_p) = \begin{cases} [-\gamma_1 \cdots - \gamma_{i_p-1} 0\ldots 0]^T & , j = i_p \\ 0, & j \neq i_p \end{cases}$$

with γ_j's as in (2.18). Let $f_{pj}(z)$ denote the j^{th} column of $\Theta_p(z)$ and

$$\mathcal{L}_{lk} = \{\sum_{j=1}^{l}[0\ I]c_{pj}f_{pj}(z_k) \mid c_{pj} \in C^{2m} \cap \mu_p^{\perp V}\}$$

$$\tilde{\mathcal{L}}_{i_pk} = \bigvee_{j=1}^{p}\{[0\ I]f_{pi_j}(z_k)\},$$

where $\bigvee_{j=1}^{p}\{v_j\}$ denotes the linear span of vectors $v_1, v_2, \ldots, v_p$ and

$$\mu_p^{\perp V} = \{y \in C^{2m} \mid y^T V z = 0,\ \forall z \in \bigvee_{j=1}^{p}\{U_j e_j\}\}.$$

Define

$$(2.21)\qquad i_{p+1} := min\{l \,|\, dim(\frac{\mathcal{L}_{lk}}{\tilde{\mathcal{L}}_{i_p k}}) \geq 1\}, \quad k \in \underline{n}.$$

Here, it is worthwhile to note that there exists i_{p+1} satisfying (2.21) for $p < m$ by Lemma 4.6 in [BK]. Now, we claim there exists a set of complex numbers $\{\epsilon_j\}_{j=1}^{i_{p+1}-1}$ for which

$$rank[0\ I][\tilde{\theta}_{i_1}(z_k)\ldots\tilde{\theta}_{i_{p+1}}(z_k)] = p+1, \quad k \in \underline{n},$$

where

$$\tilde{\theta}_{i_{p+1}}(z) := f_{p,i_{p+1}}(z) + \sum_{j=1}^{i_{p+1}-1} \epsilon_j f_{pj}(z).$$

We can choose such $\{\epsilon_j | j = 1, \cdots, i_{p+1} - 1\}$ by following the algorithm given in STEP III of the proof of Lemma 4.4.3 of [K1] with $\mathcal{L}_{i_{p+1}k}$, n, p in places of $\mathfrak{N}_k$, m, n, respectively.

Let

$$U := U_0 U_1 \ldots U_{m-1},$$
$$\Delta := U \cdot [e_{i_1} e_{i_2} \ldots e_{i_m}],$$

and $\tilde{\Theta}_I(z) := \Theta(z)\Delta$, where $\{e_j\}_{j=1}^{2m}$ denotes the usual standard basis for C^{2m}. Then, Δ is a full-rank $2m \times m$ upper echelon constant matrix with leading 1's occuring in rows $i_1 < i_2 < \cdots < i_m$ since U is an upper triangular matrix with 1's on its main diagonal by the construction. Thus, $\tilde{\Theta}_I(z)$ is given by

$$\tilde{\Theta}_I(z) = [\tilde{\theta}_{i_1}(z), \ldots, \tilde{\theta}_{i_m}(z)]$$

from (2.18) and (2.20). Upon setting $\Delta = [Q_{10}^T\ Q_{20}^T]^T$, we complete the proof.

COROLLARY 2.4. *Let $\Theta(z)$, σ, and $\mathcal{Q}_{10}\,\mathcal{Q}_{20}$ be as in Theorem 2.3. A solutions of (1.1)-(1.3) denoted by F_{min} is given by*

$$F_{min} = (\Theta_{11}\mathcal{Q}_{10} + \Theta_{12}\mathcal{Q}_{20})(\Theta_{21}\mathcal{Q}_{10} + \Theta_{22}\mathcal{Q}_{20})^{-1},$$

with

$$\delta(F_{min}) = n + \sum_{j=1}^{m} \alpha_{i_j},$$

where α_{i_j} is the ${i_j}^{th}$ element of the ordered set $\{-\alpha_1, \cdots, -\alpha_t, \underbrace{0, \cdots, 0}_{(2m-2t)\text{ times}}, \alpha_t, \cdots, \alpha_1\}$ with α_j's the observ-ablility indices of the pair $(C|_{Ker\Gamma}, \rho_\pi A_\pi)$ for ρ_π, a projection onto $Ker\Gamma$.

The above result is straitforward from (2.16) and the construction of $\mathcal{Q}_{10}, \mathcal{Q}_{20}$ in Theorem 2.3.

COROLLARY 2.5. *Suppose Γ given by (2.3) is invertible, then*

$$\Theta(z) = I + C_\pi(zI - A_\pi)^{-1}\Gamma^{-1}B_\zeta$$

satisfies (2.4)-(2.6). In this case, $\delta(F_{min}) = n$.

Proof. Upon recalling the invertibility of Γ implies that $\alpha_j = 0, \quad j = 1, \cdots, t$, for the $\alpha_j{}'s$ in Corollary 2.4, the result follows.

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