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SOME PROPERTIES OF THE CLASSES OF MATRICES IN THE LINEAR COMPLEMENTARITY PROBLEMS

YOUNG-CHEN LEE

Dept. of Mathematics, Honam University, Kwangju 506-090, Korea.

Abstract We are concerned with three classes of matrices that are relevant to the linear complementary problem. We prove that within the class of P_0-matrices, the Q-matrices are precisely the regular matrices and we show that the same characterizations hold for an L-matrix as well, and that the symmetric copositive-plus Q-matrices are precisely those which are strictly copositive.

1. Introduction

B.C.Eaves, in 1971, showed the relations among matrices in linear complementary problems. And in a recent paper [1], Aganagic and Cottle have established a constructive characterization for such matrices.

DEFINITION 1. A Q-matrix is a real square matrix M for which the linear complementary problem, (q, M), of finding a vector x such that

$$q + Mx \geq 0, \quad x \geq 0 \text{ and } x^t(q + Mx) = 0$$

has a solution for every vector q. The class of Q-matrices is denoted by Q.

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DEFINITION 2. A P_0-matrix is a real square with nonnegative principal minors.

R denotes a class of regular matrices M, i.e., the following system is inconsistent

$$M_i.x + t = 0, \qquad \text{if} \qquad x_i > 0,$$

$$M_i.x + t \geq 0, \qquad \text{if} \qquad x_i = 0,$$

$$0 \neq x \geq 0, \qquad t \geq 0.$$

where M_i is the i-th row of M.

DEFINITION 3. The class of S-matrices, denoted by S, consists of those real square matrices M for which there is a vector $x \geq 0$ such that $Mx > 0$. Obiviously, $Q \subseteq S$.

DEFINITION 4. A real square matrix M is said to be semi-monotone if for every $0 \neq x \geq 0$, there exists an index k such that $x_k > 0$ and $(Mx)_k \geq 0$. The class of semi-monotone matrices is denoted by L_1[3]. P_0-matrices are certainly semi-monotone [4].

DEFINITION 5. Real square matrices M such that $x^t Mx \geq 0$ for all $x \geq 0$ are called copositive matrices. The class of L_2-matrices, denoted by L_2, consists of those real square matrices M satisfying the condition: for every $0 \neq x \geq 0$ with $Mx \geq 0$ and $x^t Mx = 0$, there exist nonnegative diagonal matrices D_1 and D_2 such that $D_2 x \neq 0$ and $(D_1 M + M^t D_2)x = 0$. We define $L = L_1 \cap L_2$.

This class L was inroduced by *Eaves*[3] who showed that if M is an L-matrix, then it is a Q-matrix if and only if it is a S-matrix.

DEFINTION 6. A copositive matrix M is called copositive-plus if $x^t Mx = 0$ and $x \geq 0$ imply $(M + M^t)x = 0$ and strictly copositive if $x^t Mx = 0$ and $x \geq 0$ imply $x = 0$.

Clearly, copositive-plus and strictly copositive matrices belong to L.

DEFINITION 7. A flat point of M [2] is a vector x such that $x^tMx = 0$ and $(M + M^t)x = 0$.

2. Prelininaries

We shall say that the real square matrix M belongs to R_0 if and only if the system

$$\begin{aligned} M_{i.}z &= 0, \quad \text{if } z_i > 0, \\ M_{i.}z &\geq 0, \quad \text{if } z_i = 0, \\ 0 &\neq z \geq 0 \end{aligned} \tag{1}$$

is inconsistant. It is easy to see that R_0 is Carcia's [5] and $E^*(0)$ of matrices for which $(0, M)$ has the unique solution $z = 0$.

Notice that $R_0 \supset R$. In general, these two classes are not equal.

EXAMPLE 2.1. The matrix

$$M = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

belongs to R_0 but not to R. The following lemma shows that within P_0 there is no difference between these two classes.

P_0-matrices have a further characterization which we find useful in this investigation.

LEMMA 2.1. *If $M \in P_0 \cap R^{n\times n}, q^tx < 0$ and $M^tx < 0$ for $x \geq 0$ in the linear complementarity problem (q, M), then (q, M) is infeasible.*

Proof. $(q + Mx)^tx < 0$, $\forall x \geq 0$, hence $x^tM^tx \leq 0$, i.e., $q + Mx \not\geq 0$. Therefore (q, M) is infeasible.

LEMMA 2.2. *If $M \in R^{n\times n} \cap P_0 \cap R_0$, then $M \in R$.*

Proof. Suppose there exists a solution z, t to the system in Definition 2. If $t = 0$ then z satisfies Lemma 2.1 which is impossible since $M \in R_0$. If $t > 0$, then $z_i(Mz)_i < 0$ for all i such that $z_i > 0$, and there is such i since $0 \neq z \geq 0$. This is also impossible since $M \in P_0$. Hence, the system in Definition 2 has no solution and $M \in R$.

COROLLARY 2.1. *If $M \in R^{n\times n} \cap P_0 \cap R_0$, then $M \in Q$.*

Proof. The hypothesis implies $M \in R$. But $R \subset Q$ by Karamardian's theorem [6. Theorem 4.1.].

LEMMA 2.3. *If $M \in R^{n\times n} \cap P_0 \cap Q$, then $M \in R_0$.*

Proof. Suppose the system (1) has a solution $\bar{z}$. Let

$$\alpha = \{i : \bar{z}_i > 0\} \text{ and } \bar{\alpha} = \{i : \bar{z}_i = 0\}.$$

By assumption $\alpha = \phi$. Choose a vector q with $q_\alpha < 0$ and $q_{\bar{\alpha}} > 0$. Let $\bar{z}$ be a solution to the problem (q, M). We claim that for $\lambda > 0$ sufficiently small

$$(\bar{z} - \lambda\bar{z})_i(M(\bar{z} - \lambda\bar{z}))_i < 0 \text{ if } (\bar{z} - \lambda\bar{z})_i \neq 0. \tag{2}$$

Indeed if $i \in \bar{\alpha}$ and $(\bar{z} - \lambda\bar{z})_i \neq 0$ then $(\bar{z} - \lambda\bar{z})_i = -\lambda\bar{z}_i < 0$. Hence, by complementarity, we have $(q + M\bar{z})_i = 0$ which implies $(M(\bar{z} - \lambda\bar{z}))_i \geq \lambda q_i > 0$. This proves (2) for $i \in \bar{\alpha}$. On the other hand, if $i \in \alpha$, then

$$(M(\bar{z} - \lambda\bar{z}))_i \geq \lambda q_i < 0.$$

Consequently, if we choose $\lambda > 0$ which satisfies that $\bar{z}_\alpha - \lambda\bar{z}_\alpha > 0$, then it follows that (2) holds for $i \in \alpha$ as well. Nevertheless, this isimpossible by Lemma 2.1. This contradicton establishes the lemma.

These Lemma 2.2 and 2.3 bring us directly to the following.

THEOREM 2.1. *If $M \in P_0$, then the following are equivalent:*
(1) $M \in R_0$,
(2) $M \in R$, and
(3) $M \in Q$.

We establish theorem which shows that Lemma 2.2 is valid if P_0 is replaced by the larger class L.

LEMMA 2.4. *Let $M \in L_1 \cap R_0$. Then $M \in R$, thus $M \in Q$.*

Proof. Suppose that the system in Definition 1 has a solution $\bar{x}$ for $t = 0$. We must have $t > 0$. Hence for $x_i > 0$, we have $M_i x < 0$. This contradicts the assumption that $M \in L_1$. The contradiction establishes the lemma.

COROLLARY 2.2. *Let $M \in R_0$ be copositive. Then $M \in R$, thus $M \in Q$.*

LEMMA 2.5. *Let $M \in L_2 \cap Q$. Then $M \in R_0$.*

Proof. Suppose that the systems in Definition 1 has a solution $\bar{x}$ for $t = 0$. We must have $x^t M\bar{x} = 0$. By assumption, there exist nonnegative diagonal matrices D_1 and D_2 such that $D_2\bar{x} = 0$ and $(D_1 M + M^t D_2)\bar{x} = 0$. Hence $(x^t D_2)M = -(M\bar{x})^t D_1 \leq 0$. Consequently, if we choose $q \leq 0$ with $q_i < 0$ for $(D_2\bar{x})_i \neq 0$, the problem (q, M) is infeasible. This contradiction establishes the lemma. Combining Lemma 2.4 and 2.5 we deduce;

3. Main Results

Among the principal results in this paper, we show that the same characterizations hold for an L-matrix as well, and that the symmetric copositive-plus Q-matrices are precisely those which are strictly copositive.

THEOREM 3.1. *Let $M \in L$. Then the following are equivalent.*
(a) $M \in Q$,
(b) $M \in R$,
(c) $M \in R_0$, and
(d) $M \in S$.

THEOREM 3.2. *Let $M \in Q$ be copositive. Then the only flat point of M which is also a solution to (O, M) is the zero vector.*

Proof. Suppose that $x \geq 0$ is a nonzero flat point of M that is also a solutions to (O, M). Let

$$\alpha = \{i : x_i > 0\} \quad \text{and} \quad \bar{\alpha} = \{j : x_j = 0\}.$$

Since $Mx \geq 0$ and $x^tMx = 0$, we must have $(Mx)_\alpha = 0$. It is easy to see that

$$(x - \theta u)^t M(x + \theta u) = \theta^2 u^t M u.$$

If $u_{\bar{\alpha}} \geq 0$, the left side of the above equation is nonnegative for sufficiently small $\theta > 0$. Thus $u^tMu \geq 0$ provided that $u_{\bar{\alpha}} \geq 0$. This implies that for all θ,

$$(x - \theta u)^t M(x + \theta u) \geq 0 \quad \text{if} \quad u_{\bar{\alpha}} \geq 0. \tag{3}$$

Choose a vector q such that $q_\alpha < 0$ and $q_{\bar{\alpha}} > 0$. Let z be a solution to (q, M). It is then easy to show that if $\lambda > 0$ is sufficiently small so that $x_\alpha - \lambda z_\alpha > 0$, then $(x - \lambda z)_i(M(x - \lambda z))_i < 0$ for $(x - \lambda z)_i \neq 0$. Hence it follows that for such a λ, we have $(x - \lambda z)^t M(x - \lambda z) < 0$ which contradicts (3). The contradiction establishes the theorem.

COROLLARY 3.1. *Let $M \in Q$ be symmetric and copositive. Then the following implication is valid:*

$$\textit{If } Mx = 0, \quad x \geq 0 \textit{ implies } x = 0. \tag{4}$$

Proof. In fact, any vector $x \geq 0$ satisfying $Mx = 0$ is a flat point of M which is also a solution to (O, M). By theorem 3.2 the only such vector is zero.

REMARK. *The implication (4) is weaker than the statement that (O, M) has a unique solution. Nevertheless, Corollary 3.1 does not follow from Theorem3.1 either*

COROLLARY 3.2. *Let $M \in Q$ be symmetric and copositive-plus. Then M is strictly copositive.*

Proof. Let $x \geq 0$ be such that $x^tMx = 0$. Since M is symmmtric and copositive-plus, it follows that $Mx = 0$. Hence by Corollary 3.1, we must have $x = 0$. Consequently, M is strictly copositive.

Combining Theorem 3.1 and Corollary 3.2, we reduce:

THEOREM 3.3. *Let M be copositive-plus. The following are equivalent:*

(a) $M \in Q$,

(b)$M \in R$,

(c) $M \in R_0$, and

(d) $M \in S$.

If in addition, M is symmetric, then any one of the above is equivalent to:

(e) M is strictly copositive, and

(f) the implication (4) holds.

REMARK. Corollary 3.2 and Theorem 3.3 also follow directly from Theorem 3.1.

4. Conclusion

We have presented the characterizations and relations of linear programming problems, quadratic programming problems and complemmentarity problem. And also, we have suggested some properties of the classes of matrices in the complmentarity problems.

We might also pose the question that Theorem 3.1 without condition (d) will remain valid if L is replaced by L_1. The difficulty lies in establishing the inclusion $M \in (L_1 \cap Q \cap R_0)$ or providing a counterexample.

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