A CHARACTERIZATION OF SOME REAL HYPERSURFACES IN A COMPLEX HYPERBOLIC SPACE

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0. Introduction

We denote by $M_n(c)$ a complete and simply connected complex $n$-dimensional Kahlerian manifold of constant holomorphic sectional curvature $4c$, which is called a complex space form. Such an $M_n(c)$ is bi-holomorphically isometric to a complex projective space $P_n \mathbb{C}$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n \mathbb{C}$, according as $c > 0$, $c = 0$ or $c < 0$.

In this paper, we consider a real hypersurface $M$ in $M_n(c)$. Typical examples of $M$ in $P_n \mathbb{C}$ are the six model spaces of type $A_1$, $A_2$, $B$, $C$, $D$ and $E$, and the ones of $M$ in $H_n \mathbb{C}$ are the four model spaces of type $A_0$, $A_1$, $A_2$ and $B$ (cf. Theorem A in §1), which are all given as orbits under certain Lie subgroups of the group consisting of all isometries of $P_n \mathbb{C}$ or $H_n \mathbb{C}$. Denote by $(\phi, \xi, \eta, g)$ the almost contact metric structure of $M$ induced from the almost complex structure of $M_n(c)$, and by $A$ the shape operator of $M$. The structure vector $\xi$ is said to be principal if $A\xi = \alpha \xi$, where $\alpha = \eta(A\xi)$. Many differential geometers have studied $M$ from various points of view. Berndt [1] and Takagi [14] investigated the homogeneity of $M$. According to Takagi’s classification theorem and Berndt’s one, the principal curvatures and their multiplicities of a homogeneous real hypersurface in $M_n(c)$ are given. Moreover, it is very interesting to give a characterization of homogeneous real hypersurfaces of $M_n(c)$. Let $\mathcal{L}_\xi$ be the Lie derivative in the direction of $\xi$. Then Okumura [13] and Montiel-Romero [12] proved the fact in $P_n \mathbb{C}$ and $H_n \mathbb{C}$, respectively that $M$ is locally congruent to one of homogeneous ones of type $A$ if and only if $\xi$ is an infinitesimal isometry, that is, $\mathcal{L}_\xi g = 0$, where type $A$ means type $A_1$ or $A_2$ in $P_n \mathbb{C}$ and type $A_0$, $A_1$
or $A_2$ in $H_n\mathbb{C}$. Motivated by these results, Maeda-Udagawa [11] studied the condition \( \mathcal{L}_\xi \phi = 0 \) and Ki-Kim-Lee [3] investigated the condition \( \mathcal{L}_\xi A = 0 \). Recently, Kimura and Maeda [10] completely classified $M$ in $P_n\mathbb{C}$ satisfying $\mathcal{L}_\xi S = 0$, where $S$ denotes the Ricci tensor of $M$.

The purpose of the present paper is to investigate $M$ of $H_n\mathbb{C}$ which satisfies $\mathcal{L}_\xi S = 0$ under the condition that $A\xi$ is principal.

1. Preliminaries

We begin with recalling the basic properties of real hypersurfaces of a complex space form. Let $N$ be a unit normal vector field on a neighborhood of a point $p$ in $M$ and $J$ the almost complex structure of $M_n(c)$. For a local vector field $X$ on a neighborhood of $p$, the images of $X$ and $N$ under the transformation $J$ can be represented as

\[ JX = \phi X + \eta(X)N, \quad JN = -\xi, \]

where $\phi$ defines a skew-symmetric transformation on the tangent bundle $TM$ of $M$, while $\eta$ and $\xi$ denote a 1-form and a vector field on the neighborhood of $p$, respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$, where $g$ denotes the induced Riemannian metric on $M$. By the properties of the almost complex structure $J$, the set $(\phi, \xi, \eta, g)$ of tensors satisfies

\begin{align*}
\phi^2 &= -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\
(1.1) &
\end{align*}

where $I$ denotes the identity transformation. Accordingly, this set $(\phi, \xi, \eta, g)$ defines the almost contact metric structure on $M$. Furthermore, the covariant derivatives of the structure tensors are given by

\begin{align*}
(\nabla_X \phi)Y &= \eta(Y)AX - g(AX, Y)\xi, \\
(1.2) &
\nabla_X \xi &= \phi AX, \\
(1.3) &
\end{align*}

where $\nabla$ is the Riemannian connection of $g$. Since the ambient space is of constant holomorphic sectional curvature $4c$, the equations of Gauss and Codazzi are respectively given as follows:

\begin{align*}
R(X, Y)Z &= c\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \\
&- g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\
(1.4) &
+ g(AY, Z)AX - g(AX, Z)AY
\end{align*}
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\[(\nabla_X A)Y - (\nabla_Y A)X \]
\[= c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},\]

where $R$ denotes the Riemannian curvature tensor of $M$. The Ricci tensor $S'$ of $M$ is the tensor of type $(0,2)$ given by $S'(X,Y) = \text{tr}\{Z \to R(Z,X)Y\}$. But it may be also regarded as a tensor of type $(1,1)$ and denoted by $S : TM \to TM$; it satisfies $S'(X,Y) = g(SX,Y)$. From the Gauss equation and (1.1), the Ricci tensor $S$ is given by

\[S = c\{(2n+1)I - 3h \otimes \xi\} + hA - A^2,\]

where $h$ is the trace of $A$. Moreover, using (1.3), we get

\[ (\nabla_X S)Y = -3c\{g(\phi AX,Y)\xi + \eta(Y)\phi AX\} + (Xh)AY \]
\[+ (Xh)AY + (hI - A)(\nabla_X A)Y - (\nabla_X A)AY. \]

Now we quote the following in order to prove our results.

**Theorem A** [1]. Let $M$ be a real hypersurface of $H_n \mathbb{C}$. Then $M$ has constant principal curvatures and $\xi$ is principal if and only if $M$ is locally congruent to one of the following:

- $A_0$. a horosphere in $H_n \mathbb{C}$,
- $A_1$. a geodesic hypersphere $H_0 \mathbb{C}$ or a tube over a hyperplane $H_{n-1} \mathbb{C}$,
- $A_2$. a tube over a totally geodesic $H_k \mathbb{C}$ $(1 \leq k \leq n - 2)$,
- $B$. a tube over a totally real hyperbolic space $H_n \mathbb{R}$.

**Theorem B** [4]. Let $M$ be a real hypersurface of $H_n \mathbb{C}(n \geq 3)$. If $\xi$ is principal and $M$ satisfies $\mathcal{L}_\xi S = 0$, then $M$ is locally congruent to type $A$.

2. **Real hypersurfaces in $M_n(c)$ satisfying $\mathcal{L}_\xi S = 0$**

We denote by $M_n(c)$ a complex space form with the metric of constant holomorphic sectional curvature $4c$ and $M$ a real hypersurface in $M_n(c), c \neq 0$. In this section, we suppose that the Ricci tensor $S$
satisfies the condition $\mathcal{L}_\xi S = 0$. The following discussion in the case where $c > 0$ is indebted to Kimura and maeda [10]:

From (1.3), for any $X \in TM$ we have

$$(\mathcal{L}_\xi S)X = [\xi, SX] - S[\xi, X]$$

$$= (\nabla_\xi S)X - \nabla SX\xi + S\nabla X\xi$$

$$= (\nabla_\xi S)X - \phi ASX + S\phi AX.$$ 

Then we see that "$\mathcal{L}_\xi S = 0$" is equivalent to

$$(2.1) \quad \nabla_\xi S = \phi AS - S\phi A.$$ 

Since $g((\nabla SX)X, Y) = g((\nabla SX)Y, X)$ for any $X, Y \in TM$, the equation (2.1) shows

$$(2.2) \quad (\phi A - A\phi)S = S(\phi A - A\phi).$$ 

From (1.6) it follows that

$$(2.3) \quad \phi S - S\phi = h(\phi A - A\phi) - (\phi A^2 - A^2\phi).$$ 

Here we hope to calculate $||\phi S - S\phi||^2$, which is equivalent to $tr(\phi S - S\phi)^2$ because $\phi S - S\phi$ is symmetric. From (2.3), we get

$$(2.4) \quad tr(\phi S - S\phi)^2 = htr(\phi A - A\phi)(\phi S - S\phi) - tr(\phi A^2 - A^2\phi)(\phi S - S\phi).$$ 

In general, we get

$$(2.5) \quad tr(\phi A - A\phi)(\phi S - S\phi) = 2tr\phi A\phi S - trA\phi^2 S - tr\phi AS\phi.$$ 

Taking the trace of (2.2), we find

$$(2.6) \quad tr\phi^2 AS - 2tr\phi S\phi A + tr\phi^2 SA = 0.$$ 

Combining (2.5) with (2.6), we obtain

$$(2.7) \quad tr(\phi A - A\phi)(\phi S - S\phi) = 0.$$
On the other hand, we find
\[(2.8) \quad tr(\phi A^2 - A^2 \phi)(\phi S - S \phi) = 2tr\phi A^2 \phi S - trA^2 \phi^2 S - tr\phi A^2 S \phi.\]
From (2.2) it follows that
\[
\phi A\{(\phi A - A\phi)S - S(\phi A - A\phi)\} = 0,
\]
which implies
\[(2.9) \quad tr\phi ASA\phi = tr\phi A^2 \phi S.\]
Then combining (2.8) with (2.9) we have
\[(2.10) \quad tr(\phi A^2 - A^2 \phi)(\phi S - S \phi) = 2tr\phi^2 ASA - tr\phi^2 S A^2 - tr\phi^2 A^2 S.\]
Thus substituting (2.7) and (2.10) into (2.4) and using (1.1) and (1.6), we can see that
\[(2.11) \quad tr(\phi S - S \phi)^2 = -\frac{3}{2} c(\beta - \alpha^2),\]
where we have put \(\beta = \eta(A^2 \xi)\) and \(\alpha = \eta(A\xi)\). Taking account of (1.1), we find
\[(2.12) \quad \|\phi A\xi\|^2 = \beta - \alpha^2.\]
Hence from (2.11) and (2.12), we have
\[
tr(\phi S - S \phi)^2 = -\frac{3}{2} c\|\phi A\xi\|^2
\]
or
\[
\|\phi S - S \phi\|^2 + \frac{3}{2} c\|\phi A\xi\|^2 = 0.
\]
Consequently, the condition "\(\mathcal{L}_\xi S = 0\)" implies the fact that \(\phi S = S \phi\) and \(\xi\) is principal in the case where \(c > 0\) and that \(\phi S = S \phi\) if and only if \(\xi\) is principal in the case where \(c < 0\). Here we note that Kimura and Maeda [10] proved a local classification theorem for real hypersurfaces in \(P_n \mathbb{C}\) which satisfy \(\mathcal{L}_\xi S = 0\). Thus because of Theorem B, it is seems to be interested to consider real hypersurfaces in \(H_n \mathbb{C}\) satisfying \(\mathcal{L}_\xi S = 0\) under the weaker condition than one that \(\xi\) is principal.
3. Real hypersurfaces in $H_n C$ satisfying $\mathcal{L}_\xi S = 0$

Let $M$ be a real hypersurface in a complex hyperbolic space $H_n C$ endowed with the Bergmann metric of constant holomorphic sectional curvature $-4$. In this section, we assume that $M$ satisfies $\mathcal{L}_\xi S = 0$ and $A\xi$ is principal. The second assumption means

$$A^2 \xi = \lambda A\xi,$$

where $\lambda = \eta(A^3 \xi)$. For simplicity we put $U = \nabla_\xi \xi$. Then we have $U = \phi A\xi$, which together with (1.1) implies

$$\phi U = -A\xi + \alpha \xi$$

and so $g(\phi U, \xi) = 0$. Thus we define $\phi U$ by $\phi U = -\mu W$, where $W$ is a unit vector field orthogonal to $\xi$ and $\mu$ is a smooth function on $M$. Namely, we have

$$A\xi = \alpha \xi + \mu W.$$

Here we note that this and $U = \mu \phi W$ give $g(U, W) = 0$. Moreover, it follows from (1.6) and (3.1) that

$$S\xi = -2(n - 1)\xi + (h - \lambda)A\xi,$$

(3.5) \quad $SU = -(2n + 1)U + hAU - A^2 U.$

From (3.1) and (3.3) we find

$$AW = \gamma A\xi,$$

where $\gamma \mu = \lambda - \alpha$. Thus (1.6) combined with (3.1) and (3.6) gives us

$$SW = -(2n + 1)W + \gamma(h - \lambda)A\xi.$$ 

From (2.2), we find

$$(\phi A - A\phi)S\xi = S(\phi A - A\phi)\xi,$$

which, together with (1.1), (3.4), (3.5) and the definition of $U$, yields
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(3.8) \[ A^2 U = (2h - \lambda)AU + (\lambda^2 - \lambda h - 3)U. \]

Also, from (2.2) we get
\[ (\phi A - A\phi)SW = S(\phi A - A\phi)W, \]
which, together with (1.1), (1.6), (3.5) \sim (3.8) and the definition of $W$, leads to

(3.9) \[ \{2(\lambda - h)^2 - 3\}AU = \{\lambda(\lambda - h)^2 + 3(h - 2\lambda + \alpha)\}U. \]

On the other hand, differentiating (3.2) covariantly in the direction of $X$ and making use of (1.1), (1.2) and (1.3), we obtain
\[ g(4X, U)\xi - \phi(\nabla_X A)\xi + A\phi AX - d\alpha(X)\xi - \alpha\phi AX. \]

Taking the inner product of this and $\xi$ and using (1.1) and (1.3), we have

(3.10) \[ g((\nabla_X A)\xi, \xi) = 2g(AU, X) + d\alpha(X). \]

Moreover, differentiating (3.1) covariantly in the direction of $X$, we get

(3.11) \[ (\nabla_X A)A\xi + A(\nabla_X A)\xi + A^2\phi AX \]
\[ = d\lambda(X)A\xi + \lambda(\nabla_X A)\xi + \lambda A\phi AX. \]

If we take the inner product of this and $\xi$ and make use of (3.1), (3.10) and the fact that $g((\nabla_X A)\xi, Y) = g((\nabla_X A)Y, \xi)$ for any $X, Y \in TM$, then we find

(3.12) \[ g((\nabla_X A)\xi, A\xi) = \frac{1}{2}d(\lambda\alpha)(X) + \lambda g(AU, X). \]

From (3.11), replacing $X$ by $\xi$ and taking the inner product of this result and $\xi$, we have

(3.13) \[ \frac{1}{2}d(\lambda\alpha)(X) + g(U, X) + 3g(A^2U, X) + d\alpha(AX) \]
\[ = d\lambda(\xi)g(A\xi, X) + 2\lambda g(AU, X) + \lambda d\alpha(X), \]

where we have used (1.5), (3.10) and (3.12).

Let $M_0$ be the set of consisting of points $x$ in $M$ such that $(\lambda - h)(x) = 0$. On the subset $M_0$, from (3.9) it is seen that

(3.14) \[ AU = (\lambda - \alpha)U \quad \text{on} \quad M_0. \]

Then, by using (3.14) the equation (3.8) turns out to be $\{\alpha(\lambda - \alpha) - 3\}U = 0$ on $M_0$. 

Lemma 3.1. Let $M$ be a real hypersurface of $H_n\mathbb{C}$. Assume that it satisfies $L_\xi S = 0$ and $A_\xi$ is principal. If $U \neq 0$, then $\text{Int}(M_0) = \emptyset$, where $\text{Int}(M_0)$ denotes the interior of $M_0 = \{x \in M \mid (\lambda - h)(x) = 0\}$.

Proof. We assume that the interior of $M_0$ is not empty. Since we have supposed that $U \neq 0$, from the above equation it follows that $\alpha(\lambda - \alpha) = 3$. By means of (3.1), it is clear that $\beta = \lambda\alpha$, which together with (2.12) gives us $g(U, U) = 3$. Thus, using (3.8) and (3.14), the equation (3.13) is reformed as

$$ad\lambda(\xi)g(A\xi, X) = (3\lambda - 8\alpha)g(U, X) - 3\alpha(X) + ad\alpha(AX).$$

Replacing $X$ by $U$ into this equation and making use of (3.14), we obtain $3(3\lambda - 8\alpha) = 0$, which yields $3\lambda = 8\alpha$. Thus we can see that $\alpha = 3/\sqrt{5}$, $\lambda = 8/\sqrt{5}$ and $\mu = \sqrt{3}$.

On the other hand, since $g((\phi S - S\phi)U, W) = -g(SU, \phi W) - g(SW, \phi U)$, using (3.5), (3.7) and (3.8), we get $g((\phi S - S\phi)U, W) = 3g(\phi U, W)$, which together with (3.2) implies $g((\phi S - S\phi)U, W) = -3\sqrt{3}$. Then it is clear that $\|\phi S - S\phi + \sqrt{3}(W \otimes U + U \otimes W)\|^2 = 0$, where we have used (2.11) and (2.12). Thus we can see that $\phi S - S\phi = -\sqrt{3}(W \otimes U + U \otimes W)$. Since $\phi U = -\sqrt{3}W$, we obtain $\phi S - S\phi = \phi U \otimes U + U \otimes \phi U$. Combining this with (2.3), we have

$$\lambda \phi A - \phi A^2 - \lambda A\phi + A^2\phi = \phi U \otimes U + U \otimes \phi U,$$

which, together with (3.1), (3.2) and (3.14), shows that

$$\lambda \phi A^2 - \phi A^3 - \lambda A\phi A + A^2\phi A = (\alpha - \lambda)\{A\xi \otimes U + U \otimes A\xi\} + 3U \otimes \xi.$$

Substituting $X$ by $\xi$ into (1.7), we get

$$(\nabla_\xi S)X = 3g(U, X)\xi + 3\eta(X)U + \lambda(\nabla_\xi A)X - A(\nabla_\xi A)X - (\nabla_\xi A)AX,$$

which, together with (1.6) and (2.1), leads to

$$\lambda \phi A^2X - \phi A^3X - \lambda A\phi AX + A^2\phi AX = 3g(U, X)\xi + \lambda(\nabla_\xi A)X - A(\nabla_\xi A)X - (\nabla_\xi A)AX.$$
From (3.16) and (3.17), it is seen that

\[(\alpha - \lambda)\{A^2 \xi \otimes U + U \otimes A\xi\}(X) + 3U \otimes \xi(X)\]
\[= 3g(U,X)\xi + \lambda(\nabla \xi A)X - A(\nabla \xi A)X - (\nabla \xi A)AX.\]

By using the Codazzi equation (1.5), the equation (3.17) is reformulated as

\[\lambda \phi A^2 X - \phi A^3 X - \lambda A \phi AX + A^2 \phi AX\]
\[= 3g(U,X)\xi + \lambda(\nabla X A)\xi - \lambda \phi X - A(\nabla X A)\xi + A \phi X - (\nabla \xi A)AX,\]

which together with (3.11) yields

\[(\nabla X A)A\xi - (\nabla \xi A)AX\]
\[= \lambda \phi A^2 X - \phi A^3 X - 3g(X,U)\xi + \lambda \phi X - A \phi X.\]

Transforming this by \(\phi\) and taking account of (1.1), we get

\[\phi\{(\nabla X A)A\xi - (\nabla \xi A)AX\}\]
\[= A^3 X - \lambda A^2 X - \lambda X + \lambda \eta(X)\xi - \phi A \phi X.\]

Differentiating (3.15) covariantly along \(M_0\) and making use of (3.1), we obtain

\[(\nabla X \phi)(\lambda A - A^2)Y + \phi(\lambda(\nabla X A)Y - (\nabla X A)AY - A(\nabla X A)Y)\]
\[- (\lambda(\nabla X A) - (\nabla X A)A - A(\nabla X A))\phi Y - (\lambda A - A^2)(\nabla X \phi)Y\]
\[= \{\phi \nabla X U \otimes U + \phi U \otimes \nabla X U + \nabla X U \otimes \phi U + U \otimes \phi \nabla X U\]
\[+ (\lambda - \alpha)g(U,X)(\xi \otimes U + U \otimes \xi)\}(Y).\]

Taking the skew symmetric part for \(X\) and \(Y\) of this and then replacing \(X\) by \(\xi\) into the obtained result, then we get

\[\lambda\{X - \eta(X)\xi\} - \phi(\nabla \xi A)AX + \phi(\nabla X A)A\xi + \phi A \phi X\]
\[= g(U,X)\phi \nabla \xi U + g(\nabla \xi U, X)\phi U - g(\nabla X U, \xi)\phi U\]
\[+ g(\phi U, X)\nabla \xi U + g(\phi \nabla \xi U, X)U - (\lambda - \alpha)g(U,X)U,\]
where we have used (1.1), (1.2), (1.5) and (3.1). Combining this and (3.19) and taking account of the fact that \( g(\nabla_X U, \xi) = -g(\nabla_X \xi, U) = (\alpha - \lambda)g(A\xi, X) \), we get

\[
3g(U, \phi X)\xi + (\lambda - \alpha)g(U, \phi X)A\xi \nonumber \\
= g(U, X)\phi \nabla_\xi U + g(\nabla_\xi U, X)\phi U + (\lambda - \alpha)g(A\xi, X)\phi U \\
+ g(\phi U, X)\nabla_\xi U + g(\phi \nabla_\xi U, X)U. 
\tag{3.20}
\]

On the other hand, differentiating \( U \) covariantly in the direction of \( X \) and making use of (1.2), (1.3) and (3.1), we get

\[
\nabla_X U = \alpha AX - \lambda g(X, A\xi)\xi + \phi(\nabla_X A)\xi + \phi A\phi AX,
\]

which implies that

\[
\nabla_\xi U = \alpha A\xi - \lambda \alpha \xi + \phi(\nabla_\xi A)\xi + \phi AU.
\]

Then, by means of (3.14), we have

\[
\phi \nabla_\xi U = (2\alpha - \lambda)U - (\nabla_\xi A)\xi.
\]

Substituting the last two equations into (3.20) and taking the inner product of this result and \( U \), we can see that

\[
(6\alpha - 4\lambda)g(U, X) = 0,
\]

where we have used (3.14). Thus, we obtain \( 3(6\alpha - 4\lambda) = 0 \). Hence \( 3\alpha = 2\lambda \). Since \( 3\lambda = 8\alpha \) on the subset \( M_0 \), we get \( \alpha = 0 \). This contradicts the fact that \( \alpha = 3/\sqrt{5} \). Consequently, we conclude that \( \text{Int}(M_0) = \emptyset \).

The following is immediate from Lemma 3.1.

**Lemma 3.2.** Let \( M \) be a real hypersurface of \( H_n \mathbb{C} \). If it satisfies \( \mathcal{L}_\xi S = 0 \) and \( A\xi \) is principal such that \( \eta(A^3\xi) = \text{tr}A \), then \( \xi \) is principal.
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Remark 1. In general, "$\xi$ is principal" implies "$A\xi$ is principal". But the converse is not true.

Remark 2. The structure vector $\xi$ is principal with respect to $S$ if the Ricci tensor $S$ satisfies $S\xi = \sigma \xi$ for some function $\sigma$ on $M$. Under the same assumption as Lemma 3.2, we have $\xi$ is principal with respect to $S$. In fact, since $\lambda = \eta(A^3\xi) = trA = h$, taking account of (3.4) we have $S\xi = -2(n-1)\xi$.

Remark 3. A ruled real hypersurface does not satisfy the condition that $A\xi$ is principal. In fact, let $M$ be a ruled real hypersurface in a complex space form $M_n(c)$. Then $M$ satisfies

$$A\xi = \alpha \xi + \beta V(\beta \neq 0),$$

$$AV = \beta \xi,$$

$$AX = 0$$

for any vector $X$ orthogonal to $\xi$ and $V$, where $V$ is a unit orthogonal to $\xi$, and $\alpha$ and $\beta$ are smooth functions on $M$. Assume that $M$ satisfies the condition that $A\xi$ is principal, that is, $A^2\xi = \lambda A\xi$. Then using the above properties of $M$, we get $A^2\xi = A(\alpha \xi + \beta V) = (\alpha^2 + \beta^2)\xi + \alpha \beta V$ and $A^2\xi = \lambda (A\xi) = \alpha \lambda \xi + \beta \lambda V$. Thus comparing to these two equations, we have $\alpha = \lambda$ and $\beta = 0$. This contradicts the fact that $\beta \neq 0$.

From Lemma 3.2 and Theorem B we have the following.

Theorem 3.3. Let $M$ be a real hypersurface of $H_n \mathbb{C}(n \geq 3)$. If $A\xi$ is principal such that $\eta(A^3\xi) = trA$ and $M$ satisfies $\mathcal{L}_\xi S = 0$, then $M$ is locally congruent to type $A$.

For a real hypersurface of $H_n \mathbb{C}$ satisfying the condition "$\mathcal{L}_\xi S = 0"$, we see that $\phi S = S\phi$ if and only if $\xi$ is principal. Thus we get the following.

Theorem 3.4. Let $M$ be a real hypersurface of $H_n \mathbb{C}(n \geq 3)$. If $M$ satisfies $\mathcal{L}_\xi S = 0$ and $\phi S = S\phi$, then $M$ is locally congruent to type $A$. 
References