THE BOOK–SHORE TYPE LAW OF A GAUSSIAN PROCESS WITH STATIONARY INCREMENTS

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1. Introduction and results

Let $a_T (0 < T < \infty)$ be a nondecreasing function of $T$ for which
(i) $0 < a_T \leq T$,
(ii) $T/a_T$ is nondecreasing.

For instance, we can choose $a_T$ as $1, \log T, T^\theta, (0 < \theta < 1), T/(\log T)^r, (0 < r < \infty)$ and $cT (0 < c \leq 1)$, etc.

Under these conditions on $a_T$, Csörgő and Révész [4] obtained the following theorem for a standard Wiener process $\{W(t); t \geq 0\}$:

THEOREM A. If $a_T (0 < T < \infty)$ satisfies the conditions (i) and (ii), then

\begin{align}
\limsup_{T \to \infty} \sup_{0 \leq t \leq T - a_T} \frac{|W(t + a_T) - W(t)|}{\beta_T \sqrt{a_T}} &= 1 \quad \text{a.s.} \\
\limsup_{T \to \infty} \sup_{0 \leq s \leq a_T} \sup_{0 \leq t \leq T - a_T} \frac{|W(t + s) - W(t)|}{\beta_T \sqrt{a_T}} &= 1 \quad \text{a.s.}
\end{align}

where $\beta_T = \sqrt{2[\log(T/a_T) + \log \log T]}$. If, in addition, we have also
(iii) $\lim_{T \to \infty} \{\log T - \log a_T\}/\log \log T = \infty$,

then we have

\begin{align}
\lim_{T \to \infty} \sup_{0 \leq t \leq T - a_T} \frac{|W(t + a_T) - W(t)|}{\beta_T \sqrt{a_T}} &= 1 \quad \text{a.s.} \\
\lim_{T \to \infty} \sup_{0 \leq s \leq a_T} \sup_{0 \leq t \leq T - a_T} \frac{|W(t + s) - W(t)|}{\beta_T \sqrt{a_T}} &= 1 \quad \text{a.s.}
\end{align}

On the other hand, Book and Shore [1] extended the result (1.2) of the above Csörgő–Révész theorem as follows:

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THEOREM B. If $a_T (0 < T < \infty)$ satisfies the above conditions (i), (ii) and further

(iii)' $\lim_{T \to \infty} (\log T - \log a_T) / \log \log T = r, \quad 0 \leq r \leq \infty,$

then

$$\liminf_{T \to \infty} \sup_{0 \leq t \leq T - a_T} \frac{|W(t + a_T) - W(t)|}{\beta_T \sqrt{a_T}} = \frac{r}{1 + r} \quad \text{a.s.}$$

For the standard Wiener process $\{W(t); t \geq 0\}$, the Strassen's law of iterated logarithm in [6] implies that for any $0 < c < 1$

$$\limsup_{T \to \infty} \sup_{0 \leq t \leq T - cT} \frac{|W(t + cT) - W(t)|}{\sqrt{2c \log \log T}} = 1 \quad \text{a.s.} \tag{1.3}$$

For $0 < c < 1$ if we set $a_T = cT$ in (1.1), we get the result (1.3). Clearly, $a_T = cT (0 < c < 1)$ fails to satisfy the condition (iii) of Theorem A, but it satisfies the condition (iii)' of Theorem B. Thus the Strassen's law of iterated logarithm is complemented as follows:

$$\liminf_{T \to \infty} \sup_{0 \leq t \leq T - cT} \frac{|W(t + cT) - W(t)|}{\sqrt{2c \log \log T}} = 0 \quad \text{a.s.}$$

We are going to extend Theorems A and B to a Gaussian process with stationary increments. Let $\{X(t): 0 \leq t < \infty\}$ be a almost surely continuous Gaussian process with $X(0) = 0$, $E\{X(t)\} = 0$ and stationary increments: $E\{(X(t) - X(s))^2\} = \sigma^2(|t - s|)$, where $\sigma(y)$ is a function of $y \geq 0$ (for example, if $\{X(t): 0 \leq t < \infty\}$ is a standard Wiener process, then $\sigma(t) = \sqrt{t}$). Further assume that $\sigma(t), t > 0$, is a nondecreasing continuous, regularly varying function with exponent $\gamma (0 < \gamma < 1)$ at infinity (or zero). A positive function $q(t), t > 0$, is said to be regularly varying with exponent $\gamma > 0$ at $a(a = \infty$ or $0$) if, for all $x > 0$, one has

$$\lim_{t \to a} \frac{q(xt)}{q(t)} = x^\gamma.$$

Let us define continuous parameter processes $X_1(T), X_2(T), \cdots$, ...
$X_6(T)$ by

\[
X_1(T) = \sup_{0 \leq s \leq a_T} \sup_{0 \leq t \leq T-s} \frac{|X(t + s) - X(t)|}{\beta_T \sigma(a_T)},
\]

\[
X_2(T) = \sup_{0 \leq s \leq a_T} \sup_{0 \leq t \leq T-s} \frac{X(t + s) - X(t)}{\beta_T \sigma(a_T)},
\]

\[
X_3(T) = \sup_{0 \leq s \leq a_T} \sup_{0 \leq t \leq T-s} \frac{|X(t + s) - X(t)|}{\beta_T \sigma(a_T)},
\]

\[
X_4(T) = \sup_{0 \leq s \leq a_T} \sup_{0 \leq t \leq T-a_T} \frac{X(t + s) - X(t)}{\beta_T \sigma(a_T)},
\]

\[
X_5(T) = \sup_{0 \leq t \leq T-a_T} \frac{|X(t + a_T) - X(t)|}{\beta_T \sigma(a_T)},
\]

\[
X_6(T) = \sup_{0 \leq t \leq T-a_T} \frac{X(t + a_T) - X(t)}{\beta_T \sigma(a_T)},
\]

respectively. Clearly, $X_1(T)$ is the largest process and $X_6(T)$ is the smallest one of all $X_i(T), i = 1, \ldots, 6$.

In this paper we shall investigate almost sure limiting values of $X_i(T), i = 1, 2, \cdots , 6$, under varying conditions on $a_T$. Thus we are concerning only with behavior of functions near at infinity. We often use the letter $c$ for a positive absolute constant which may be different from line to line if necessary.

The following theorem is an extension of Theorem A to a Gaussian process, which is proved in Csáki et al. [3] and Choi [2].

**THEOREM C.** Let $a_T$ be a nondecreasing function of $T$ such that

(i) $0 < a_T \leq T$,

(ii) $T/a_T$ is nondecreasing.

Let the Gaussian process \{X(t); 0 \leq t < \infty\} in the above statements satisfy the condition which for any $a \leq b \leq c \leq d$

(iii) $E \{(X(b) - X(a))(X(d) - X(c))\} \leq 0$.

Then

\[
\limsup_{T \to \infty} X_1(T) = 1, \quad \text{a.s.},
\]

where $i = 1, 2, \cdots , 6$. Moreover, if we have also

(iv) $\lim_{T \to \infty} (\log T - \log a_T)/\log \log T = \infty$.
and if either the condition (iii) it holds or

(v) \( \sigma^2(t) \) is twice continuously differentiable which satisfies

\[
|\sigma''(t)| \leq c \sigma(t)/t^2, \quad t > 0,
\]

where \( c \) is a positive constant, then

(1.4) \[
\lim_{T \to \infty} X_i(T) = 1, \quad \text{a.s.,}
\]

where \( i = 1, 2, \cdots, 6 \).

Note that the condition (iv) of Theorem C is satisfied in cases of \( a_T = 1, (\log \log T)^\beta (0 < \beta < \infty), (\log T)^\beta (0 < \beta < \infty) \) and \( T^\theta (\log T)^\alpha (0 < \theta < 1, -\infty < \alpha < \infty) \), etc. But in case of \( a_T = T/(\log T)^r (0 < r < \infty) \), it is not satisfied. Thus we investigate this case:

**THEOREM 1.** Let \( a_T \) be a nondecreasing function of \( T \) such that

(i) \( 0 < a_T \leq T/(\log T)^r \) for all \( 0 < r < \infty \),

(ii) \( T/a_T \) is nondecreasing.

Assume that the above Gaussian process \( \{X(t); 0 \leq t < \infty\} \) satisfies the condition (iii) of Theorem C. Then

\[
\liminf_{T \to \infty} X_i(T) \geq \sqrt{\frac{r}{1 + r}}, \quad \text{a.s.,}
\]

where \( i = 1, 2, \cdots, 6 \).

The following theorem complements its lack for gaps in \( T/(\log T)^r < a_T \leq T, 0 < r < \infty \) and exactly yields the "liminf" value. Theorem 2 is an extension of Theorem B for Wiener processes, and it gives the same value as the result (1.4) of Theorem C only when \( r = \infty \) in Theorem 2. Theorem 1 also needs to prove Theorem 2.

**THEOREM 2.** Let \( a_T \) be a nondecreasing function of \( T \) for which

(i) \( 0 < a_T \leq T \),

(ii) \( T/a_T \) is nondecreasing,

(iii) \( \lim_{T \to \infty} (\log T - \log a_T)/\log \log T = r, \quad 0 \leq r \leq \infty \).
Assume that the above Gaussian process \( \{X(t); 0 \leq t < \infty\} \) satisfies the condition which, for \( t > 0 \),

(iv) \( \sigma(t) = t^\gamma, \quad 0 < \gamma \leq 1/2. \)

Then we have

\[
\liminf_{T \to \infty} X_i(T) = \sqrt{\frac{r}{1 + r}} \quad \text{a.s.,}
\]

where \( i = 1, 2, \ldots, 6 \) if \( r > 0 \), and \( i = 1, 3, 5 \) if \( r = 0 \).

We note that the condition (iii) of Theorem C is weaker than that (iv) of Theorem 2, but the condition (iii) of Theorem 2 contains that (iv) of Theorem C.

2. Proofs

For proving our Theorem 1, we shall make use of the following lemma:

**Lemma 1** (Slepian [5]). Suppose that \( \{X_i : i = 1, 2, \ldots, n\} \) and \( \{Y_i : i = 1, 2, \ldots, n\} \) are jointly standardized normal random variables with

\[
\text{covariance} (X_i, X_j) \leq \text{covariance} (Y_i, Y_j), \quad i \neq j.
\]

Then for any real number \( u_n \),

\[
P\{X_i \leq u_n; i = 1, 2, \ldots, n\} \leq P\{Y_j \leq u_n; j = 1, 2, \ldots, n\}.
\]

**Proof of Theorem 1.** Considering the order of magnitude of \( X_i(T), \ i = 1, 2, \ldots, 6, \) it suffices to prove

\[
\liminf_{T \to \infty} X_6(T) \geq \sqrt{\frac{r}{1 + r}} \quad \text{a.s.}
\]

For given \( T > 0 \) large enough, let us define a positive integer \( n_T \) by \( n_T = \lfloor T/a_T \rfloor \) where \( \lfloor y \rfloor \) denotes the greatest integer not exceeding \( y \). By the assumption (i) of \( a_T \), the integers \( n_T \) are increasing and \( n_T \to \infty \) as \( T \to \infty \). For \( j = 1, 2, \ldots, n_T \), define incremental random variables

\[
Z_T(j) = X(ja_T) - X((j - 1)a_T).
\]
From the condition (iii) it follows that for $i \neq j$

covariance $(Z_T(i), Z_T(j)) \leq 0$.

Applying Lemma 1 for $X_j = Z_T(j)/\sigma(a_T)$, $j = 1, 2, \ldots, n_T$, we have for any $0 < \epsilon < 1$

$$P\{X_6(T) < \sqrt{(1-\epsilon)r/(1+r)}\}$$

$$= P\left\{ \sup_{0 \leq t \leq T-a_T} \frac{X(t+a_T) - X(t)}{\sigma(a_T)} < u_T \right\}$$

$$\leq P\left\{ \sup_{1 \leq j \leq n_T} \frac{Z_T(j)}{\sigma(a_T)} < u_T \right\}$$

$$\leq \{\Phi(u_T)\}^{n_T}$$

where $u_T = \sqrt{(1-\epsilon)r/(1+r)}\sqrt{2}\{\log(T/a_T) + \log\log T\}$ and $\Phi(\cdot)$ denotes the standard normal distribution function. Since, for large $T$

$$\{\Phi(u_T)\}^{n_T} \leq \exp(-c(T/a_T)\log T)^{-1(1-\epsilon)r/(1+r)n_T},$$

we have

$$P\{X_6(T) < \sqrt{(1-\epsilon)r/(1+r)}\} \leq \exp(-c(\log T)^{er}).$$

Let $0 < \alpha < 1$ and set $T_k = \exp(k^\alpha)$, $k \in N$, where $N$ is a set of positive integers. Then the above inequality yields

$$P\{X_6(T_k) < \sqrt{(1-\epsilon)r/(1+r)}\} \leq \exp(-ck^\alpha r).$$

Using the Borel-Cantelli lemma, we obtain

$$\liminf_{k \to \infty} X_6(T_k) \geq \sqrt{\frac{r}{1+r}} \quad \text{a.s.}$$

For given $T_k$, let $T$ be in $T_k \leq T \leq T_{k+1}$, $k \in N$. Then by the similar techniques as in the proof of Lemma 4.6 of Choi [2], we have

$$\liminf_{T \to \infty} X_6(T) \geq \liminf_{k \to \infty} X_6(T_k) \quad \text{a.s.}$$

This proves Theorem 1.

In proving Theorem 2 we shall use a form of the modulus of continuity for Gaussian processes (cf. Lemma 2), which is an extension of Lévy's modulus of continuity for Wiener processes.
**Lemma 2** [3]. (Moduli of continuity for a Gaussian process)

Assume that the condition (iii) of Theorem C holds. Then

\[
\lim_{h \to 0} \sup_{0 \leq u \leq 1 - h} \frac{X(u + h) - X(u)}{\sqrt{2\log(1/h)\sigma(h)}} = 1, \tag{2.1}
\]

\[
\lim_{h \to 0} \sup_{0 \leq u \leq 1 - h} \frac{|X(u + h) - X(u)|}{\sqrt{2\log(1/h)\sigma(h)}} = 1, \tag{2.2}
\]

\[
\lim_{h \to 0} \sup_{0 \leq u \leq 1 - h} \frac{X(u + v) - X(u)}{\sqrt{2\log(1/h)\sigma(h)}} = 1, \tag{2.3}
\]

\[
\lim_{h \to 0} \sup_{0 \leq u \leq 1 - h} \frac{|X(u + v) - X(u)|}{\sqrt{2\log(1/h)\sigma(h)}} = 1, \tag{2.4}
\]

\[
\lim_{h \to 0} \sup_{0 \leq u \leq 1 - h} \frac{X(u + v) - X(u)}{\sqrt{2\log(1/h)\sigma(h)}} = 1, \tag{2.5}
\]

\[
\lim_{h \to 0} \sup_{0 \leq u \leq 1 - h} \frac{|X(u + v) - X(u)|}{\sqrt{2\log(1/h)\sigma(h)}} = 1 \tag{2.6}
\]

hold almost surely, where \(0 < h' < h < 1\).

The proof of Theorem 2 applies the similar techniques as the proof of Book-Shore [1].

**Proof of Theorem 2.** When \(r = \infty\), we have already proved in Theorem C. The only part of the proof is the “liminf” part when \(0 \leq r < \infty\). Since \(a_T/T\) is nonincreasing, either \(a_T/T \to 0\) or \(a_T/T \to \delta(0 < \delta \leq 1)\) as \(T \to \infty\). First suppose the case \(a_T/T \to \delta(0 < \delta \leq 1)\). Then \(a_T \geq \delta T\) for all large \(T\), and we must be in a case when \(r = 0\) because in the condition (iii)

\[
0 \leq \lim_{T \to \infty} \frac{\log(T/a_T)}{\log \log T} \leq \lim_{T \to \infty} \frac{\log(1/\delta)}{\log \log T} = 0.
\]

Let us denote \(U(t) \overset{d}{=} V(t)\) if \(U(t)\) has the same distribution as \(V(t)\). By the condition (iv),

\[
\sigma(a_T)X(t/a_T) \overset{d}{=} X(t).
\]
Thus, in case \( X_1(T) \), we have

\[
0 \leq X_1(T) = \sup_{0 \leq s \leq a_T} \sup_{0 \leq t \leq T-s} \frac{|X(t+s) - X(t)|}{\sqrt{2(\log(T/a_T) + \log \log T)} \sigma(a_T)} \\
= \sup_{0 \leq s \leq a_T} \sup_{0 \leq t \leq T-s} \frac{\sigma(a_T)|X((t+s)/a_T) - X(t/a_T)|}{\sqrt{2(\log(T/a_T) + \log \log T)} \sigma(a_T)} \\
\leq \sup_{0 \leq p \leq 1} \sup_{0 \leq q \leq (T/a_T)-p} \frac{|X(q+p) - X(q)|}{\sqrt{2 \log \log T}} \\
\leq \sup_{0 \leq p \leq 1} \sup_{0 \leq q \leq (1/6)-p} \frac{|X(q+p) - X(q)|}{\sqrt{2 \log \log T}} \to 0 \quad \text{a.s.,}
\]

as \( T \to \infty \) by the a.s. continuity of Gaussian process. So \( X_1(T) \to 0 \) in probability as \( T \to \infty \) and hence there exists a subsequence \( \{T_k : 1 \leq k < \infty \} \) such that \( X_1(T_k) \) converges almost surely to zero as \( k \to \infty \). It follows that

\[
\liminf_{T \to \infty} X_1(T) = 0 \quad \text{a.s.}
\]

Also \( X_3(T) \) and \( X_6(T) \) are proved by the same way as \( X_1(T) \). In the remainder of the proof, we shall consider only the case when \( a_T/T \to 0 \) as \( T \to \infty \). Then there are two cases: \( r > 0 \) or \( r = 0 \). First consider the case \( r > 0 \). This does not imply \( a_T/T \to \delta \) for some \( \delta > 0 \), and the \( a_T \)'s in this case are contained in the set \( \{a_T : 0 < a_T \leq T/(\log T)^r, 0 < r < \infty \} \). Thus from Theorem 1

\[
(2.7) \quad \liminf_{T \to \infty} X_i(T) \geq \sqrt{\frac{r}{1+r}}, \quad i = 1, 2, \ldots, 6, \quad \text{a.s.}
\]

Now let us prove

\[
\liminf_{T \to \infty} X_i(T) \leq \sqrt{\frac{r}{1+r}}, \quad i = 1, 2, \ldots, 6, \quad \text{a.s.}
\]
Set $B_T = \sqrt{1 + \left\{\log \log T / \log(T/a_T)\right\}}$. Then $B_T \to \sqrt{(1 + r)/r}$ as $T \to \infty$ by the condition (iii). Since $\sigma(T)X(t/T) \overset{d}{=} X(t)$, we have, in case $X_2(T)$,

$$X_2(T) = \sup_{0 \leq s \leq a_T} \sup_{0 \leq t \leq T-s} \frac{X(t+s) - X(t)}{\sqrt{2 \log(T/a_T)B_T \sigma(a_T)}} \overset{d}{=} M_2(T)$$

(2.8)

$$:= \sup_{0 \leq s \leq a_T} \sup_{0 \leq t \leq T-s} \frac{\sigma(T)\{X((t+s)/T) - X(t/T)\}}{\sqrt{2 \log(T/a_T)B_T \sigma(a_T)}}$$

$$= \sup_{0 \leq s \leq a_T} \sup_{0 \leq t \leq T-s} \frac{X((t/T) + (s/T)) - X(t/T)}{\sqrt{2 \log(T/a_T)B_T \sigma(a_T/T)}}.$$  

Because we are in the case $h = a_T/T \to 0$ as $T \to \infty$, we have, from Lemma 2 ((2.3) or (2.5))

$$\lim\sup_{h \downarrow 0} \sup_{0 \leq u \leq h} \sup_{0 \leq \nu \leq 1-v} \frac{X(u + \nu) - X(u)}{\sqrt{2 \log(1/h)\sigma(h)}} = 1 \quad \text{a.s.}$$

Thus in (2.8)

$$\lim_{T \to \infty} M_2(T) = \sqrt{\frac{r}{1+r}} \quad \text{a.s.}$$

This implies that

$$\lim_{T \to \infty} X_2(T) = \sqrt{\frac{r}{1+r}} \quad \text{in probability.}$$

Therefore we can find a subsequence $\{T_k : 1 \leq k < \infty\}$ such that

$$\lim_{k \to \infty} X_2(T_k) = \sqrt{\frac{r}{1+r}} \quad \text{a.s.}$$

Thus

$$\liminf_{T \to \infty} X_2(T) \leq \sqrt{\frac{r}{1+r}} \quad \text{a.s.}$$
By (2.7) and (2.9), we have
\[ \liminf_{T \to \infty} X_2(T) = \sqrt{\frac{r}{1 + r}} \quad \text{a.s.} \]

Also, as for the others \( X_i(T), i = 1, 3, 4, 5, 6 \), it is easily proved by the same method as \( X_2(T) \). Consider the next case when \( r = 0 \). Clearly,
\[ \liminf_{T \to \infty} X_i(T) \geq 0, \quad i = 1, 3, 5, \quad \text{a.s.} \]

If we define \( \frac{1}{0} = \infty \), then by the same method as above, we can deduce
\[ \lim_{T \to \infty} M_i(T) = 0, \quad i = 1, 3, 5, \quad \text{a.s.} \]

and
\[ \liminf_{T \to \infty} X_i(T) \leq 0, \quad i = 1, 3, 5, \quad \text{a.s.} \]

By (2.10) and (2.11) we have, for \( r = 0 \),
\[ \lim_{T \to \infty} X_i(T) = 0, \quad i = 1, 3, 5, \quad \text{a.s.} \]

Thus the proof is complete.

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References


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