APPLICATIONS OF RUSCHEWEYH DERIVATIVES

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1. Introduction

Let $A(n)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n = 1, 2, 3, \ldots)$$

which are analytic in the unit disk $U = \{z : |z| < 1\}[8,9]$. An univalent function $f(z)$ belonging to $A(n)$ is said to be starlike of order $\gamma$, $0 \leq \gamma < 1$, if it satisfies

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \gamma$$

for all $z \in U$. We denote this class by $S^*(n, \gamma)$.

An univalent function $f(z)$ belonging to $A(n)$ is called convex of order $\gamma$, $0 \leq \gamma < 1$, if it satisfies

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \gamma$$

for all $z \in U$. We denote this class by $C(n, \gamma)$.

Let $f \in A(n)$ and $g \in S^*(n, \gamma)$, $0 \leq \gamma < 1$. Then we define $f \in K(n, \beta, \gamma)$ if and only if

$$\text{Re} \left( \frac{zf'(z)}{g(z)} \right) > \beta,$$
where $0 \leq \beta < 1$ and $0 \leq \gamma < 1$. Such functions are called close-to-convex functions of order $\beta$ type $\gamma$.

Let $f \in A(n)$ and $g \in C(n, \gamma)$, $0 \leq \gamma < 1$. Then we define $f \in C^*(n, \beta, \gamma)$ if and only if

$$Re \left( \frac{(zf'(z))'}{g'(z)} \right) > \beta,$$

where $0 \leq \beta < 1$.

Let $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$ in $A(n)$. Then the Hadamard product (or convolution) $f \ast g(z)$ of $f(z)$ and $g(z)$ is defined by

$$f \ast g(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k \quad (n = 1, 2, 3, \ldots).$$

By using the Hadamard product, we define, for $\alpha \geq -1$,

$$D^\alpha f(z) = \frac{z}{(1 - z)^{1+\alpha}} \ast f(z)$$

for $f \in A(n)$, $D^\alpha f(z)$ is called the Ruscheweyh derivative and was introduced by Ruscheweyh[11].

We easily note that, for $\alpha \geq -1$,

$$D^\alpha(zf'(z)) = z(D^\alpha f(z))'.$$

2. Main results

In proving our results, we shall need the following lemmas due to Miller and Mocanu [5, 6], and Fukui and Sakaguchi [2].

**Lemma 2.1** [5, 6]. Let $\psi(u, v)$ be a complex function,

$$\psi : D \rightarrow C, \quad D \subset C \times C,$$

where $C$ is a complex plane,

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the function $\psi(u, v)$ satisfies the following conditions:

i) $\psi(u, v)$ is continuous in $D$,

ii) $(1, 0) \in D$ and $Re\{\psi(1, 0)\} > 0$,

iii) $Re\{\psi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ with $v_1 \leq -n(1 + u_2^2)/2$.

Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ be analytic in the unit disk $U$ such that $(p(z), z'p(z)) \in D$ for all $z \in U$. If $Re\{\psi(p(z), zp'(z))\} > 0 \quad (z \in U)$, then $Re\{p(z)\} > 0 \quad (z \in U)$.
LEMMA 2.2 [2]. For a real number \( \alpha (\alpha > -1) \), we have

\[
z(D^\alpha f(z))' = (\alpha + 1)D^{\alpha+1}f(z) - \alpha D^\alpha f(z).
\]

Now we consider the new classes:

\[ S_\alpha^*(n, \gamma) = \{ f \in A(n) : D^\alpha f \in S^*(n, \gamma), \ \alpha \geq -1 \} . \]

\[ C_\alpha(n, \gamma) = \{ f \in A(n) : D^\alpha f \in C(n, \gamma), \ \alpha \geq -1 \} . \]

\[ K_\alpha(n, \beta, \gamma) = \{ f \in A(n) : D^\alpha f \in K(n, \beta, \gamma), \ \alpha \geq -1 \} . \]

\[ C_\alpha^*(n, \beta, \gamma) = \{ f \in A(n) : D^\alpha f \in C^*(n, \beta, \gamma), \ \alpha \geq -1 \} . \]

Above all classes is equal to the classes of Noor[7] when \( n = 1 \), respectively.

We study some properties of these classes and an integral operator for these classes.

Applying the above Lemma 2.1 and Lemma 2.2, we have the following theorem.

**Theorem 2.3.** For \( \alpha \geq 0 \), we get \( S_\alpha^*(n, \gamma) \subset S_{\alpha+1}^*(n, \gamma) \).

**Proof.** Let us define the function \( h(z) \) by

\[
(2.1) \quad \frac{z(D^\alpha f(z))'}{D^\alpha f(z)} = \gamma + (1 - \gamma)h(z),
\]

where \( h(z) = 1 + c_nz^n + c_{n+1}z^{n+1} + \cdots \) is analytic in \( U \). Hence, from Lemma 2.2, we get

\[
\frac{D^{\alpha+1}f(z)}{D^\alpha f(z)} = \frac{1}{\alpha + 1} \left( \frac{z(D^\alpha f(z))'}{D^\alpha f(z)} + \alpha \right)
\]

\[
= \frac{1}{\alpha + 1} \left( (1 - \gamma)h(z) + \gamma + \alpha \right)
\]

or

\[
(2.2) \quad D^{\alpha+1}f(z) = \frac{1}{\alpha + 1} \left( (1 - \gamma)h(z) + \gamma + \alpha \right) D^\alpha f(z).
\]
Differentiating both sides of (2.2) logarithmically and multiplying $z$ to both sides of that equation, we have
\[
\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)} = \frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)} + \frac{(1 - \gamma)zh'(z)}{(1 - \gamma)h(z) + \gamma + \alpha}
\]
\[
= (1 - \gamma)h(z) + \gamma + \frac{(1 - \gamma)zh'(z)}{(1 - \gamma)h(z) + \gamma + \alpha}.
\]

If $f \in S^*_\alpha(n, \gamma)$, then we have
\[
\text{Re} \left( (1 - \gamma)h(z) + \frac{(1 - \gamma)zh'(z)}{(1 - \gamma)h(z) + \gamma + \alpha} \right)
\]
\[
= \text{Re} \left( \frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)} - \gamma \right) > 0.
\]
(2.3)

Defining the function $\psi(u, v)$ by
\[
\psi(u, v) = (1 - \gamma)u + \frac{(1 - \gamma)v}{(1 - \gamma)u + \gamma + \alpha}
\]
where $u = h(z)$ and $v = zh'(z)$, we have
i) $\psi(u, v)$ is continuous in $D = \left( C - \left\{ \frac{x + \alpha}{\gamma + 1} \right\} \right) \times C$,
ii) $(1, 0) \in D$ and $\text{Re}\psi(1, 0) = 1 - \gamma > 0$,
iii) for all $(iu_2, v_1)$ such that $v_1 \leq -n(1 + u_2^2)/2$,
\[
\text{Re}\psi(iu_2, v_1) = \frac{(1 - \gamma)v_1(\gamma + \alpha)}{(\gamma + \alpha)^2 + (1 - \gamma)^2u_2^2} \leq -\frac{1}{2} \frac{(1 - \gamma)n(1 + u_2^2)(\gamma + \alpha)}{(\gamma + \alpha)^2 + (1 - \gamma)^2u_2^2} < 0.
\]

Therefore, the function $\psi(u, v)$ satisfies the conditions in Lemma 2.1. This implies that $\text{Re}(h(z)) > 0$ ($z \in U$), which is equivalent to
\[
\text{Re} \left( \frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)} \right) > \gamma \quad (z \in U).
\]
(2.5)

Hence $f \in S^*_\alpha(n, \gamma)$. 
COROLLARY 2.4. For $\alpha \geq 0$, we get $C_{\alpha+1}(n, \gamma) \subset C_\alpha(n, \gamma)$.

Proof. We easily note that

\begin{equation}
(2.6) \quad f \in C(n, \gamma) \text{ if and only if } zf' \in S^*(n, \gamma).
\end{equation}

By Theorem 2.3 and (2.6), we have that

\[ D^{\alpha+1}(zf'(z)) = z(D^{\alpha+1}f(z))' \in S^*(n, \gamma) \]

if $f \in C_{\alpha+1}(n, \gamma)$. Thus

\[ zf' \in S^*_{\alpha+1}(n, \gamma) \subset S^*_\alpha(n, \gamma). \]

Hence

\[ z(D^\alpha f(z))' = D^\alpha(zf'(z)) \in S^*(n, \gamma). \]

Therefore $f \in C_\alpha(n, \gamma)$.

THEOREM 2.5. For $\alpha \geq 0$, we have

\begin{equation}
(2.7) \quad K_{\alpha+1}(n, \beta, \gamma) \subset K_\alpha(n, \beta, \gamma).
\end{equation}

Proof. Assume that $f \in K_{\alpha+1}(n, \beta, \gamma)$. Then there exists a function $k \in S^*(n, \gamma)$ such that

\begin{equation}
(2.8) \quad \text{Re} \left( \frac{z(D^{\alpha+1}f(z))'}{k(z)} \right) > \beta.
\end{equation}

Letting $k(z) = D^{\alpha+1}g(z)$, we have $g \in S^*_\alpha(n, \gamma) \subset S^*_\alpha(n, \gamma)$, by Theorem 2.3. Hence we have

\[ \text{Re} \left( \frac{z(D^\alpha g(z))'}{D^\alpha g(z)} \right) > \gamma \]

or

\begin{equation}
(2.9) \quad \frac{z(D^{\alpha+1}g(z))'}{D^\alpha g(z)} = (1 - \gamma)h(z) + \gamma,
\end{equation}
where $Re(h(z)) > 0$ ($z \in U$). Now we set

$$\frac{z(D^\alpha f(z))'}{D^\alpha g(z)} = (1 - \beta)p(z) + \beta$$

or

(2.10) $$z(D^\alpha f(z))' = D^\alpha g(z)\{(1 - \beta)p(z) + \beta\},$$

where $p(z) = 1 + p_nz^n + \cdots$. From Lemma 2.2, (1.8) and (2.10), we have

$$\frac{z(D^{\alpha+1} f(z))'}{D^{\alpha+1} g(z)} = \frac{D^{\alpha+1}(z f'(z))}{D^{\alpha+1} g(z)}$$

$$= \frac{1}{\alpha + 1}z(D^\alpha(z f'(z))') + \frac{\alpha}{\alpha + 1}D^\alpha(z f'(z))$$

$$= \frac{z(D^\alpha(z f'(z))') + \alpha \frac{D^\alpha(z f'(z))}{D^\alpha g(z)}}{D^\alpha g(z)}$$

Differentiating both sides of (2.10), we have

$$(z(D^\alpha f(z)))' = (1 - \beta)p'(z)D^\alpha g(z) + (D^\alpha g(z))'((1 - \beta)p(z) + \beta)$$

Hence, from (1.8) we get

$$\frac{z(D^\alpha(z f'(z))')}{D^\alpha g(z)}$$

(2.12) $$= (1 - \beta)zp'(z) + ((1 - \beta)p(z) + \beta)\frac{z(D^\alpha g(z))'}{D^\alpha g(z)}$$

$$= (1 - \beta)zp'(z) + ((1 - \beta)p(z) + \beta)((1 - \gamma)h(z) + \gamma)$$

From (2.11) and (2.12), we have

$$\frac{z(D^{\alpha+1} f(z))'}{D^{\alpha+1} g(z)} = (1 - \beta)p(z) + \beta + \frac{(1 - \beta)zp'(z)}{(1 - \gamma)h(z) + \gamma + \alpha}$$
or

\[ Re \left( (1 - \beta)p(z) + \frac{(1 - \beta)zp'(z)}{(1 - \gamma)h(z) + \gamma + \alpha} \right) \]

(2.13)

\[ = Re \left( \frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}g(z)} - \beta \right) > 0. \]

Define the function \( \psi(u, v) \) by

\[ \psi(u, v) = (1 - \beta)u + \frac{(1 - \beta)v}{(1 - \gamma)h(z) + \gamma + \alpha}. \]

(2.14)

It is clear that the function \( \psi(u, v) \) defined in \( D = C \times C \) by (2.14) satisfies conditions (i) and (ii) of Lemma 2.1. To verify condition (iii),

\[ Re(\psi(zi_2, v_1)) = \frac{(1 - \beta)v_1((1 - \gamma)h_1 + \gamma + \alpha)}{((1 - \gamma)h_1 + \gamma + \alpha)^2 + (1 - \gamma)h_2^2} \]

where \( h(z) = h_1 + ih_2 \) and \( Re h(z) = h_1 > 0 \). By putting \( v \leq \frac{-n(1+u_2^2)}{2} \),

\[ Re(\psi(zi_2, v_1)) \leq \frac{(1 - \beta)n(1 + u_2^2)((1 - \gamma)h_1 + \gamma + \alpha)}{2((1 - \gamma)h_1 + \gamma + \alpha)^2 + (1 - \gamma)h_2^2} < 0. \]

Therefore, the function \( \psi(u, v) \) satisfies the conditions in Lemma 2.1. This implies that \( Re(z) > 0 \) \( (z \in U) \), which is equivalent to

\[ Re \left( \frac{z(D^{\alpha}f(z))'}{D^{\alpha}g(z)} \right) > \gamma \quad (z \in U). \]

Hence \( f \in K_\alpha(n, \beta, \gamma) \).

From Theorem 2.5, (1.8) and the definition of \( C^*_\alpha(n, \beta, \gamma) \), we have

**COROLLARY 2.6.** For \( \alpha \geq 0 \), we have \( C^*_\alpha(n, \beta, \gamma) \subset C^*_\alpha(n, \beta, \gamma) \)

**Proof.** We note that

(2.15) \( f \in C^*(n, \beta, \gamma) \) if and only if \( zf' \in K(n, \beta, \gamma) \).
By Theorem 2.5 and (2.15), we have that
\[ D^{\alpha+1}(zf'(z)) = z(D^{\alpha+1}f(z))' \in K(n, \beta, \gamma) \]
if \( f \in C_{\alpha+1}^*(n, \beta, \gamma) \). Hence
\[ zf' \in K_{\alpha+1}(n, \beta, \gamma) \subset K_{\alpha}(n, \beta, \gamma). \]

Therefore
\[ z(D^\alpha f(z))' = D^\alpha(zf'(z)) \in K(n, \beta, \gamma). \]

It follows that
\[ f \in C_{\alpha}^*(n, \beta, \gamma). \]

Next we define the integral operator \( I_{n,c}(f) \) as
\[ (2.16) \quad I_{n,c}(f) = \frac{c+1}{zc} \int_0^z t^{c-1}f(t)dt \]
for \( f \in A(n) \). The operator \( I_{1,c} \) was studied by Noor[7], Owa and Chen[10]. The operator \( I_{1,m} \) (when \( m \) is positive integers) was studied by Bernardi[1] and \( I_{1,1} \) was investigated by Libera[3] and Livingston[4].

**Lemma 2.7.** If \( f \in A(n) \), then for \( \alpha > -1 \), \( D^{\alpha+1}I_{n,\alpha}(f) = D^\alpha f \).

**Proof.** By simple calculating of (2.16), we get
\[ I_{n,\alpha}(f) = z + \sum_{k=n+1}^{\infty} \frac{\alpha + 1}{\alpha + k} a_k z^k. \]

From (1.7), we have
\[ D^{\alpha+1}I_{n,\alpha}(f) = z + \sum_{k=n+1}^{\infty} \frac{\prod_{j=1}^{k-1}(j + \alpha + 1) \alpha + 1}{(k - 1)! \alpha + k} a_k z^k \]
\[ = z + \sum_{k=n+1}^{\infty} \frac{\prod_{j=1}^{k-1}(j + \alpha)}{(k - 1)!} a_k z^k = D^\alpha f. \]

With the aid of Theorem 2.3 and Lemma 2.7, we have
Theorem 2.8. If \( f \in S^*_\alpha(n, \gamma) \) with \( \alpha \geq 0, 0 \leq \beta < 1 \) and \( 0 \leq \gamma < 1 \), then \( I_{n,\alpha}(f) \) also belongs to \( S^*_\alpha(n, \gamma) \).

Proof. If \( f \in S^*_\alpha(n, \gamma) \), then \( D^\alpha f \in S^*(n, \gamma) \). By Lemma 2.7, we have
\[
D^{\alpha+1}I_{n,\alpha}(f) \in S^*(n, \gamma).
\]
From Theorem 2.3,
\[
I_{n,\alpha}(f) \in S^*_{\alpha+1}(n, \gamma) \subset S^*_\alpha(n, \gamma).
\]

Using Theorem 2.5 and Lemma 2.7, we have the following.

Theorem 2.9. If \( f \in K_\alpha(n, \beta, \gamma) \) with \( \alpha \geq 0, 0 \leq \beta < 1 \) and \( 0 \leq \gamma < 1 \), then \( I_{n,\alpha}(f) \) also belongs to \( K_\alpha(n, \alpha, \beta) \).

Finally, we state the similar results for the classes \( C^*_\alpha(n, \gamma) \) and \( C^*_\alpha(n, \beta, \gamma) \) from Theorem 2.8 and Theorem 2.9.

Corollary 2.10. If \( f \in C^*_\alpha(n, \gamma) \) with \( \alpha \geq 0, 0 \leq \beta < 1 \) and \( 0 \leq \gamma < 1 \), then \( I_{n,\alpha}(f) \) also belongs to \( C^*_\alpha(n, \gamma) \).

Corollary 2.11. If \( f \in C^*_\alpha(n, \beta, \gamma) \) with \( \alpha \geq 0, 0 \leq \beta < 1 \) and \( 0 \leq \gamma < 1 \), then \( I_{n,\alpha}(f) \) also belongs to \( C^*_\alpha(n, \alpha, \beta) \).

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