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HYPERBOLIC CURVATURE AND CONVEX CURVES

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1. Introduction

We begin with a brief introduction to hyperbolic regions in the complex plane **C**. For a general discussion of hyperbolic regions we refer the reader to [1], [5], and [7]. A region Ω in **C** is called hyperbolic if the complement of Ω with respect to **C** contains at least two points. Let D be the open unit disk in **C**. If a region Ω is hyperbolic, then, by the uniformization theorem [3, p.39], there is a holomorphic universal covering projection φ of D onto Ω. The collection of all holomorphic universal covering projections of D onto Ω consists of the functions $\varphi \circ T$, where $T \in Aut(D)$, the group of conformal automorphisms of D. The hyperbolic metric on D in **C** is defined by

$$\lambda_D(z)\,|dz| = \frac{2\,|dz|}{1-|z|^2}.$$

The density $\lambda_\Omega(z)$ of the hyperbolic metric $\lambda_\Omega(z)\,|dz|$ on a hyperbolic region Ω is obtained from

$$\lambda_\Omega(\varphi(z))\,|\varphi'(z)| = \lambda_D(z),$$

where φ is any holomorphic universal covering projection of D onto Ω. We note that the density of the hyperbolic metric is independent of the choice of the holomorphic universal covering projection. The hyperbolic metric is invariant under holomorphic covering projections: If $f : \Omega \to \Delta$ is a holomorphic covering projection, then

$$\lambda_\Delta(f(z))\,|f'(z)|\,|dz| = \lambda_\Omega(z)\,|dz|.$$

In particular, the hyperbolic metric is a conformal invariant.

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Let us recall the definition of hyperbolic curvature. For more details, see [4], [8], and [9]. If γ is a C^2 curve in a hyperbolic region Ω with parametrization $z = z(t)$, then the hyperbolic curvature of γ at the point $z = z(t)$ is given by

$$K_\Omega(z,\gamma) = \frac{1}{\lambda_\Omega(z)}\left[K_e(z,\gamma) + 2\mathrm{Im}\left\{\frac{\partial \log \lambda_\Omega(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\}\right],$$

where

$$K_e(z,\gamma) = \frac{1}{|z'(t)|}\mathrm{Im}\left[\frac{z''(t)}{z'(t)}\right]$$

denotes the euclidean curvature of γ at $z = z(t)$. Since $\lambda_D(z) = 2/\left(1-|z|^2\right)$, we have

$$K_D(z,\gamma) = \frac{1}{2}\left(1-|z|^2\right)K_e(z,\gamma) + \mathrm{Im}\left[\frac{\overline{z(t)}z'(t)}{|z'(t)|}\right]$$

for a C^2 curve $\gamma : z = z(t)$ in D. Because the hyperbolic metric is invariant under holomorphic covering projections, the same is true of the hyperbolic curvature: For a holomorphic covering projection $f : \Omega \to \Delta$ and a C^2 curve in Ω, we have

$$K_\Omega(z,\gamma) = K_\Delta(f(z), f\circ\gamma).$$

In particular, hyperbolic curvature is a conformal invariant. A C^2 curve with parametrization $z = z(t)$, $t \in [a,b]$ is said to be convex provided that

$$\frac{d}{dt}\arg z'(t) = \mathrm{Im}\left[\frac{z''(t)}{z'(t)}\right] \geq 0,\ t\in[a,b].$$

Note that γ is convex if and only if $K_e(z,\gamma) \geq 0$ for all $z \in \gamma$.

In this paper, we investigate relationships between the convexity of all images of a curve under conformal mappings of the open unit disk D and the hyperbolic curvature of that curve.

2. Main results

Suppose f is holomorphic and univalent in the open unit disk D and normalized by $f(0) = f'(0) = 1$; say

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Then deBranges' Theorem [2] asserts that $|a_n| \leq n$ for $n = 2, 3, \cdots$ with equality if and only if

$$f(z) = \frac{z}{(1 - e^{i\theta} z)^2} = z + \sum_{n=2}^{\infty} n e^{i(n-1)\theta} z^n, \ \theta \in \mathbf{R}.$$

It is well known that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is convex, then $|a_n| \leq 1$ for $n = 2, 3, \cdots$ with equality if and only if

$$f(z) = \frac{z}{1 - e^{i\theta} z} = z + \sum_{n=2}^{\infty} e^{i(n-1)\theta} z^n, \ \theta \in \mathbf{R}.$$

The following result is well known (see [6] and [9]). We include a proof for the convenience of the reader

THEOREM 1. *Let Ω be a hyperbolic region in* $\mathbf{C}$.
(i) If Ω is simply connected, then for $z \in \Omega$

$$\left| \frac{\partial \log \lambda_\Omega(z)}{\partial z} \right| \leq \lambda_\Omega(z).$$

(ii) If Ω is convex, then for $z \in \Omega$

$$\left| \frac{\partial \log \lambda_\Omega(z)}{\partial z} \right| \leq \frac{1}{2} \lambda_\Omega(z).$$

Proof. We begin by establishing a general formula. Fix $a \in \Omega$ and let $z = \varphi(w)$ be a holomorphic universal covering projection of $(D, 0)$ onto (Ω, a). From the identity

$$\lambda_\Omega(\varphi(w)) |\varphi'(w)| = \frac{2}{1 - |w|^2}, \ w \in D$$

we obtain

$$\log \lambda_\Omega (\varphi (w)) + \frac{1}{2} \log \varphi' (w) + \frac{1}{2} \log \overline{\varphi' (w)} = \log 2 - \log (1 - w\overline{w}).$$

We apply the operator $\partial/\partial w$ to both sides of this identity and obtain

$$\frac{\partial \log \lambda_\Omega (\varphi (w))}{\partial z} \varphi' (w) + \frac{1}{2} \frac{\varphi'' (w)}{\varphi' (w)} = \frac{\overline{w}}{1 - w\overline{w}}.$$

For $w = 0$, this gives

$$\frac{\partial \log \lambda_\Omega (a)}{\partial z} \varphi' (0) = -\frac{1}{2} \frac{\varphi'' (0)}{\varphi' (0)},$$

so that

$$\left| \frac{\partial \log \lambda_\Omega (a)}{\partial z} \right| = \frac{1}{4} \frac{|\varphi'' (0)|}{|\varphi' (0)|} \lambda_\Omega (a) \tag{1}$$

since $\lambda_\Omega (a) = 2/ |\varphi' (0)|$. If Ω is simply connected, then $\varphi : (D, 0) \to (\Omega, a)$ is a conformal mapping. The function

$$z = \psi (w) = [\varphi (w) - \varphi (0)] / \varphi' (0) = w + a_2 w^2 + \cdots$$

is a normalized univalent function in D, so $|a_2| \leq 2$, or

$$\left| \frac{\varphi'' (0)}{\varphi' (0)} \right| = \left| \frac{\psi'' (0)}{\psi' (0)} \right| \leq 4.$$

This inequality in conjunction with (1) establishes the inequality in (*i*). If Ω is convex, then the function $[\varphi (w) - \varphi (0)] / \varphi' (0) = w + a_2 w^2 + \cdots$ is a normalized convex univalent function in D, so $|a_2| \leq 1$, or $|\varphi''(0)/\varphi'(0)| \leq 2$. This inequality in conjunction with (1) establishes the inequality in (*ii*).

THEOREM 2. *Let γ be a curve in a hyperbolic region Ω. (i) If Ω is simply connected and $K_\Omega (z, \gamma) \geq 2$ for all $z \in \gamma$, then the curve γ is convex. The constant 2 is sharp. (ii) If Ω is convex and $K_\Omega (z, \gamma) \geq 1$ for all $z \in \gamma$, then the curve γ is convex. The constant 1 is sharp.*

Proof. (*i*) From the definition of hyperbolic curvature, we obtain

$$
\begin{aligned}
K_e(z,\gamma) &= K_\Omega(z,\gamma)\lambda_\Omega(z) - 2\mathrm{Im}\left\{\frac{\partial \log \lambda_\Omega(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\} \\
&\geq K_\Omega(z,\gamma)\lambda_\Omega(z) - 2\left|\frac{\partial \log \lambda_\Omega(z)}{\partial z}\right|.
\end{aligned} \tag{2}
$$

Since Ω is simply connected and $K_\Omega(z,\gamma) \geq 2$, it follows from Theorem 1 that

$$K_e(z,\gamma) \geq 2\lambda_\Omega(z) - 2\lambda_\Omega(z) = 0$$

Thus, the curve γ is convex. For the sharpness of the constant 2, let $\Omega = \mathbf{C} - (-\infty, -1/4]$ and $\gamma : z(t) = -it$, $t \in \mathbf{R}$. Clearly, $K_e(0,\gamma) = 0$. We shall show that $K_\Omega(0,\gamma) = 2$. The function

$$z = \varphi(w) = \frac{w}{(1-w)^2} = w + 2w^2 + \cdots$$

is a conformal mapping of D onto Ω. In particular, $\lambda_\Omega(0) = 2/\varphi'(0) = 2$. We have

$$\frac{\partial \log \lambda_\Omega(0)}{\partial z} = -\frac{1}{2}\frac{1}{\varphi'(0)}\frac{\varphi''(0)}{\varphi'(0)} = -2$$

Consequently,

$$
\begin{aligned}
K_\Omega(0,\gamma) &= \frac{1}{\lambda_\Omega(0)}\left[K_e(0,\gamma) + 2\mathrm{Im}\left\{\frac{\partial \log \lambda_\Omega(0)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\}\right] \\
&= -2\mathrm{Im}\left\{\frac{z''(t)}{|z'(t)|}\right\} = -2\mathrm{Im}\{-i\} = 2.
\end{aligned}
$$

(*ii*) If Ω is convex and $K_\Omega(z,\gamma) \geq 1$, then, from (2) and Theorem 1, we obtain $K_e(z,\gamma) \geq 0$. We note that if $\Omega = \{z : \mathrm{Im}(z) > 0\}$ and $\gamma : z(t) = iy_0 + t$, $y_0 > 0$, then $K_\Omega(z(t),\gamma) = 1$ and $K_e(z(t),\gamma) = 0$. This completes the proof of (*ii*).

Flinn and Osgood [4] established the condition on the hyperbolic metric $K_\Omega(z,\gamma)$ insures that $f \circ \gamma$ is convex for any conformal mapping f of Ω. We give a new proof of their result in the case $\Omega = D$.

THEOREM 3. *Let γ be a curve in the open unit disk D. (i) $K_D(z,\gamma) \geq 2$ for all $z \in \gamma$ if and only if the curve $\varphi \circ \gamma$ is convex for every conformal mapping φ of D. (ii) $K_D(z,\gamma) \geq 1$ for all $z \in \gamma$ if and only if the curve $\varphi \circ \gamma$ is convex for every convex conformal mapping φ of D.*

Proof. (i) Suppose $K_D(z,\gamma) \geq 2$ for all $z \in \gamma$. Let $\varphi : D \to \Omega$ be a conformal mapping. Then by conformal invariance of hyperbolic curvature

$$K_\Omega(\varphi(z), \varphi \circ \gamma) = K_D(z,\gamma) \geq 2.$$

Because Ω is simply connected, Theorem 2 yields $\varphi \circ \gamma$ is convex. Conversely, suppose $\varphi \circ \gamma$ is convex for every conformal mapping of D. Because of the invariance of hyperbolic curvature under $Aut(D)$, there is no harm in assuming that $z = 0$. Because hyperbolic curvature is invariant under rotations of D, there is no harm in assuming that $-i$ is the unit tangent to γ at the origin, that is, $-i = z'(t_0) / |z'(t_0)|$, where $z(t_0) = 0$. Let

$$w = \varphi(z) = \frac{z}{(1-z)^2} = z + 2z^2 + \cdots$$

as in Theorem 2. Then φ is a conformal mapping of D onto $\Omega = \mathbf{C} - (-\infty, -1/4]$ and $w'(t_0) = \varphi'(0)z'(t_0) = z'(t_0)$. Therefore,

$$\begin{aligned} K_\Omega(0, \varphi \circ \gamma) &= \frac{1}{\lambda_\Omega(0)}\left[K_e(0,\varphi \circ \gamma) + 2\mathrm{Im}\left\{\frac{\partial \log \lambda_\Omega(0)}{\partial w}\frac{w'(t_0)}{|w'(t_0)|}\right\}\right] \\ &\geq \frac{2}{\lambda_\Omega(0)}\mathrm{Im}\left\{\frac{\partial \log \lambda_\Omega(0)}{\partial w}\frac{w'(t_0)}{|w'(t_0)|}\right\} = 2. \end{aligned}$$

Because hyperbolic curvature is a conformal invariant,

$$K_D(0,\gamma) = K_\Omega(0, \varphi \circ \gamma) \geq 2.$$

This completes the proof of (i).

(ii) Suppose $K_D(z,\gamma) \geq 1$ for all $z \in \gamma$. Let φ be a convex conformal mapping of D onto a convex region Ω. Since hyperbolic curvature is a conformal invariant,

$$K_\Omega(\varphi(z), \varphi\circ\gamma) = K_D(z,\gamma) \geq 1.$$

Because Ω is convex, Theorem 2 yields $\varphi\circ\gamma$ is convex. Conversely, suppose $\varphi\circ\gamma$ is convex for every conformal mapping of D. As in (i), we may assume that $z = 0$ and that $-i = z'(t_0)/|z'(t_0)|$, where $z(t_0) = 0$. We note that

$$w = \varphi(z) = \frac{z}{1-z} = z + z^2 + \cdots$$

maps D conformally onto the convex region $\Omega = \left\{w : \mathrm{Re}(w) > -\frac{1}{2}\right\}$. Now $\lambda_\Omega(0) = 2/\varphi'(0) = 2$ and

$$\frac{\partial \log \lambda_\Omega(0)}{\partial w} = -\frac{1}{2}\frac{1}{\varphi'(0)}\frac{\varphi''(0)}{\varphi'(0)} = -1$$

Because $K_e(0, \varphi\circ\gamma) \geq 0$, we obtain

$$K_\Omega(0, \varphi\circ\gamma) \geq \frac{2}{\lambda_\Omega(0)}\mathrm{Im}\left\{\frac{\partial \log \lambda_\Omega(0)}{\partial w}\frac{w'(t_0)}{|w'(t_0)|}\right\} = \mathrm{Im}\{i\} = 1$$

since $w'(t_0) = \varphi'(0)z'(t_0) = z'(t_0)$. This yields $K_D(0,\gamma) \geq 1$ since hyperbolic curvature is a conformal invariant.

Let γ be a positively oriented circle in the open unit D with center 0 and radius $r \in (0,1)$. A parametrization of γ is $z = z(t) = re^{it}$, $0 \leq t \leq 2\pi$. We have

$$\begin{aligned} K_D(z,\gamma) &= \frac{1}{2}\left(1-|z|^2\right)K_e(z,\gamma) + \mathrm{Im}\left[\frac{\overline{z(t)}z'(t)}{|z'(t)|}\right] \\ &= \frac{1-r^2}{2}\frac{1}{r} + r = \frac{1}{2}\left(r + \frac{1}{r}\right). \end{aligned}$$

Note that $r + \frac{1}{r} > 2$. Since hyperbolic curvature is invariant under $Aut(D)$, it follows that any circle in D has hyperbolic curvature strictly larger than 1.

The hyperbolic distance between the points a and b on the open unit disk D is

$$d_h(a,b) = 2\tanh^{-1}\left|\frac{a-b}{1-a\bar{b}}\right|.$$

The hyperbolic disk in D with center $a \in D$ and hyperbolic radius ρ, $0 < \rho \le \infty$, is defined by

$$D_h(a,\rho) = \{z \in D : d_h(a,z) < \rho\}.$$

We note that the hyperbolic disk $D_h(a,\rho)$ is a euclidean disk $D(c,r) = \{z : |z-c| < r\}$, where

$$c = \frac{1-\left(\tanh\frac{\rho}{2}\right)^2}{1-\left(\tanh\frac{\rho}{2}\right)^2|a|^2}a,$$

$$r = \tanh\frac{\rho}{2}\frac{1-|a|^2}{1-\left(\tanh\frac{\rho}{2}\right)^2|a|^2}.$$

If $a = 0$, then $r = \tanh\frac{\rho}{2}$. So hyperbolic curvature of any hyperbolic circle in D with hyperbolic radius ρ is $\frac{1}{2}\left(r + \frac{1}{r}\right) = \coth\rho$.

THEOREM 4. *Let $D_h(a,\rho)$ denote the hyperbolic disk in D with center a and hyperbolic radius ρ. If φ is a conformal mapping of D and $0 < \rho \le \frac{1}{2}\log 3$, then $\varphi(D_h(a,\rho))$ is convex.*

Proof. Let γ denote the positively oriented boundary of $D_h(a,\rho)$. The hyperbolic curvature of γ is

$$K_D(z,\gamma) = \coth(\rho) \ge \coth\left(\frac{1}{2}\log 3\right) = 2.$$

Theorem 3 gives $\varphi \circ \gamma$ is a convex closed curve. Therefore, its interior $\varphi(D_h(a,\rho))$ is convex.

Let γ be the positively oriented circle $\{z : |z-a| = 1-a\}$, where $0 < a < 1$. This circle is internally tangent to the unit circle at the point 1. A parametrization for γ is

$$z(t) = a + (1-a)e^{it},\ 0 \le t \le 2\pi.$$

Note that $z(0) = z(2\pi) = 1 \notin D$. A straightforward calculation yields $K_D(z,\gamma) \doteq 1$. Because hyperbolic curvature is a conformal invariant, hyperbolic curvature of any oricycle, that is, a circle internally tangent to the unit circle, is always 1.

THEOREM 5. *Let Δ be any disk in the open unit disk D. If φ is a conformal mapping of D onto a convex region, then $\varphi(\Delta)$ is convex.*

Proof. Let γ denote the positively oriented boundary of Δ. Then either γ is a circle in D and so $K_D(z,\gamma) > 1$ or γ is an oricycle in D and $K_D(z,\gamma) = 1$. Thus, $K_D(z,\gamma) \geq 1$ in all cases. Theorem 3 yields $\varphi \circ \gamma$ is convex. This completes the proof.

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