WEAK TYPE \((\phi, \phi)\) INEQUALITY FOR GENERALIZED MAXIMAL OPERATORS ON SPACES OF HOMOGENEOUS TYPE

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1. Introduction

Let \(f\) be a locally integrable function on \(\mathbb{R}^n\). We define the maximal operator

\[
\mathcal{M}f(x,t) = \sup_Q \frac{1}{|Q|} \int_Q |f(x)| \, dx, \quad x \in \mathbb{R}^n, \quad t \geq 0,
\]

where the supremum is taken over the cubes \(Q\) in \(\mathbb{R}^n\), containing \(x\) and having side length at least \(t\). It is well known that this maximal operator \(\mathcal{M}\) controls Poisson integral. Several authors was studied this maximal operators \((3),(8),(9))\).

For given positive measure \(v\) on \(\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t \geq 0\}\), Carleson \([1]\) showed that \(\mathcal{M}\) is bounded from \(L^p(\mathbb{R}^n, dx)\) into \(L^p(\mathbb{R}^{n+1}_+, dv)\) if and only if \(v\) satisfies the "Carleson condition" \(\sup_{x \in Q} \frac{v(\tilde{Q})}{|Q|} \leq C\), where \(\tilde{Q}\) denotes the cubes in \(\mathbb{R}^{n+1}_+\) with the cube \(Q\) as its base. Later, Fefferman-Stein \([3]\) proved that \(\mathcal{M}\) is bounded from \(L^p(\mathbb{R}^n, w(x)dx)\) into \(L^p(\mathbb{R}^{n+1}_+, dv)\) if \(\sup_{x \in Q} \frac{v(\tilde{Q})}{|Q|} \leq Cw(x)\) a.e. \(x\), where \(w\) is a weight in \(\mathbb{R}^n\). Also, Ruiz\([6]\) found the condition

\[
\frac{\mu(\tilde{Q})}{|Q|} \left( \frac{1}{|Q|} \int_Q v^{1-p'}(x) \, dx \right)^{p-1} \leq C
\]

to be necessary and sufficient for the boundedness of the operator \(\mathcal{M}\) from \(L^p(\mathbb{R}^n, v(x)dx)\) into \(\text{weak} - L^p(\mathbb{R}^{n+1}_+, \mu)\). In 1993, Jie-Cheng Chen \([2]\) found the conditions on \((\mu,v)\) for \(\mathcal{M}\) to be bounded from

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$L_\phi (R^n, v(x)dx)$ to weak $- L_\phi (R^{n+1}, d\mu)$. On the other hand, Sueiro [9] studied a certain maximal operator defined on space of homogeneous type $X$.

In this paper, we extend the result of Jie-Cheng Chen in [2]. So we can get the conditions for the maximal operator $\mathcal{M}_{\Omega, \gamma}$ to be bounded from $L_\phi (X, v(x)dx)$ to weak $- L_\phi (X \times [0, \infty), d\mu)$.

2. Preliminaries

**Definition 2.1.** Let $X$ be a topological space and let $d : X \times X \to [0, \infty)$ be a map satisfying:

(i) $d(x, x) = 0; d(x, y) > 0 \quad \text{if} \quad x \neq y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq K [d(x, y) + d(y, z)]$, where $K$ is some fixed constant. Assume further that

(iv) the balls $B(x, r) = \{y : d(x, y) < r\}$ form a basis of open neighborhood at $x \in X$ and that $\mu$ is a Borel measure on $X$ such that

(v) $0 < \mu(B(x, 2r)) \leq A \mu(B(x, r)) < \infty$, where $A$ is some fixed constant. Then the triple $(X, d, \mu)$ is called a space of homogeneous type ([9]).

**Remark 2.2.** Properties (iii) and (v) will be referred to as the "triangle inequality" and the "doubling property", respectively. Note that the condition (v) is equivalent with the fact that for every $c > 0$, there exists $A_c < \infty$ such that $\mu(B(x, cr)) \leq A_c \mu(B(x, r))$ for all $x \in X$ and $r > 0$.

**Definition 2.3.** Assume for each $x \in X$, we are given a set $\Omega_x \subset X \times [0, \infty)$. Let $\Omega$ denote the family $\{\Omega_x : x \in X\}$. For each $t \geq 0$ and $\alpha > 0$, set

$$\Omega_{(x, t)} = \Omega_x \cap (X \times [t, \infty))$$

and

$$\mathcal{R}_\alpha (x, t) = \{(y, r) \in X \times [0, \infty) : \Omega_{(y, r)}(t) \cap B(x, \alpha t) \neq \phi\},$$

where $\Omega_{(y, r)}(t) = \{z \in X : (z, t) \in \Omega_{(y, r)}\}$ is the cross-section of $\Omega_{(y, r)}$ at height $t$. A nonnegative and locally integrable function $w$ on $X$ is called a weight.
Now, we recall the basic terminology and results concerning \(N\)-function which will be used in this paper.

An \(N\)–function is a continuous and convex function \(\phi : [0, \infty) \to \mathbb{R}\) with \(\phi(0) > 0\), \(s > 0\), \(s^{-1}\phi(s) \to 0\) for \(s \to 0\) and \(s^{-1}\phi(s) \to \infty\) for \(s \to \infty\). An \(N\)-function \(\phi\) has the representation \(\phi(s) = \int_0^s \varphi\), where \(\varphi : [0, \infty) \to \mathbb{R}\) is continuous from the right, non-decreasing such that \(\varphi(s) > 0\), \(s > 0\), \(\varphi(0) = 0\) and \(\varphi(s) \to \infty\) for \(s \to \infty\). This \(\varphi\) will be called the density function of \(\phi\). Associated to \(\phi\) we have the function \(\rho : [0, \infty) \to \mathbb{R}\) defined by \(\rho(t) = \sup\{s : \phi(s) \leq t\}\).

We will call \(\rho\) the generalized inverse of \(\phi\). Also, the \(N\)-function \(\psi\) defined by \(\psi(t) = \int_0^t \rho\) is called the complementary \(N\)–function of \(\phi\). An \(N\)-function \(\phi^*\) is said to satisfy the \(\Delta_2\)-condition in \([0, \infty)\) if 
\[
\sup_{s > 0} \frac{\phi(2s)}{\phi(s)} < \infty.
\]
If \(\varphi\) is the density function of \(\phi\), then \(\phi\) satisfies \(\Delta_2\) if and only if there exists a constant \(\alpha > 1\) such that \(s\phi(s) < \alpha\varphi(s), s > 0\).

If \((\mathcal{Y}, \mathcal{M}, \mu)\) is a \(\sigma\)-finite measure space. We denote by \(\mathcal{M}\) the space of \(\mathcal{M}\)-measurable and \(\mu\) are finite functions from \(\mathcal{Y}\) to \(\mathbb{R}\) (or to \(C\)). If \(\phi\) is an \(N\)-function, then the Orlicz spaces \(L^\phi(\mu) = L^\phi(\mathcal{Y}, \mathcal{M}, \mu)\) and \(L^{\phi^*}(\mu) = L^{\phi^*}(\mathcal{Y}, \mathcal{M}, \mu)\) are defined by
\[
L^\phi(\mu) = \{f \in \mathcal{M} : \int_X \phi(|f|)d\mu < \infty\}
\]
\[
L^{\phi^*}(\mu) = \{f \in \mathcal{M} : fg \in L_1(\mu) \text{ for all } g \in L^\psi\}
\]
respectively, where \(\psi\) is the complementary \(N\)-function of \(\phi\). Then the Orlicz space \(L^{\phi^*}(\mu)\) is a Banach space with the norms \(\|f\|_\phi = \sup\{\int_\mathcal{Y} |fg|d\mu : g \in S_\psi\}\), where \(S_\psi = \{g \in L^\psi : \int_\mathcal{Y} \psi(|g|)d\mu \leq 1\}\), and \(\|f\|_{\psi^*} = \inf\{\lambda > 0 : \int_\mathcal{Y} \phi(\frac{|f|}{\lambda})d\mu \leq 1\}\), which are called the Orlicz norm and Luxemburg norm, respectively.

In fact both norms are equivalent, actually \(\|f\|_{\psi^*} \leq \|f\|_\phi \leq 2\|f\|_{\psi^*}\) H"older inequality asserts that for every \(f \in L^{\phi^*}(\mu)\) and every \(g \in L^\psi(\mu)\), we have
\[
\|fg\|_1 \leq \|f\|_{\psi^*}\|g\|_{\psi^*},
\]
where \(\phi\) and \(\psi\) are complementary \(N\)-functions. The proof of above results can be found in [4],[5].

For a nonnegative measure \(\nu\) on \(X \times [0, \infty)\) and a weight \(w\) on \(X\), we define the generalized maximal operator on space of homogenous type.
DEFINITION 2.4. Assume that we have a family \( \{ Q_x : x \in X \} \). For \( f \in L^1_{loc}(X, d\mu) \) and \( x \in X, t \geq 0 \), set

\[
M_{\Omega, \gamma} f(x) = \sup_{(y,s) \in \Omega(x,t)} \frac{\gamma(\mu(B(y,s)))}{\mu(B(y,s))} \int_{B(y,s)} |f(x)| d\mu,
\]

where \( \gamma : (0, \infty) \to (0, \infty) \) is essentially nondecreasing. i.e., there is a positive constant \( C \) for which \( \gamma(t) \leq C \gamma(s) \) for \( t \leq s \) and

\[
(2.1) \quad \lim_{t \to \infty} \frac{\gamma(t)}{t} = 0.
\]

REMARK 2.5. In above, the condition (2.1) is necessary to rule out examples such as \( \gamma(t) = t^m, m > 1 \). In this case, \( M_{\Omega, \gamma} f(x) = \infty \) for all \( x \), if we consider \( f \equiv \delta \), the point mass at the origin.

DEFINITION 2.6. Let \( f \in L^1_{loc}(X, d\mu) \) and \( \lambda > 0 \). An operator \( M_{\Omega, \gamma} \) is of weak type \((\phi, \phi)\) with respect to \((v, w)\) if there is a constant \( C \) so that

\[
v\{(x,t) \in X \times [0, \infty) : M_{\Omega, \gamma} f > \lambda \} \leq \frac{C}{\phi(\lambda)} \int_X \phi(|f|) v d\mu.
\]

DEFINITION 2.7. Let \( \varphi \) be the density function of the \( N \)-function \( \phi \) and \( \rho \) be the generalized inverse of \( \varphi \). A pair \((v, w)\) is said to satisfy the condition \( A_\phi(\Omega) \) if there are constants \( C \) and \( \alpha > 0 \) such that

\[
(\frac{1}{\mu(B)} \int_{B(x,r)} \varepsilon v d\mu) \varphi(\frac{1}{\mu(B)} \int_B \rho(\frac{1}{\varepsilon w}) d\mu) \leq C
\]

for every ball \( B(x, r) \) in \( X \) and every \( \varepsilon > 0 \).

In this paper, we shall always assume that \( \phi \), together with its complementary \( N \)-function \( \rho \), satisfy \( \Delta_2 \)-condition. Also, the letter \( C \) denotes a constant which need not be the same at each occurrence.
3. Results

**Lemma 3.1** [9]. Let $E$ be a bounded subset of $X$ and for each $x \in X$. Let $r(x)$ be a positive number for each $x \in E$. Then there is a sequence of disjoint balls $B(x_t, r(x_t)), x_t \in E$ such that the balls $B(x_t, 4Kr(x_t))$ cover $E$, where $K$ is the constant in the triangle inequality. Furthermore, every $x \in E$ is contained in some ball $B(x_t, 4Kr(x_t))$ satisfying $r(x) \leq 2r(x_t)$.

**Lemma 3.2** [2]. For any $N$-function $\phi$, $t \leq \varphi(\rho(t))$ and $\phi(t) \leq t\varphi(t)$. If $\phi$ satisfies $\Delta_2$-condition, then $\varphi(\rho(t)) \leq C_\phi t$ and $\phi(t) \geq t\varphi(t)/C_\phi$.

**Theorem 3.3.** Assume that $\Omega$ satisfies the condition that if $x \in X, (y, r) \in \Omega_\delta$ and $k \geq r$, then $(y, r) \in \Omega_\delta$. Then $\mathcal{M}_{\Omega, \gamma}$ is of weak type $(\phi, \phi)$ with respect to $(v, w)$ if and only if $(v, w)$ satisfies $A_\phi(\Omega)$.

**Proof.** Suppose that $\mathcal{M}_{\Omega, \gamma}$ is of weak type $(\phi, \phi)$ with respect to $(v, w)$. If $(x_0, t) \in \mathcal{R}_\alpha(x, t)$, then $\Omega_{(x_0, t)}(r) \cap B(x, \alpha r) \neq \emptyset$. And so we can choose $y \in \Omega_{(x_0, t)}(r) \cap B(x, \alpha r)$. From the triangle inequality,

$$B(x, r) \subseteq B(y, K(\alpha + 1)r) \subseteq B(x, (K^2\alpha + K\alpha + K^2)r).$$

Let $f$ be a nonnegative measurable function on $X$. Let

$$f_{B(y, r), \gamma} = \gamma(\mu(B(y, r))) \int_{B(y, r)} f \, d\mu.$$

Since $(y, K(\alpha + 1)r) \in \Omega_{(x_0, t)}$ by the hypothesis, we have

$$f_{B(y, K(\alpha + 1)r), \gamma} \leq \mathcal{M}_{\Omega, \gamma}(f \cdot \chi_{B(y, K(\alpha + 1)r)})(x_0, t).$$

Putting $\lambda = f_{B(y, K(\alpha + 1)r), \gamma}$ and $E_\lambda = \{\mathcal{M}_{\Omega, \gamma}(f \cdot \chi_{B(y, K(\alpha + 1)r)}) > \lambda\}$, then the previous argument shows that $\mathcal{R}_\alpha(x, r) \subseteq E_\lambda$ and so

$$\nu(\mathcal{R}_\alpha(x, r)) \leq \nu(E_\lambda) \leq \frac{C}{\phi(\lambda)} \int_{B(y, K(\alpha + 1)r)} \phi(|f|)w \, d\mu.$$
Hence if we invoke (1) and the doubling property of $\mu$,

\[
v(\mathcal{R}_\alpha(x, r)) \phi\left(\frac{1}{\mu(B(x, r))} \int_{B(x, r)} f \, d\mu \right) \\
\leq v(\mathcal{R}_\alpha(x, r)) \phi\left(\frac{1}{\mu(B(x, K(\alpha+1) r))} \int_{B(x, K(\alpha+1) r)} f \, d\mu \right) \\
\leq C \int_{B(x, K(\alpha+1) r)} \phi(|f|) \, w \, d\mu.
\]

Replacing $f$ by $\rho(\frac{1}{w}) \cdot \chi_{B(x, r)}$ and using Lemma 3.2,

\[
v(\mathcal{R}_\alpha(x, r)) \phi\left(\frac{1}{\mu(B(x, r))} \int_{B(x, r)} \rho(\frac{1}{w}) \, d\mu \right) \leq C [C \phi \int_{B(x, r)} \rho(\frac{1}{w}) \, d\mu].
\]

So $(v, w)$ satisfies $A_\phi(\Omega)$ condition with constant $\alpha C$. Conversely, suppose $(v, w)$ satisfies the condition $A_\phi(\Omega)$. We follow the idea of Sueiro [9]. For each $\lambda > 0$, define

\[E_\lambda = \{(x, t) \in X \times [0, \infty) : M_{\Omega, \gamma} f(x, t) > \lambda\},\]

\[E_\lambda' = \{x \in X : \sup_{r > 0} \frac{\gamma(\mu(B(x, r)))}{\mu(B(x, r))} \int_{B(x, r)} |f| \, d\mu > \lambda\}.
\]

And for each $x \in E_\lambda'$, if we put

\[r(x) = \sup\{r > 0 : \frac{\gamma(\mu(B(x, r)))}{\mu(B(x, r))} \int_{B(x, r)} |f| \, d\mu > \lambda\},
\]

then there exists a finite positive real number $r(x)$ such that $\frac{\gamma(\mu(B(x, r)))}{\mu(B(x, r(x)))} \int_{B(x, r(x))} |f| \, d\mu \geq \lambda$. Assume for a moment that $E_\lambda'$ is bounded. Then by Lemma 3.1, there exists a sequence of balls $\{B(x_i, r_i)\}$, where $x_i \in E_\lambda'$, $r_i = r(x_i)$ such that $E_\lambda' \subset \cup_i B(x_i, 4Kr_i)$ and $\frac{\gamma(\mu(B(x, r)))}{\mu(B(x, r))} \int_{B(x, r)} |f| \, d\mu \geq \lambda$. Now we want to verify $E_\lambda \subset \cup_i \mathcal{R}_\alpha(x_i, 4Kr_i)$. To do this, let $(x, t) \in E_\lambda$. Then $\frac{\gamma(\mu(B(x, r)))}{\mu(B(y, r))} \int_{B(y, r)} |f| \, d\mu > \lambda$ for some $(y, r) \in \Omega(x, t)$. So $y \in E_\lambda'$ and $t \leq r \leq r(y)$. By Lemma 3.1,
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\(y \in B(x, 4Kr_i)\) for some \(i\) such that \(r(y) \leq 2r_i\). Hence we may assume \(\alpha < 2K\). Consequently, \(t \leq r \leq r(y) \leq 2r_i < \frac{4Kr_i}{\alpha}\) and so

\((y, \frac{4Kr_i}{\alpha}) \in \Omega_{(x,t)}\). Since \(y \in B(x, \alpha(\frac{4K}{\alpha})r_i)\), it follows that

\[ y \in \Omega_{(x,t)}(\frac{4Kr_i}{\alpha}) \cap B(x, \alpha(\frac{4K}{\alpha})r_i), \]

and thus \((x, t) \in \mathcal{R}_\alpha(x_i, \frac{4Kr_i}{\alpha})\), as so it holds. Now by Hölder inequality, we have

\[ \int_{B(x,r)} |f|d\mu \leq 2\|f\chi_{B(x,r)}\|_{\phi, \varepsilon w} \|\varepsilon w\|^{-1} \chi_{B(x,r)}\|_{\phi, \varepsilon w}. \]

On the other hand, for every \(\eta > 0\), we get

\[ \int_X \psi(\eta \varepsilon w)^{-1} \chi_{B(x,r)} \varepsilon w d\mu \leq \int_{B(x,r)} \eta^{-1} \rho((\eta \varepsilon w)^{-1}) d\mu \]

\[ \leq \eta^{-1} \mu(B) \rho(C \mu(B)(\eta \varepsilon v(\mathcal{R}_\alpha))^{-1}), \]

where \(C\) is constant in the \(A_\phi(\Omega)\) condition for \((v, w)\) (\(B\) and \(\mathcal{R}_\alpha\) denote \(B(x, r)\) and \(\mathcal{R}_\alpha(x_i, \frac{4Kr_i}{\alpha})\), respectively). Therefore, putting \(\eta = C \mu(B) \phi^{-1}(1/\varepsilon v(\mathcal{R}_\alpha))\) and taking account that \(s \leq \phi^{-1}(s)\psi^{-1}(s), s \geq 0\), we have

\[ \int_X \psi(\eta \varepsilon w)^{-1} \chi_{B} \varepsilon w d\mu \leq \frac{1}{C \phi^{-1}(1/\varepsilon v(\mathcal{R}_\alpha))} \rho\left(\frac{1}{\varepsilon v(\mathcal{R}_\alpha) \phi^{-1}(1/\varepsilon v(\mathcal{R}_\alpha))}\right) \]

\[ \leq \alpha C^{-1} \varepsilon v(\mathcal{R}_\alpha) \psi^{-1}(\frac{1}{\varepsilon v(\mathcal{R}_\alpha)}), \]

\[ \leq \alpha C^{-1}, \]

where \(\alpha > 1\) is such that \(s \rho(s) \leq \alpha \psi(s), s \geq 0\). We may assume that \(C \geq \alpha\) and therefore

\[ \|\varepsilon w\|^{-1} \chi_{B}\|_{\psi, \varepsilon w} \leq C \mu(B) \phi^{-1}(1/\varepsilon v(\mathcal{R}_\alpha)). \]

Now, it follows that

\[ \frac{1}{\mu(B)} \int_B |f|d\mu \leq 2C \phi^{-1}(1/\varepsilon v(\mathcal{R}_\alpha)) \|f\chi_B\|_{\phi, \varepsilon w}. \]
Then taking \( \varepsilon = (\int_B \phi(|f|)wd\mu)^{-1} \), we have \( ||f\chi_B||_{(\phi),\varepsilon,w} = 1 \) and

\[
\frac{1}{\mu(B)} \int_B |f|d\mu \leq 2C\phi^{-1}(\frac{1}{v(\mathcal{R}_\alpha)}) \int_B \phi(|f|)wd\mu.
\]

So \( v(\mathcal{R}_\alpha) \leq C \int_B \phi(|f|)wd\mu / \phi(\frac{1}{\mu(B)} \int_B |f|d\mu) \). Therefore we have

\[
v(E_\lambda) \\
\leq \sum_i v(\mathcal{R}_\alpha(x_i, \frac{4Kr_i}{\alpha})) \\
\leq C \sum_i \int_{B(x_i, \frac{4Kr_i}{\alpha})} \phi(|f|)wd\mu / \phi(\frac{1}{\mu(B(x_i, \frac{4Kr_i}{\alpha}))}) \int_{B(x_i, \frac{4Kr_i}{\alpha})} |f|d\mu \\
\leq C \frac{1}{\phi(\lambda)} \int_{\sum_i B(x_i, \frac{4Kr_i}{\alpha})} \phi(|f|)wd\mu \\
\leq C \frac{1}{\phi(\lambda)} \int_X \phi(|f|)wd\mu.
\]

Next, \( E'_\lambda \) is unbounded. Fix \( a \in X \) and \( r > 0 \), we consider

\[
E'_\lambda = \{(x,t) : \mathcal{M}_\Omega f(x,t) > \lambda \text{ and } y \in E'_\lambda \cap B(a,r)\}
\]

for some \( y \in \Omega(x,\eta)(r) \) and apply the Lemma 3.1 to the balls \( \{B(y,r(x)) : y \in E'_\lambda \cap B(a,r)\} \). Letting \( r \to \infty \), we obtain the same estimate as before.

**Remark 3.4.** Let \( dv(x,t) = u(x)dx \otimes d\delta_0(t) \), where \( \delta_0(t) \) is the Dirac mass on \([0,\infty)\) (i.e., concentrated on \( X \times \{0\}\)). Set \( S_\alpha(x,r) = \{x_0 \in X : \Omega_{x_0}(r) \cap B(x,\alpha r) \neq \emptyset\} \), where \( \Omega_{x_0}(r) \) is the cross section of \( \Omega_{x_0} \) at height \( r \) ([9]). Then \( v(\mathcal{R}_\alpha(x,r)) = \mu(S_\alpha(x,r)) \). If \( \tilde{Q} \) is instead of \( \mathcal{R}_\alpha \) and \( X = \mathbb{R}^n \), then \( A_\phi(\Omega) \) condition reduces to the condition \( A^+_\phi([2]) \). So we can get the following:

**Corollary 3.5** [2]. For an \( N \)-function \( \Phi \) satisfying the \( \Delta_2 \)-condition, a nonnegative measure \( \mu \) on \( \mathbb{R}^{n+1}_+ \) and a weight \( v \) on \( \mathbb{R}^n \), the following inequality holds:

\[
\mu(\{(x,t) : \mathcal{M}(f)(x,t) > \eta\}) \leq \frac{C}{\Phi(\eta)} \int_{\mathbb{R}^n} \Phi(|f|)v(x)dx
\]
every $\eta > 0$ if and only if $(u, v) \in A_\Phi^+$. i.e.,

$$\sup_{Q, \varepsilon > 0} \varphi\left(\psi\left(\frac{1}{\varepsilon v}\right)Q\right) \cdot \frac{\varepsilon \mu(Q)}{|Q|} < \infty,$$

where

$$\mu(Q) = \int_Q d\mu, \ (g)Q = |Q|^{-1} \int_Q g(x)dx.$$

References

1. L. Caleson, Interpolation by bounded analytic functions and the corona problem, Annals of Math 76 (1962), 547-559

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