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WEAK TYPE (ϕ,ϕ) INEQUALITY FOR GENERALIZED MAXIMAL OPERATORS ON SPACES OF HOMOGENEOUS TYPE

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1. Introduction

Let f be a locally integrable function on R^n. We define the maximal operator

$$\mathcal{M}f(x,t)=\sup_Q \frac{1}{|Q|}\int_Q |f(x)|dx, \quad x\in R^n, \quad t\geq 0,$$

where the supremum is taken over the cubes Q in R^n, containing x and having side length at least t. It is well known that this maximal operator $\mathcal{M}$ controls Poisson integral. Several authors was studied this maximal operators ([3],[8],[9]).

For given positive measure v on $\overline{R^{n+1}_+}=\{(x,t) : x\in R^n, t\geq 0\}$, Carleson [1] showed that $\mathcal{M}$ is bounded from $L^p(R^n,dx)$ into $L^p(\overline{R^{n+1}_+},dv)$ if and only if v satisfies the "Carleson condition" $\sup_{x\in Q} \frac{v(\widetilde{Q})}{|Q|}\leq C$, where $\widetilde{Q}$ denotes the cubes in $\overline{R^{n+1}_+}$ with the cube Q as its base. Later, Fefferman-Stein [3] proved that $\mathcal{M}$ is bounded from $L^p(R^n,w(x)dx)$ into $L^p(\overline{R^{n+1}_+},dv)$ if $\sup_{x\in Q}\frac{v(\widetilde{Q})}{|Q|}\leq Cw(x)$ *a.e.* x, where w is a weight in R^n. Also, Ruiz[6] found the condition

$$\frac{\mu(\widetilde{Q})}{|Q|}(\frac{1}{|Q|}\int_Q v^{1-p'}(x)dx)^{p-1}\leq C$$

to be necessary and sufficient for the boundedness of the operator $\mathcal{M}$ from $L^p(R^n,v(x)dx)$ into $weak-L^p(R^{n+1}_+,\mu)$ In 1993, Jie-Cheng Chen [2] found the conditions on (μ,v) for $\mathcal{M}$ to be bounded from

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$L_\Phi(R^n, v(x)dx)$ to $weak - L_\Phi(R^{n+1}, d\mu)$. On the other hand, Sueiro [9] studied a certain maximal operator defined on space of homogeneous type X.

In this paper, we extend the result of Jie-Cheng Chen in [2]. So we can get the conditions for the maximal operator $\mathcal{M}_{\Omega,\gamma}$ to be bounded from $L_\Phi(X, v(x)dx)$ to $weak - L_\Phi(X \times [0,\infty), d\mu)$.

2. Preliminaries

DEFINITION 2.1. Let X be a topological space and let $d : X \times X \to [0,\infty)$ be a map satisfying;

(i) $d(x,x) = 0; d(x,y) > 0 \quad if \quad x \neq y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \leq K[d(x,y) + d(y,z)]$, where K is some fixed constant. Assume further that

(iv) the balls $B(x,r) = \{y : d(x,y) < r\}$ form a basis of open neighborhood at $x \in X$ and that μ is a Borel measure on X such that

(v) $0 < \mu(B(x,2r)) \leq A\mu(B(x,r)) < \infty$, where A is some fixed constant. Then the triple (X, d, μ) is called a space of homogeneous type ([9]).

REMARK 2.2. Properties (iii) and (v) will be referred to as the "triangle inequality" and the "doubling property", respectively. Note that the condition (v) is equivalent with the fact that for every $c > 0$, there exists $A_c < \infty$ such that $\mu(B(x,cr)) \leq A_c\mu(B(x,r))$ for all $x \in X$ and $r > 0$.

DEFINITION 2.3. Assume for each $x \in X$, we are given a set $\Omega_x \subset X \times [0,\infty)$. Let Ω denote the family $\{\Omega_x : x \in X\}$. For each $t \geq 0$ and $\alpha > 0$, set

$$\Omega_{(x,t)} = \Omega_x \cap (X \times [t,\infty))$$

and

$$\mathcal{R}_\alpha(x,t) = \{(y,r) \in X \times [0,\infty) : \Omega_{(y,r)}(t) \cap B(x,\alpha t) \neq \phi\},$$

where $\Omega_{(y,r)}(t) = \{z \in X : (z,t) \in \Omega_{(y,r)}\}$ is the cross-section of $\Omega_{(y,r)}$ at height t. A nonnegative and locally integrable function w on X is called a *weight*.

Now, we recall the basic terminology and results concerning N-function which will be used in this paper.

An $N-function$ is a continuous and convex function $\phi:[0,\infty)\to R$ with $\phi(s)>0, s>0, s^{-1}\phi(s)\to 0$ for $s\to 0$ and $s^{-1}\phi(s)\to\infty$ for $s\to\infty$. An N-function ϕ has the representation $\phi(s)=\int_0^s\varphi$, where $\varphi:[0,\infty)\to R$ is continuous from the right, nondecresing such that $\varphi(s)>0, s>0, \varphi(0)=0$ and $\varphi(s)\to\infty$ for $s\to\infty$. This φ will be called the *density function* of ϕ. Associated to ϕ we have the function $\rho:[0,\infty)\to R$ defined by $\rho(t)=\sup\{s:\phi(s)\le t\}$. We will call ρ the *generalized inverse* of ϕ. Also, the N-function ψ defined by $\psi(t)=\int_0^t\rho$ is called the *complementary* $N-function$ of ϕ. An N- function ϕ is said to satisfy the Δ_2-condition in $[0,\infty)$ if $\sup_{s>0}\frac{\phi(2s)}{\phi(s)}<\infty$. If φ is the density function of ϕ, then ϕ satisfies Δ_2 if and only if there exists a constant $\alpha>1$ such that $s\phi(s)<\alpha\varphi(s), s>0$.

If $(\mathcal{Y},\mathcal{M},\mu)$ is a σ-finite measure space. We denote by M the space of $\mathcal{M}$-measurable and μ are finite functions from $\mathcal{Y}$ to R(or to C). If ϕ is an N-function, then the *Orlicz spaces* $L_\phi(\mu)\equiv L_\phi(\mathcal{Y},\mathcal{M},\mu)$ and $L^*_\phi(\mu)\equiv L^*_\phi(\mathcal{Y},\mathcal{M},\mu)$ are defined by

$$L_\phi(\mu)=\{f\in M:\int_X\phi(|f|)d\mu<\infty\}$$

$$L^*_\phi(\mu)=\{f\in\mathcal{M}:fg\in L_1(\mu)\ for\ all\ g\in L_\psi\}$$

respectively, where ψ is the complementary N-function of ϕ. Then the Orlicz space $L^*_\phi(\mu)$ is a Banach space with the norms $||f||_\phi=\sup\{\int_{\mathcal{Y}}|fg|d\mu : g\in S_\psi\}$, where $S_\psi=\{g\in L_\psi:\int_{\mathcal{Y}}\psi(|g|)d\mu\le 1\}$, and $||f||_{(\phi)}=\inf\{\lambda>0:\int_{\mathcal{Y}}\phi(\frac{|f|}{\lambda})d\mu\le 1\}$, which are called the *Orlicz norm* and *Luxemburg norm*, respectively.

In fact both norms are equivalent, actually $||f||_{(\phi)}\le||f||_\phi\le 2||f||_{(\phi)}$ Hölder inequality asserts that for every $f\in L^*_\phi(\mu)$ and every $g\in L^*_\psi(\mu)$, we have

$$||fg||_1\le||f||_{(\phi)}||g||_{(\psi)},$$

where ϕ and ψ are complementary N-functions. The proof of above results can be found in [4],[5].

For a nonnegative measure v on $X\times[0,\infty)$ and a weight w on X, we define the generalized maximal operator on space of homogenous type.

DEFINITION 2.4. Assume that we have a family $\{\Omega_x : x \in X\}$. For $f \in L^1_{loc}(X, d\mu)$ and $x \in X, t \geq 0$, set

$$\mathcal{M}_{\Omega,\gamma} f(x) = \sup_{(y,s)\in\Omega(x,t)} \frac{\gamma(\mu(B(y,s)))}{\mu(B(y,s))} \int_{B(y,s)} |f(x)| d\mu,$$

where $\gamma : (0,\infty) \to (0,\infty)$ is essentially nondecresing. i.e., there is a positive constant C for which $\gamma(t) \leq C\gamma(s)$ for $t \leq s$ and

$$\lim_{t\to\infty} \frac{\gamma(t)}{t} = 0. \tag{2.1}$$

REMARK 2.5. In above, the condition (2.1) is necessary to rule out examples such as $\gamma(t) = t^m, m > 1$. In this case, $\mathcal{M}_{\Omega,\gamma} f(x) = \infty$ for all x, if we consider $f \cong \delta$, the point mass at the origin.

DEFINITION 2.6. Let $f \in L^1_{loc}(X, d\mu)$ and $\lambda > 0$. An operator $\mathcal{M}_{\Omega,\gamma}$ is of weak type (ϕ, ϕ) with respect to (v, w) if there is a constant C so that

$$v\{(x,t) \in X \times [0,\infty) : \mathcal{M}_{\Omega,\gamma} f > \lambda\} \leq \frac{C}{\phi(\lambda)} \int_X \phi(|f|) w d\mu.$$

DEFINITION 2.7. Let φ be the density function of the N-function ϕ and ρ be the generalized inverse of φ. A pair (v, w) is said to satisfy the condition $A_\phi(\Omega)$ if there are constants C and $\alpha > 0$ such that

$$(\frac{1}{\mu(B)} \int_{\mathcal{R}_\alpha(x,r)} \varepsilon v d\mu) \varphi(\frac{1}{\mu(B)} \int_B \rho(\frac{1}{\varepsilon w}) d\mu) \leq C$$

for every ball $B(x, r)$ in X and every $\varepsilon > 0$.

In this paper, we shall always assume that ϕ, together with its complementary N-function , satisfy Δ_2-condition. Also, the letter C denotes a constant which need not be the same at each occurrrence.

3. Results

LEMMA 3.1 [9]. *Let E be a bounded subset of X and for each $x \in X$. Let $r(x)$ be a positive number for each $x \in E$. Then there is a sequence of disjoint balls $B(x_i, r(x_i)), x_i \in E$ such that the balls $B(x_i, 4Kr(x_i))$ cover E, where K is the constant in the triangle inequality. Futhermore, every $x \in E$ is contained in some ball $B(x_i, 4Kr(x_i))$ satisfying $r(x) \le 2r(x_i)$.*

LEMMA 3.2 [2]. *For any N-function $\phi, t \le \varphi(\rho(t))$ and $\phi(t) \le t\varphi(t)$. If ϕ satisfies Δ_2-condition, then $\varphi(\rho(t)) \le C_\phi t$ and $\phi(t) \ge t\varphi(t)/C_\phi$.*

THEOREM 3.3. *Assume that Ω satisfies the condition that if $x \in X, (y,r) \in \Omega_x$ and $k \ge r$, then $(y,r) \in \Omega_x$. Then $\mathcal{M}_{\Omega,\gamma}$ is of weak type (ϕ,ϕ) with respect to (v,w) if and only if (v,w)satisfies $A_\phi(\Omega)$.*

Proof. Suppose that $\mathcal{M}_{\Omega,\gamma}$ is of weak type (ϕ,ϕ) with respect to (v,w). If $(x_0,t) \in \mathcal{R}_\alpha(x,t)$, then $\Omega_{(x_0,t)}(r) \cap B(x,\alpha r) \ne \phi$. And so we can choose $y \in \Omega_{(x_0,t)}(r) \cap B(x,\alpha r)$. From the triangle inequality,

$$(1) \qquad B(x,r) \subset B(y, K(\alpha+1)r) \subset B(x, (K^2\alpha + K\alpha + K^2)r).$$

Let f be a nonnegative measurable function on X. Let

$$f_{B(y,r),\gamma} = \frac{\gamma(\mu(B(y,r)))}{\mu(B(y,r))}\int_{B(y,r)} f d\mu.$$

Since $(y, K(\alpha+1)r) \in \Omega_{(x_0,t)}$ by the hypothesis, we have

$$f_{B(y,K(\alpha+1)r),\gamma} \le \mathcal{M}_{\Omega,\gamma}(f \cdot \chi_{B(y,K(\alpha+1)r)})(x_0,t).$$

Putting $\lambda = f_{B(y,K(\alpha+1)r),\gamma}$ and $E_\lambda = \{\mathcal{M}_{\Omega,\gamma}(f\cdot\chi_{B(y,K(\alpha+1)r)}) > \lambda\}$, then the previous argument shows that $\mathcal{R}_\alpha(x,r) \subset E_\lambda$ and so

$$v(\mathcal{R}_\alpha(x,r)) \le v(E_\lambda) \le \frac{C}{\phi(\lambda)}\int_{B(y,K(\alpha+1)r)} \phi(|f|) w d\mu.$$

Hence if we invoke (1) and the doubling property of μ,

$$\begin{aligned} & v(\mathcal{R}_\alpha(x,r))\phi(\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f d\mu) \\ \leq\ & v(\mathcal{R}_\alpha(x,r))\phi(\frac{1}{\mu(B(x,r))}\int_{B(x,K(\alpha+1)r)} f d\mu) \\ \leq\ & C\int_{B(x,K(\alpha+1)r)} \phi(|f|) w d\mu. \end{aligned}$$

Replacing f by $\rho(\frac{1}{w})\cdot\chi_{B(x,r)}$ and using Lemma 3.2,

$$v(\mathcal{R}_\alpha(x,r))\phi(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}\rho(\frac{1}{w})d\mu) \leq C[C_\phi\int_{B(x,r)}\rho(\frac{1}{w})d\mu].$$

So (v,w) satisfies $A_\phi(\Omega)$ condition with constant αC. Conversely, suppose (v,w) satisfies the condition $A_\phi(\Omega)$. We follow the idea of Sueiro [9]. For each $\lambda > 0$, define

$$E_\lambda = \{(x,t)\in X\times[0,\infty) : \mathcal{M}_{\Omega,\gamma}f(x,t) > \lambda\},$$

$$E'_\lambda = \{x\in X : \sup_{r>0}\frac{\gamma(\mu(B(x,r)))}{\mu(B(x,r))}\int_{B(x,r)}|f|d\mu > \lambda\}.$$

And for each $x\in E'_\lambda$, if we put

$$r(x) = \sup\{r>0 : \frac{\gamma(\mu(B(x,r)))}{\mu(B(x,r))}\int_{B(x,r)}|f|d\mu > \lambda\},$$

then there exists a finite positive real number $r(x)$ such that $\frac{\gamma(\mu(B(x,r)))}{\mu(B(x,r(x)))}\int_{B(x,r(x))}|f|d\mu \geq \lambda$. Assume for a moment that E'_λ is bounded. Then by Lemma 3.1, there exists a sequence of balls $\{B(x_i,r_i)\}$, where $x_i\in E'_\lambda, r_i = r(x_i)$ such that $E'_\lambda \subset \cup_i B(x_i, 4Kr_i)$ and $\frac{\gamma(\mu(B(x,r)))}{\mu(B(x_i,r_i))}\int_{B(x_i,r_i)}|f|d\mu \geq \lambda$. Now we want to verify $E_\lambda \subset \cup_i\mathcal{R}_\alpha(x_i,\frac{4Kr_i}{\alpha})$. To do this, let $(x,t)\in E_\lambda$. Then $\frac{\gamma(\mu(B(x,r)))}{\mu(B(y,r))}\int_{B(y,r)}|f|d\mu > \lambda$ for some $(y,r)\in\Omega_{(x,t)}$. So $y\in E'_\lambda$ and $t\leq r\leq r(y)$. By Lemma 3.1,

$y \in B(x_i, 4Kr_i)$ for some i such that $r(y) \le 2r_i$. Hence we may assume $\alpha < 2K$. Consequently, $t \le r \le r(y) \le 2r_i < \frac{4K}{\alpha} r_i$ and so $(y, \frac{4K\alpha_i}{\alpha}) \in \Omega_{(x,t)}$. Since $y \in B(x_i, \alpha(\frac{4K}{\alpha})r_i)$, it follows that

$$y \in \Omega_{(x,t)}(\frac{4Kr_i}{\alpha}) \cap B(x_i, \alpha(\frac{4K}{\alpha})r_i),$$

and thus $(x,t) \in \mathcal{R}_\alpha(x_i, \frac{4Kr_i}{\alpha})$, as so it holds. Now by Hölder inequality, we have

$$\int_{B(x,r)} |f| d\mu \le 2||f\chi_{B(x,r)}||_{(\phi),\varepsilon w}||(\varepsilon w)^{-1}\chi_{B(x,r)}||_{(\psi),\varepsilon w}.$$

On the other hand, for every $\eta > 0$, we get

$$\begin{aligned}\int_X \psi(\eta\varepsilon w)^{-1}\chi_{B(x,r)})\varepsilon w d\mu &\le \int_{B(x,r)} \eta^{-1}\rho((\eta\varepsilon w)^{-1})d\mu \\ &\le \eta^{-1}\mu(B)\rho(C\mu(B)(\eta\varepsilon v(\mathcal{R}_\alpha))^{-1}),\end{aligned}$$

where C is constant in the $A_\phi(\Omega)$ condition for (v,w) (B and $\mathcal{R}_\alpha$ denote $B(x,r)$ and $\mathcal{R}_\alpha(x_i, \frac{4Kr_i}{\alpha})$), respectively). Therfore, putting $\eta = C\mu(B)\phi^{-1}(1/\varepsilon v(\mathcal{R}_\alpha))$ and taking account that $s \le \phi^{-1}(s)\psi^{-1}(s), s \ge 0$, we have

$$\begin{aligned}\int_X \psi(\eta\varepsilon w)^{-1}\chi_B)\varepsilon w d\mu &\le \frac{1}{C\phi^{-1}(1/\varepsilon v(\mathcal{R}_\alpha))}\rho(\frac{1}{\varepsilon v(\mathcal{R}_\alpha)\phi^{-1}(1/\varepsilon v(\mathcal{R}_\alpha))}) \\ &\le \alpha C^{-1}\varepsilon v(\mathcal{R}_\alpha)\psi(\frac{1/\varepsilon v(\mathcal{R}_\alpha)}{\phi^{-1}(1/\varepsilon v(\mathcal{R}_\alpha)}) \\ &\le \alpha C^{-1},\end{aligned}$$

where $\alpha > 1$ is such that $s\rho(s) \le \alpha\psi(s), s \ge 0$. We may assume that $C \ge \alpha$ and therefore

$$||(\varepsilon w)^{-1}\chi_B||_{(\psi),\varepsilon w} \le C\mu(B)\phi^{-1}(1/\varepsilon v(\mathcal{R}_\alpha)).$$

Now, it follows that

$$\frac{1}{\mu(B)}\int_B |f| d\mu \le 2C\phi^{-1}(1/\varepsilon v(\mathcal{R}_\alpha))||f\chi_B||_{(\phi),\varepsilon w}.$$

Then taking $\varepsilon = (\int_B \phi(|f|)wd\mu)^{-1}$, we have $\|f\chi_B\|_{(\phi),\varepsilon w} = 1$ and

$$\frac{1}{\mu(B)}\int_B |f|d\mu \le 2C\phi^{-1}(\frac{1}{v(\mathcal{R}_\alpha)}\int_B \phi(|f|)wd\mu).$$

So $v(\mathcal{R}_\alpha) \le C\int_B \phi(|f|)wd\mu/\phi(\frac{1}{\mu(B)}\int_B |f|d\mu)$. Therefore we have

$$\begin{aligned}
&v(E_\lambda)\\
&\le \sum_i v(\mathcal{R}_\alpha(x_i, \frac{4Kr_i}{\alpha}))\\
&\le C\sum_i \int_{B(x_i,\frac{4Kr_i}{\alpha})} \phi(|f|)wd\mu/\phi(\frac{1}{\mu(B(x_i,\frac{4Kr_i}{\alpha}))}\int_{B(x_i,\frac{4Kr_i}{\alpha})} |f|d\mu)\\
&\le C\frac{1}{\phi(\lambda)}\int_{\sum_i B(x_i,\frac{4Kr_i}{\alpha})} \phi(|f|)wd\mu\\
&\le C\frac{1}{\phi(\lambda)}\int_X \phi(|f|)wd\mu.
\end{aligned}$$

Next, E'_λ is unbounded. Fix $a \in X$ and $r > 0$, we consider

$$E'_\lambda = \{(x,t) : \mathcal{M}_\Omega f(x,t) > \lambda \ \ and \ \ y \in E'_\lambda \cap B(a,r)\}$$

for some $y \in \Omega_{(x,t)}(r)$ and apply the Lemma 3.1 to the balls $\{B(y, r(x)) : y \in E'_\lambda \cap B(a,r)\}$. Letting $r \to \infty$, we obtain the same estimate as before.

REMARK 3.4. Let $dv(x,t) = u(x)dx \otimes d\delta_0(t)$, where $\delta_0(t)$ is the Dirac mass on $[0,\infty)$ (i.e., concentrated on $X \times \{0\}$). Set $\mathcal{S}_\alpha(x,r) = \{x_0 \in X : \Omega_{x_0}(r) \cap B(x,\alpha r) \ne \emptyset\}$, where $\Omega_{x_0}(r)$ is the cross section of Ω_{x_0} at height r ([9]). Then $v(\mathcal{R}_\alpha(x,r)) = \mu(\mathcal{S}_\alpha(x,r))$. If $\tilde{Q}$ is instead of $\mathcal{R}_\alpha(x,r)$ and $X = R^n$, then $A_\phi(\Omega)$ condition reduces to the condition A^+_Φ([2]). So we can get the following:

COROLLARY 3.5 [2]. *For an N-function Φ satisfying the $\triangle_2$-condition, a nonnegative measure μ on R^{n+1}_+ and a weight v on R^n, the following inequality holds:*

$$\mu(\{(x,t) : \mathcal{M}(f)(x,t) > \eta\}) \le \frac{C}{\Phi(\eta)}\int_{R^n} \Phi(|f|)v(x)dx$$

every $\eta > 0$ *if and only if* $(u, v) \in A_\Phi^+$. *i.e.*,

$$\sup_{Q,\varepsilon>0} \varphi((\psi(\frac{1}{\varepsilon v}))_Q) \cdot \frac{\varepsilon\mu(\widetilde{Q})}{|Q|} < \infty,$$

where

$$\mu(\widetilde{Q}) = \int_{\widetilde{Q}} d\mu, \ (g)_Q = |Q|^{-1} \int_Q g(x)dx.$$

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