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ON (∗)–IDEALS AND POSITIVE IMPLICATIVE IDEALS IN BCI–ALGEBRAS

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The study of ideals forms an important part of the theory of BCI-algebra. Since K. Iséki [6] generalized the notion of ideals in BCK-algebras to BCI-algebras, several classes of ideals in BCI-algebras have been occurred, for instance, closed ideals [2], strong ideals [1], p–ideals [17], positive implicative ideals [3], and so on. In [4] and [9] closed ideals, strong ideals, and p–ideals were further investigated. In particular, it is shown that in a BCI-algebra the notion of strong ideals and closed p–ideals coincide. As a continuation of [4], [9] and [14], we now will deeply study further properties of (∗)–ideals and positive implicative ideals and clarify the relation of the two classes of ideals.

Let X be a nonempty set. Let $*$ be a binary operation on X and 0 is a constant of X. An algebra $< X; *, 0 >$ of type (2,0) is said to be a BCI-algebra if for all $x, y, z \in X$,

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A binary relation $\leq$ on X can be defined by putting $x \leq y$ if and only if $x * y = 0$. Then $< X; \leq >$ is a partially ordered set with a minimal element 0.

In any BCI-algebra X the following properties hold.

(1) $(x * y) * z = (x * z) * y$,

(2) $x * 0 = x$,

(3) $(x * z) * (y * z) \leq x * y$,

(4) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

A BCI-algebra X with the condition $0 \leq x$ for all $x \in X$ is called a BCK-algebra.

A nonempty subset I of a BCI-algebra X is called an ideal of X if

(i) $0 \in I$,

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(ii) $x \in I$ whenever $x * y \in I$ and $y \in I$.

Every ideal I of X satisfies

(iii) $x \leq y$ and $y \in I$ imply $x \in I$.

An ideal I of a BCI-algebra X is said to be closed if $0 * x \in I$ for all $x \in I$. An ideal I of a BCI-algebra X is closed if and only if I is a subalgebra of X. Every ideal of a BCK-algebra is always closed. An element a in a BCI-algebra X is said to be an atom if for all $x \in X$, $x * a = 0$ implies $x = a$. Let $L(X)$ be the set of all atoms of a BCI-algebra X. For any $a \in L(X)$, the set $\{x \in X \mid a \leq x\}$ is called the branch of a BCI-algebra X and denoted by $V(a)$. The branch $V(0)$ is the BCK-part of X, which is denoted by $B(X) = \{x \in X | 0 \leq x\}$. For all x in a BCI-algebra X, we denote $a_x = 0 * (0 * x) \in L(X)$. In the sequel, we will use the following properties:

(5) $L(X)$ is a subalgebra of X,

(6) $L(X) = \{0 * (0 * x) | x \in X\} = \{0 * x | x \in X\}$,

(7) If $x, y \in X$, then x and y belong to the same branch if and only if $x * y \in B(X)$ and so $x * a_x \in B(X)$.

A BCI-algebra X is said to be p-semisimple if $B(X) = \{0\}$ or equivalently, $L(X) = X$.

The above concepts and results can be found in [1], [7] and [12]. Throughout this paper, X will mean a BCI-algebra unless mentioned otherwise.

Before starting to discuss $(*)$-ideals and positive implicative ideals, we give an elementary property of BCI-algebras.

THEOREM 1. *Let X be a BCI-algebra. Then for all $x \in X$ and $y \in B(X)$, $x * y \leq x$.*

Proof. Since $y \in B(X)$ implies $0 * y = 0$, we have $(x * y) * x = (x * x) * y = 0 * y = 0$, that is, $x * y \leq x$. ◇

E. H. Roh, Y. B. Jun and S. M. Wei [13] introduced the notion of $(*)$-ideals in BCI-algebras and obtain some of its properties.

DEFINITION 1 [13]. If an ideal I of a BCI-algebra X satisfies the condition

(iv) $x \in I$ *and* $a \in X - I$ *imply* $x * a \in I$,

then I is called a $(*)$-**ideal** of X.

Obviously, each ideal of a BCK-algebra is always a $(*)-$ideal, but for a proper BCI-algebra, $\{0\}$ is not a $(*)-$ideal.

THEOREM 2. *If I is an ideal of a BCI-algebra X, and $L(X) \subseteq I$, then I is a $(*)-$ideal of X.*

Proof. Assume that $x \in I$ and $y \in X - I$. Since $y * a_y \in B(X)$, by Theorem 1, we have $x * (y * a_y) \leq x$, and so $x * (y * a_y) \in I$. Since

$$\begin{aligned}((x * y) * a_y) * (x * (y * a_y)) &= ((x * (x * (y * a_y))) * y) * a_y \\ &\leq ((y * a_y) * y) * a_y \\ &= (0 * a_y) * a_y \in L(X),\end{aligned}$$

and $L(X) \subseteq I$, we have $((x * y) * a_y) * (x * (y * a_y)) \in I$. Combining $x * (y * a_y) \in I$ and using (ii), we have $(x * y) * a_y \in I$. Since $a_y \in L(X) \subseteq I$, it follows that $x * y \in I$. Therefore, I is a $(*)-$ideal of X. ◇

The converse of Theorem 2 need not be hold, as is shown in the following example.

EXAMPLE 1 [9]. Let $X = \{2^n | n = \pm 1, \pm 2, \dots\}$, and let $\div$ be the usual division. Then $< X; \div, 1 >$ is a $p-$semisimple BCI-algebra and $I = \{1, 2, 2^2, \dots\}$ is an ideal of X. Observe that for all natural numbers m and n, we have $2^{-n} \in X - I$, $1 \div 2^{-n} = 2^n \in I$ and $2^m \div 2^{-n} = 2^{m+n} \in I$. Hence I is a $(*)-$ideal of X. But $L(X) \not\subseteq I$ as $L(X) = X$. One easily sees that I is not closed, because $1 \div 2 = \frac{1}{2} \notin I$.

It is natural to ask whether or not for closed ideals the converse of Theorem 2 holds. The answer is positive.

THEOREM 3. *Let I be a closed $(*)-$ideal of a BCI-algebra X. Then $L(X) \subseteq I$.*

Proof. If $L(X) \not\subseteq I$, then there is $a \in L(X) - I$, that is, $a \in L(X)$ and $a \notin I$. But $a \in L(X)$ implies $a = 0 * (0 * a)$. By (iv), $a \notin I$ implies $0 * a \in I$. Furthermore, $a = 0 * (0 * a) \in I$, because I is closed. Therefore, we have a contradiction. Hence $L(X) \subseteq I$. ◇

By Theorem 2 and Theorem 3, we have

COROLLARY 4. *If I is a closed ideal of a BCI-algebra X, then I is a $(*)$–ideal if and only if $L(X) \subseteq I$.*

DEFINITION 2 [11]. Let $\mathbb{N}$ be the set of all natural numbers. Let X be a BCI-algebra. For $x \in X$, we define x^n by $x^1 = x$, $x^{n+1} = x * (0 * x^n)$. If there is $n \in \mathbb{N}$ such that $x^n \in B(X)$, then x is called finite periodic and $|x| = min\ \{n \in \mathbb{N} | x^n \in B(X)\}$ is the period of x. The set $P(X) = \{x \in X| \quad |x| < \infty\}$ is called the periodic part of X. If $X = P(X)$, then X is said to be periodic.

PROPOSITION 5 [11, THEOREM 11]. *Let X be a periodic BCI-algebra. Then each ideal of X is closed.*

Combining Corollary 4 and Proposition 5, we get

THEOREM 6. *If X is a periodic BCI-algebra, and I is an ideal of X, then I is a $(*)$–ideal of X if and only if $L(X) \subseteq I$.*

Since a finite BCI-algebra is periodic (see [11, Theorem 8]), we obtain

COROLLARY 7. *An ideal I of a finite BCI-algebra X is a $(*)$–ideal if and only if $L(X) \subseteq I$.*

Now, we give simpler characterizations of $(*)$–ideals and closed $(*)$–ideals.

THEOREM 8. *Suppose I is an ideal of a BCI-algebra X. Then I is a $(*)$–ideal of X if and only if $a \in X - I$ implies $0 * a \in I$.*

Proof. $(\Longrightarrow)$: Trivial.
$(\Longleftarrow)$: Suppose an ideal I satisfies that $a \in X - I$ implies $0 * a \in I$. If $x \in I$ and $a \in X - I$, then $(x * a) * x = (x * x) * a = 0 * a \in I$. By (ii), $x * a \in I$. Therefore, I is a $(*)$–ideal. $\Diamond$

THEOREM 9. *An ideal I of a BCI-algebra X is a closed $(*)$–ideal if and only if $0 * x \in I$ for all $x \in X$.*

Proof. Let I be a closed $(*)$–ideal and $x \in X$. If $x \in X - I$, then $0 * x \in I$, because I is a $(*)$–ideal. If $x \in I$, then $0 * x \in I$, because I is a closed ideal. Hence $0 * x \in I$ for all $x \in X$.

Conversely, suppose $0*x \in I$ for all $x \in X$. In other words, $L(X) \subseteq I$ by (6). It follows from the definition of closed ideals that I is closed. Then, applying Corollary 4, we get that I is a $(*)-$ideal.$\diamondsuit$

THEOREM 10. *A nonempty subset I of a BCI-algebra X is a closed $(*)-$ideal of X if and only if (i) t $0 \in I$, and (v) for all $x, y, z \in X$, $x * y \in I$ and $y \in I$ imply $x * z \in I$.*

Proof. Suppose that I satisfies (i) and (v). Assume that $z = 0$ in (v). Then I satisfies $x * y \in I$ and $y \in I$ imply $x \in I$. Hence I is an ideal of X. Let $x = y = 0$ in (v). Then $0 * z \in I$ for all $z \in X$. By Theorem 9, I is a closed $(*)-$ideal of X.

Conversely, let I be a closed $(*)-$ideal of X. If $x * y \in I$ and $y \in I$, then by closeness of I, we have $x * z \in I$ whenever $z \in I$. Thus, for all $x, y, z \in X$, $x * y \in I$ and $y \in I$ imply $x * z \in I$. $\diamondsuit$

For a subset A of X, let $(A]$ (resp. $(A]_*$) denotes the least ideal (resp. least closed $(*)-$ideal) containing A in X.

THEOREM 11. *Let A be a subset of a BCI-algebra X. Then $(A]_* = (A \cup L(X)]$.*

Proof. It follows directly from Theorem 9 and (6). $\diamondsuit$

Next, we discuss positive implicative ideals in BCI-algebras and their relation with $(*)-$ideals. The notion of positive implicative ideals in BCK-algebras was introduced by K Iséki in [5] and generalized to BCI-algebras by C. S. Hoo in [3].

DEFINITION 3 [5]. A nonempty subset I of a BCI-algebras X is called a **positive implicative ideal** of X if it is satisfies (i) $0 \in I$ and (vi) $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$

Any positive implicative ideal must be an ideal, but the converse need not be hold.

DEFINITION 4 [15]. A BCI-algebra X is called **quasi-associative** if for $x, y, z \in X$, $(x * y) * z \leq x * (y * z)$.

PROPOSITION 12 [15]. *A BCI-algebra X is quasi-associative if and only if for all $x \in X$, $0 * x \leq x$, or equivalently $(0 * x) * x = 0$.*

For a nonempty subset A of a BCI-algebra X and a fixed element a of X, the set $\{x \in X \mid x * a \in A\}$ is denoted by A_a.

THEOREM 13. *Let X be a quasi-associative BCI-algebra and let A be a positive implicative ideal of X. Then for any fixed $a \in X$, A_a is the least ideal containing A and a.*

Proof. Since X is quasi-associative, $(0 * a) * a = 0 \in A$. Combining $a * a = 0 \in A$ and by (vi), we have $0 * a \in A$, that is, $0 \in A_a$. Also, let $x * y \in A_a$ and $y \in A_a$. Then $(x * y) * a \in A$ and $y * a \in A$. By (vi), $x * a \in A$ and so $x \in A_a$. Therefore, A_a is an ideal of X.

Since X is quasi-associative, for all $x \in A$, we have $(x * a) * a \leq x * (a * a) = x * 0 = x \in A$ and so $(x * a) * a \in A$ by (iii). Observe that $a * a = 0 \in A$. Hence $x * a \in A$. Namely, $x \in A_a$. Thus $A \subseteq A_a$. Clearly, $a \in A_a$.

Suppose that I is any ideal containing A and a. If $x \in A_a$, then $x * a \in A \subseteq I$, and so $x * a \in I$. It follows from $a \in I$ that $x \in I$. Hence $A_a \subseteq I$. This means that A_a is the least ideal containing A and a. ◇

THEOREM 14. *Let A be an ideal of a BCI-algebra X. If for all $a \in X$, A_a is an ideal of X, then A is a positive implicative ideal of X.*

Proof. Let $x, y, z \in X$ be such that $(x * y) * z \in A$ and $y * z \in A$. Then $x * y \in A_z$ and $y \in A_z$. Since A_z is an ideal of X, by (ii), $x \in A_z$ and so $x * z \in A$. Hence A is a positive implicative ideal of X. ◇

By Theorem 13 and Theorem 14, we have

COROLLARY 15. *Let X be a quasi-associative BCI-algebra and A be an ideal of X. Then A is positive implicative if and only if for any $a \in X$, A_a is an ideal of X.*

The following result is a generalization of [8, Theorem 4]

THEOREM 16. *If I is an ideal of a BCI-algebra X, then the following are equivalent:*

(8) I is positive implicative,

(9) $(x * y) * y \in I$ *implies* $x * y \in I$, *for all* $x, y \in X$,
(10) $(x * y) * z \in I$ *implies* $(x * z) * (y * z) \in I$ *for all* $x, y, z \in I$.

The proof of the above result is similar to [8, Theorem 2] and omitted.

THEOREM 17. *Suppose* X *is a quasi-associative BCI-algebra and* A *is a positive implicative ideal of* X. *Then* A *is a closed* $(*)$–*ideal of* X.

Proof. For all $a \in X$, A_a is an ideal of X by Theorem 13. Since $0 \in A_a$, it follows that $0 * a \in A$. By Theorem 9, we know that A is a closed $(*)$–ideal of X. ◊

COROLLARY 18 [16]. *Let* I *be a positive implicative ideal of a quasi-associative BCI-algebra* X. *Then* $L(X) \subseteq I$.

Proof. It follows from Theorem 3 and Theorem 17.◊

By quotient algebras, we can characterize $(*)$–ideals. Let I be an ideal of a BCI-algebra X. Define a binary relation $\sim$ on X as follows:

$$x \sim y \text{ if and only if } x * y \in I \text{ and } y * x \in I.$$

Then $\sim$ is a congruence relation on X. Denote by $C_x = \{y \in X | y \sim x\}$ the equivalence class containing $x \in X$ and $X/I = \{C_x | x \in X\}$. Define $C_x * C_y = C_{x*y}$. Then C_0 is the greatest closed ideal contained in I, and $< X/I; *, C_0 >$ is a BCI-algebra, called the quotient algebra of X by I (see [7]). Then C_0 may not equal I. We can check that $C_0 = I$ if I is a closed ideal.

THEOREM 19. *Let* I *be a closed ideal of a BCI-algebra* X. *Then* I *is a* $(*)$–*ideal if and only if* $< X/I; *, C_0 >$ *is a BCK-algebra.*

Proof. $(\Longrightarrow)$: Suppose I is a closed $(*)$–ideal of X. By Theorem 9, $0 * x \in I$ for all $x \in X$, and so $C_0 * C_x = C_{0*x} = I = C_0$ for all $x \in X$. Hence $< X/I; *, C_0 >$ is a BCK-algebra.

$(\Longleftarrow)$: If $< X/I; *, C_0 >$ is a BCK-algebra, then for all $x \in X$, $C_0 * C_x = C_0$. Thus $C_{0*x} = I$, for I is closed. Hence $0 * x \in I$ By Theorem 9, I is a closed $(*)$–ideal of X. ◊

COROLLARY 20. *Let X be a quasi-associative BCI-algebra, and let I be a positive implicative ideal of X. Then $< X/I; *, C_0 >$ is a positive implicative BCK-algebra.*

Proof. By Theorem 17 and Theorem 19, it suffices to prove that $< X/I; *, C_0 >$ is positive implicative. So, we assume that $(C_x * C_y) * C_y \in \{C_0\}$. Hence $C_{(x*y)*y} = I$, and so $(x * y) * y \in I$. By (9), we have $x * y \in I$. Thus $C_x * C_y = C_{x*y} = C_0 \in \{C_0\}$. This is to say that the zero ideal $\{C_0\}$ is positive implicative in BCK-algebra X/I. By [8, Corollary 7], $< X/I; *, C_0 >$ is a positive implicative BCK-algebra. ◇

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