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# EXTENSION OF MEROMORPHIC MAPPINGS

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## 1. Introduction

Kneser[12] generalized the continuation theorem on meromorphic functions of Levi[13] and Okuda-Sakai[16] gave a complete proof of it. Fuks[5] stated that any domain of meromorphy in $C^n$ is analytically convex in the sense of Hartogs and Kajiwara-Sakai[11] proved that the envelope of meromorphy of a domain over a Stein manifold with respect to a family of meromorphic functions is $p_7$-convex in the sense of Docquier-Grauert[3] and, therefore, is a Stein manifold. Thus, Kajiwara-Sakai proved that a meromorphic function on a domain over a Stein manifold is represented by a quotient of two global meromorphic functions and solved the weak Poincaré problem affirmatively.

For domains of infinite dimension, Harita[6] obtained the same result concerning a domain of the Cartesian product of countable family of complex planes. Aurich[1,2] proved that the envelope of meromorphy over a complex Banach space is pseudoconvex. Harita[7] proved that the envelope of meromorphy of a domain over a sequentially complete complex locally convex Hausdorff space is pseudoconvex. Let $E$ be a locally convex complex linear Hausdorff space, which is either equipped with the finite open topology, or is a Fréchet space with bounded approximation property or a DFN-space. Harita[8] proved that the coincidence of holomorphy and meromorphy of a domain over the space $E$ making use the affirmative solution of the Levi problem.

On the other hand, let $E$ be a sequentially complete locally convex Hausdorff space, $M$ be a complex manifold modelled with the locally convex space $E$, $(\Omega, \varphi)$ be a Riemann domain over the complex manifold $M$ and $(\tilde{\Omega}, \tilde{\varphi})$ be the pseudoconvex hull of $(\Omega, \varphi)$ in the sense of Matsuda[14]. Harita[9] proved that any meromorphic function on $\Omega$ can be meromorphically continued to a meromorphic function on $\tilde{\Omega}$ without using any solution of the Levi problem.

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Let $M$ be a pseudoconvex complex manifold modelled with a complex Banach space, which has a Schauder basis, $(\Omega, \varphi)$ be a Riemann domain over $M$, the dimension of which may be infinite, $(\tilde{\Omega}, \tilde{\varphi})$ be the locally pseudoconvex hull of the domain $(\Omega, \varphi)$ over $M$. Let $X$ be a complex Banach manifold with the weak disc property and $f : \Omega \to X$ be a meromorphic mapping. In the present paper, we prove that $f$ is meromorphically extended to a meromorphic mapping $\tilde{f} : \tilde{\Omega} \to X$ if and only if the set of points of indeterminacy $A$ of $f$ is extended to an analytic thin set $\tilde{A}$ of the pseudoconvex hull $\tilde{\Omega}$.

## 2. Notations and preliminaries

Let $E$ be a complex Hausdorff linear space. A Hausdorff space $M$ is called a *complex manifold modelled with* the linear space $E$, if there exists a family $\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$ of pairs $(U_i, \varphi_i)$ of open sets $U_i$ of $M$ and homeomorphisms $\varphi_i$ of open sets $U_i$ onto open sets of $E$ satisfying the following conditions.

(1) For any $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, the mapping $\varphi_i \circ (\varphi_j | \varphi_j(U_i \cap U_j))^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ between open sets in $E$ are holomorphic.

(2) $\bigcup_{i \in I} U_i = M$.

$\mathcal{A}$ is called an *atlas* of $M$, and an element of $\mathcal{A}$ is called a *chart* of $M$.

Let $E$ and $F$ be complex Hausdorff linear spaces, and $M$ and $N$ be complex manifolds, which are modelled, respectively, with the linear spaces $E$ and $F$, which have atlases $\{(U_i, \varphi_i); i \in I\}$ and $\{(U'_\alpha, \varphi'_\alpha); \alpha \in A\}$ respectively. Then a mapping $f : M \to N$ is said to be ***holomorphic*** if, for any $i \in I$ and any $\alpha \in A$ with $f(U_i) \cap U'_\alpha \neq \emptyset$, the mapping $\varphi'_\alpha \circ f \circ (\varphi_i | U_i)^{-1}$ is holomorphic whenever it is defined. Particularly, a holomorphic mapping of $M$ in the complex plane $C$ is called a ***holomorphic function*** on $M$. A function $p : M \to [-\infty, \infty)$ is said to be plurisubharmonic if, for each $i \in I$, the function $p \circ \varphi_i^{-1}$ is plurisubharmonic. We denote the set of plurisubharmonic functions on $M$ by $P(M)$.

Let $E$ be a complex Hausdorff linear space and $X$ be a complex manifold modelled with the linear space $E$. A subset $A$ of $X$ is said to be *analytic* if, for any point $x$ of $A$, there exist a neighborhood $U$ of $x$ and a family $\{f_j; j \in J\}$ of holomorphic functions on $U$ such that $A \cap U = \{y \in U; f_j(y) = 0 \quad for \quad any \quad j \in J\}$. A subset $T$ of $X$ is said to be *thin* if, for any point $x$ of $T$, there exist a neighborhhod $U$ of $x$ and a family $\{f_j; j \in J\}$ of holomorphic functions on $U$ such that $T \cap U \subset \{y \in U; f_j(y) = 0 \quad for \quad any \quad j \in J\}$. Let $F$ be a complex Hausdorff linear space and $Y$ be a complex manifold modelled with the linear spaces $F$. A holomorphic mapping $\varphi : X \to Y$ called a *modification* if there exist thin sets $S, T$ of, respectively, $X$ and $Y$ such that the restriction $\varphi|_{X-S} : X - S \to Y - T$ is a biholomorphic mapping of $X - S$ onto $Y - T$.

Let $E, F$ be complex Hausdorff linear spaces and $X, Y$ be complex manifolds modelled, respectively, with the linear spaces $E, F$. Let $G$ be an analytic subset of the product manifold $X \times Y$ such that the projection $\pi : G \to X$ is a modification. Then, we say that there exists a *meromorphic mapping* $\mu : X \to Y$ and $G$ is called the *graph* of $\mu$. The intersection of all analytic sets $A$ of $X$ such that there exits a holomorphic mapping $h$ of $X - A$ into $Y$ and that the graph $G(h)$ of $h$ coincides with $\pi^{-1}(X - A) \subset G$ is called the *set of indeterminacy* of $\mu$. A meromorphic mapping $\mu$ of $X$ in the Riemann sphere $P := C \cup \{\infty\}$, such that the image of each connected component of $X$ does not coincide with $\{\infty\}$, is called a *meromorphic function*.

## 3. Pseudoconvex hull

Let $E$ be a complex Hausdorff linear space, $M$ be a complex manifold modelled with the linear space $E$. A complex manifold $M$ is said to be *pseudoconvex* if, for any compact subset $K$ of $M$, the set

$$(1) \qquad \hat{K}_P := \{x \in M; p(x) \leq \sup_{y \in K} p(y) \quad for \quad all \quad p \in P(M)\}$$

is relatively compact subset of $M$.

A pair $(\Omega, \psi)$ of a Hausdoff space $\Omega$ and a locally biholomorphic mapping $\psi$ is called a *domain over* the manifold $M$. It is said to be *locally pseudoconvex* if, for any atlas $\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$ of the manifold $M$ and for any finite dimensional linear subspace $L$ of $E$, the open set $((\varphi_i \circ \psi)^{-1}(\varphi_i(U_i) \cap L), \varphi_i \circ \psi|_{(\varphi_i \circ \psi)^{-1}(\varphi_i(U_i) \cap L)})$ is a pseudoconvex open set over the finite dimensional Hausdorff complex linear space $L$.

Let $E$ be a Hausdorff complex linear space, $M$ be a complex manifold modelled with the linear space $E$, $(\Omega, \varphi)$ be a domain over the complex manifold $M$ and $\mathcal{P}$ be the family $\{(\lambda_j, \Omega_j, \varphi_j); j \in P\}$ of triples such that each $(\Omega_j, \varphi_j)$ is a locally pseudoconvex domain over $M$ and that each $\lambda_j$ is a locally biholomorphic mapping of $\Omega$ in $\Omega_j$ with $\varphi = \varphi_j \circ \lambda_j$. We introduce a semi-order $\prec$ in $\mathcal{P}$. For $j, k \in P$, we write $(\lambda_j, \Omega_j, \varphi_j) \prec (\lambda_k, \Omega_k, \varphi_k)$ if there exits a locally biholomorphic map $\lambda_j^k : \Omega_j \to \Omega_k$ with $\varphi_j = \varphi_k \circ \lambda_j^k$.

In finite dimensional case, Kajiwara[10] defined a pseudoconvex hull of a domain over a holomorphically convex manifold. Ohgai[15] constructed the Durchshnitt $(\tilde{\Omega}, \tilde{\varphi})$ as the minimum of the family $\mathcal{P}$. By the same method of Matsuda[14], we can prove that the set $\mathcal{P}$ equipped with the semi-order $\prec$ is an inductive directed set in the sense of Eilenber-Steenrod[4]. In this way, we can prove the following theorem:

THEOREM 1. *Let $E$ be a complex Hausdorff linear space, $M$ be a pseudoconvex manifold modelled with the linear space $E$ and $(\Omega, \varphi)$ be a domain over $M$. Then there exists uniquely a minimum locally pseudoconvex domain $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\varphi})$ over $M$ among locally pseudoconvex domains larger than $(\Omega, \varphi)$.*

The triple $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\varphi})$ is called the locally psedoconvex hull of the domain $(\Omega, \varphi)$ over the manifold $M$.

## 4. Continuation of meromorphic mappings

We use the notations

$$D := \{z \in C; |z| < 1\}, \tag{2}$$

and

$$D^* := \{z \in D; z \neq 0\}. \tag{3}$$

Let $X$ be a Banach manifold and let $H(D, X)$ be the space of holomorphic mappings from $D$ into $X$ equipped with the compact open topology. The manifold $X$ is said to have the *weak disc property,* if every sequence $\{f_n; n \geq 1\}$ of $H(D, X)$ which converges in $H(D^*, X)$, converges in $H(D, X)$ too.

THEOREM 2. *Let $B$ be a complex Banach space with Schauder base, $M$ be a pseudoconvex manifold modelled with the Banach space $B$, $(\Omega, \varphi)$ be a domain over $M$ and $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\varphi})$ be the locally pseudoconvex hull of the domain $(\Omega, \varphi)$ over $M$. Let $X$ be a complex manifold which is modelled with a complex Banach space equipped with a Schauder basis, and which has the weak disc property. Let $f : \Omega \to X$ be a meromorphic mapping. Then $f$ is meromorphically extended to a meromorphic mapping $\tilde{f} : \tilde{\Omega} \to X$ of the locally pseudoconvex hull $\tilde{\Omega}$ in the manifold $X$ if and only if the set of points of indeterminacy of $f$ is extended to an analytic set of $\tilde{\Omega}$.*

*Proof of the necessity.* Suppose that $f$ is extended to a meromorphic mapping $\tilde{f}$ of $\tilde{\Omega}$ in the manifold $X$, then the set $\tilde{A}$ of points indeterminacy of $\tilde{f}$ is an analytic set in $\tilde{\Omega}$ and is an extension of the set of points of indeterminacy of $f$.

*Proof of the sufficiency.* Let $A$ be the points indeterminacy of $f$ and $\tilde{A}$ be an analytic set in $\tilde{\Omega}$ which is an extension of the points indeterminacy of $f$. We put $\Omega_A := \Omega - A$. Then the restriction $h := f|_{\Omega_A}$ is a holomorphic mapping of $\Omega_A$ in the Banach manifold $X$ with the disc property. Let $\pi : \Omega \times X \to \Omega$ be the canonical projection. The graph $G(h)$ of $h$ coincides with $\pi^{-1}(\Omega_A) \cap G$. Let $(\tilde{\lambda}_A, \tilde{\Omega}_A, \tilde{\varphi}_A)$ be the locally pseudoconvex hull of the domain $(\Omega_A, \varphi|_{\Omega_A})$ over the manifold $M$ and $\tilde{\pi} \cdot \tilde{\Omega} \times X \to \tilde{\Omega}$ be the canonical projection. According to Matsuda[14], $h$ has a holomorphic extension $\tilde{h}$ to the pseudoconvex hull $\tilde{\Omega}_A$.

Let $\overline{G(h)}$ be the closure of the graph $G(h)$ in the product space $\tilde{\Omega} \times X$. The intersection $\overline{G(h)} \cap \tilde{\pi}^{-1}(\tilde{\Omega} - \tilde{A})$ is an analytic set in the product space $\tilde{\Omega} \times X$. In other words, the closed subset $\overline{G(h)}$ is analytic in the complement of the analytic set $\tilde{A} \times X$ in the product space $\tilde{\Omega} \times X$. Moreover, the set $\tilde{\lambda}(\Omega) \times X$ is a connected nonempty open subset of the product space $\tilde{\Omega} \times X$ and the intersection $\overline{G(h)} \cap (\tilde{\lambda}(\Omega) \times X)$ is analytic across the analytic set $\tilde{\pi}^{-1}(\tilde{A})$. Hence, the set $\overline{G(h)}$ is not singular in the analytic set $\tilde{\pi}^{-1}(\tilde{A})$.

Since the closure $\overline{G(h)}$ is an analytic set in the product space $\tilde{\Omega} \times X$ and since the restriction $\tilde{\pi}|_{\overline{G(h)}}$ is a modification, it defines a meromorphic mapping $\tilde{f}$ such that $\tilde{f}|_{\tilde{\lambda}(\Omega)} = f$, that $\tilde{\pi}|_{\overline{G(h)} - \tilde{\pi}^{-1}(\tilde{A})}$ is a biholomorphic mapping onto $\tilde{\Omega} - \tilde{A}$, $\tilde{f}$ is a desired meromorphic extension of $f$ to the locally pseudoconvex hull $\tilde{\Omega}$.

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