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CONTINUATION OF KERNEL FUNCTIONS FOR INFINITE DIMENSIONAL REINHARDT DOMAINS

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1. Introduction

Sommer-Mehring[7] investigated the Kernhülle K(D) of a bounded domain D in the finite n-dimensional complex space C^n and obtained the relation

$$H(D) \subset K(D) \subset A(D) \tag{1}$$

where $H(D)$ is the envelope of holomorphy of the domain D and $A(D)$ is the open kernel of the intersection of domains of holomorphy, which contain D as a relatively compact subset.

On the other hand, Nishihara[6] investigated domains of convergence of power series in Reinhardt domains of a Fréchet space with unconditional Schauder basis. In the previous paper [2] and [3], we introduced kernel functions $K(z,\zeta)$ for domains D in a separable Hilbert space and in the previous papers [4] and [5], the author calculated domains of convergence of $K(z,\zeta)$ for polydiscs and ellipsoids.

In the present paper under the condition (6) corresponding to the condition concerning Nebenhülle of Sommer-Mehring[7], she proves that the domain of convergence of the power series at the origin of the kernel function of a complete Reinhardt domain containing the origin coincides with the logarithmically convex hull of it and extends the results of Sommer-Mehring[7] to domains of infinite dimension.

2. Abstract Wiener measures

A triple (B_1, T, B_2) of a self adjoint nuclear mapping T of a Banach space B_1 into a Banach space B_2 is called an *abstract Wiener spaces*. Gross[1] gave an *abstract Wiener measure* to the triple (B_1, T, B_2).

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When B_1 and B_2 are separable Hilbert spaces, we can regard them as the Hilbert space

$$(2) \qquad \ell^2 := \{(z_1, z_2, \cdots, z_n, \cdots); \sum_{n=1}^{\infty} |z_n|^2 < +\infty\}$$

of square summable sequences of complex numbers. Let $\{\nu_n; \nu \geq 1\}$ be a sequence of positive numbers satisfying $\sum_{n=1}^{\infty} \nu_n < +\infty$. We define a nuclear mapping

(3)

$$T : \ell^2 \to \ell^2,$$

$$\ell \ni z = (z_1, z_2, \cdots, z_n, \cdots) \rightsquigarrow T(z) := (\nu_1 z_1, \nu_2 z_2, \cdots, \nu_n z_n, \cdots) \in \ell^2$$

and regard the triple (ℓ^2, T, ℓ^2) as an abstract Wiener space. In the previous paper [2], for a domain D in the Hilbert space ℓ^2 given as (2), we defined the kernel function $K(z, w)$ for a general domain D in the space ℓ^2 and, for a Reinhardt domain D containg the origin in the space ℓ^2, we gave the following representation as Theorem 2 of [2]:

$$(4) \qquad K(z, w) = \sum_{\alpha} \frac{z^\alpha \overline{w^\alpha}}{\int_D |z^\alpha|^2 \mathrm{d}\mu_T}$$

for any $(z, w) \in D_T \times D_T$, where $D_T := D \cap T(\ell^2)$.

Let n be any positive integer and $\pi_n : C^n \to \ell^2$ be the canonical injection defined by

$$(5) \qquad C^n \ni (z_1, z_2, \cdots, z_n) \rightsquigarrow (z_1, z_2, \cdots, z_n, 0, 0, \cdots) \in \ell^2.$$

Let D be a Reinhart domain containing the origin in ℓ^2 and $\tilde{D}$ be its envelope of holomorphy. Of course, $\tilde{D}$ coincides with the logarithmic convex hull of D by the results of Nishihara[6]. We assume hereafter that, for any positive integer n, there exists a bounded Reinhardt domain D_n containing the origin in C^n, whose envelope of holomorphy is denoted by $\tilde{D}_n$, and that, for the canonical projection $p_n : \ell^2 \to C^n$, the sequence $\{p_n^{-1}(D_n); n \geq 1\}$ is monotonically decreasing and there holds

$$(6) \qquad C\tilde{D} = \quad \textit{the closure of} \quad \bigcup_{n=1}^{\infty} p_n^{-1}(C\tilde{D_n})$$

which corresponds to the condition on Nebenhülle of Sommer-Mehring [23], where the notation C denotes the complement.

Let $\mathrm{Conv}(D_z)$ be the intersection of the domain of convergence of the power series $K(z,z)$ in the variable $z \in \ell^2$ and the dense image $T(\ell^2)$, and let $\mathrm{Conv}(D_{z,w})$ be the intersection of the domain of convergence of the power series $K(z,w)$ in the variable $(z,w) \in \ell^2 \times \ell^2$ and the dense image $T(\ell^2) \times T(\ell^2)$. We have $D_T \subset \mathrm{Conv}(D_z) \subset T(\ell^2)$.

Main Theorem. *Let D be a bounded complete Reinhardt domain containing the origin in the space ℓ^2 and $\tilde{D}$ be the envelope of holomorphy of D. Under the assumption (6), we have*

$$(7) \qquad \mathrm{Conv}(D_{z,w}) = \{(z,w); z \in \tilde{D}_T, \bar{w} \in \tilde{D}_T\}.$$

Proof. Since there holds

$$(8) \qquad \Big(\sum_\alpha \Big|\frac{z^\alpha \overline{w^\alpha}}{\int_D |z^\alpha|^2 \mathrm{d}\mu_T}\Big|\Big)^2 \leq \sum_\alpha \frac{z^\alpha \overline{z^\alpha}}{\int_D |z^\alpha|^2 \mathrm{d}\mu_T} \times \sum_\alpha \frac{w^\alpha \overline{w^\alpha}}{\int_D |z^\alpha|^2 \mathrm{d}\mu_T}$$

according to the inequality of Schwarz, we have

$$(9) \qquad |K(z,w)|^2 \leq |K(z,z)| \times |K(w,w)|$$

and, hence, we have

$$(10) \qquad \mathrm{Conv}(D_z) \times \mathrm{Conv}(D_w) \subset \mathrm{Conv}(D_{z,w}).$$

In accordance with Nishihara[6], we have

$$(11) \qquad \tilde{D}_T \subset \mathrm{Conv}(D_z)$$

and, hence, we have

$$\tilde{D}_T \times \tilde{D}_T \subset \mathrm{Conv}(D_{z,w}). \tag{12}$$

According to Nishihara[6] and the above preparations, the domain of convergence $K(z,z)$ is a bounded complete logarithmically convex Reinhardt domain in the Hilbert space ℓ^2.

In order to prove the inequality reverse to (12) by the method of reduction to absurd, we assume that there were a point $(z^{(0)}, w^{(0)}) \in \mathrm{Conv}(D_{z,w})$ with $(z^0, w^{(0)}) \notin \tilde{D}_T \times \tilde{D}_T$. We may assume that $z^{(0)} \notin \tilde{D}_T$. Since there holds $(z^{(0)}, w^{(0)}) \in \mathrm{Conv}(D_{z,w})$, there exist neighborhood U and V, respectively of $z^{(0)}$ and $w^{(0)}$ in ℓ^2, such that there holds

$$(U \times V) \cap (\ell^2 \times \ell^2) \subset \mathrm{Conv}(D_{z,w}). \tag{13}$$

Since there holds $z^{(0)} \notin \tilde{D}_T$, by the assumption (6), the point $z^{(0)}$ belongs to the closure of the union $\bigcup_{n=1}^{\infty} p_n^{-1}(\mathit{complement\ of\ } \tilde{D}_n)$. Hence, there exists a positive integer n with $U \cap p_n^{-1}$ $(\mathit{complement\ of\ } \tilde{D}_n) \neq \phi$, a point of which is denoted by $z^{(1)}$. By the theory of convex sets of finite dimension, there would be a continuous real valued linear functional s_n on $\boldsymbol{R}^n$ such that we would have $s_n \leq 0$ on $\tilde{D}_n$ and $s_n(z^{(1)}) > 0$. Without loss of generality, we may assume that all coefficients of s_n are non negative integers. There would be a complex valued continuous complex linear functional $h_n(z)$ with coefficients non negative integers on the complex linear space $\boldsymbol{C}^n$ such that $\mathrm{Real}(h_n) = s_n$. We may assume that the imaginary part of $h_n(z^{(1)}) = 0$. Then we have

$$|h_n(z^{(1)})| > 1, \ \sup\{|h_n(z)|; z \in \tilde{D}_n\} \leq 1. \tag{14}$$

Since the holomorphic function $f(z)$ on D defined by

$$f(z) := \frac{1}{e^{h_n(z)} - e^{h_n(z^{(1)})}} \tag{15}$$

is D-bounded on D and, therefore, belongs to the Hilbert space $\mathcal{A}_b^2(D_T, d\mu)$. Since $K(z, w)$ is the reproducing kernel of the function space $\mathcal{A}_b^2(D_T, d\mu)$, according to [3] there holds the integral representation

$$f(z) = \int_D K(z, w) f(w) d\mu_T(w), \tag{16}$$

and the function $f(z)$ is holomorphically continued to the point $z^{(1)} \in \boldsymbol{C}^n \subset T(\ell^2)$, what conflicts with the above construction of the holomorphic function $f(z)$, which has the point $z^{(1)}$ as a singularity.

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