CONTINUATION OF KERNEL FUNCTIONS FOR INFINITE DIMENSIONAL REINHARDT DOMAINS

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1. Introduction

Sommer-Mehring[7] investigated the Kernhülle $K(D)$ of a bounded domain $D$ in the finite $n$-dimensional complex space $\mathbb{C}^n$ and obtained the relation

$$H(D) \subset K(D) \subset A(D)$$

where $H(D)$ is the envelope of holomorphy of the domain $D$ and $A(D)$ is the open kernel of the intersection of domains of holomorphy, which contain $D$ as a relatively compact subset.

On the other hand, Nishihara[6] investigated domains of convergence of power series in Reinhardt domains of a Fréchet space with unconditional Schauder basis. In the previous papers [2] and [3], we introduced kernel functions $K(z, \zeta)$ for domains $D$ in a separable Hilbert space and in the previous papers [4] and [5], the author calculated domains of convergence of $K(z, \zeta)$ for polydiscs and ellipsoids.

In the present paper under the condition (6) corresponding to the condition concerning Nebenhülle of Sommer-Mehring[7], she proves that the domain of convergence of the power series at the origin of the kernel function of a complete Reinhardt domain containing the origin coincides with the logarithmically convex hull of it and extends the results of Sommer-Mehring[7] to domains of infinite dimension.

2. Abstract Wiener measures

A triple $(B_1, T, B_2)$ of a self adjoint nuclear mapping $T$ of a Banach space $B_1$ into a Banach space $B_2$ is called an abstract Wiener spaces. Gross[1] gave an abstract Wiener measure to the triple $(B_1, T, B_2)$.
When $B_1$ and $B_2$ are separable Hilbert spaces, we can regard them as the Hilbert space

\[
\ell^2 := \{(z_1, z_2, \cdots, z_n, \cdots) ; \sum_{n=1}^{\infty} |z_n|^2 < +\infty\}
\]

of square summable sequences of complex numbers. Let $\{\nu_n; \nu \geq 1\}$ be a sequence of positive numbers satisfying $\sum_{n=1}^{\infty} \nu_n < +\infty$. We define a nuclear mapping

\[
T : \ell^2 \rightarrow \ell^2,
\]

such that $z = (z_1, z_2, \cdots, z_n, \cdots) \mapsto T(z) := (\nu_1 z_1, \nu_2 z_2, \cdots, \nu_n z_n, \cdots) \in \ell^2$

and regard the triple $(\ell^2, T, \ell^2)$ as an abstract Wiener space. In the previous paper [2], for a domain $D$ in the Hilbert space $\ell^2$ given as (2), we defined the kernel function $K(z, w)$ for a general domain $D$ in the space $\ell^2$ and, for a Reinhardt domain $D$ containing the origin in the space $\ell^2$, we gave the following representation as Theorem 2 of [2]:

\[
K(z, w) = \sum_{\alpha} \frac{z^{\alpha} w^{\alpha}}{\int_{D} |z^\alpha|^2 \mu_T}.
\]

for any $(z, w) \in D_T \times D_T$, where $D_T := D \cap T(\ell^2)$.

Let $n$ be any positive integer and $\pi_n : C^n \rightarrow \ell^2$ be the canonical injection defined by

\[
C^n \ni (z_1, z_2, \cdots, z_n) \mapsto (z_1, z_2, \cdots, z_n, 0, 0, \cdots) \in \ell^2.
\]

Let $D$ be a Reinhardt domain containing the origin in $\ell^2$ and $\tilde{D}$ be its envelope of holomorphy. Of course, $\tilde{D}$ coincides with the logarithmic convex hull of $D$ by the results of Nishihara[6]. We assume hereafter that, for any positive integer $n$, there exists a bounded Reinhardt domain $D_n$ containing the origin in $C^n$, whose envelope of holomorphy is denoted by $\tilde{D}_n$, and that, for the canonical projection $p_n : \ell^2 \rightarrow C^n$,
the sequence \( \{ p_n^{-1}(D_n); n \geq 1 \} \) is monotonically decreasing and there holds

\[
(6) \quad C \tilde{D} = \text{the closure of } \bigcup_{n=1}^{\infty} p_n^{-1}(C \tilde{D}_n)
\]

which corresponds to the condition on Nebenhüle of Sommer-Mehring [23], where the notation \( C \) denotes the complement.

Let \( \text{Conv}(D_z) \) be the intersection of the domain of convergence of the power series \( K(z, z) \) in the variable \( z \in \ell^2 \) and the dense image \( T(\ell^2) \), and let \( \text{Conv}(D_{z, w}) \) be the intersection of the domain of convergence of the power series \( K(z, w) \) in the variable \( (z, w) \in \ell^2 \times \ell^2 \) and the dense image \( T(\ell^2) \times T(\ell^2) \). We have \( D_T \subset \text{Conv}(D_z) \subset T(\ell^2) \).

**Main Theorem.** Let \( D \) be a bounded complete Reinhardt domain containing the origin in the space \( \ell^2 \) and \( \tilde{D} \) be the envelope of holomorphy of \( D \). Under the assumption (6), we have

\[
(7) \quad \text{Conv}(D_{z, w}) = \{ (z, w); z \in \tilde{D}_T, w \in \tilde{D}_T \}.
\]

**Proof.** Since there holds

\[
(8) \quad \left( \sum_{\alpha} \frac{|z^{\alpha} w^{\bar{\alpha}}|}{\int_D |z^{\alpha}|^2 d\mu_T} \right)^2 \leq \sum_{\alpha} \frac{|z^{\alpha} z^{\bar{\alpha}}|}{\int_D |z^{\alpha}|^2 d\mu_T} \times \sum_{\alpha} \frac{|w^{\alpha} w^{\bar{\alpha}}|}{\int_D |z^{\alpha}|^2 d\mu_T}
\]

according to the inequality of Schwarz, we have

\[
(9) \quad |K(z, w)|^2 \leq |K(z, z)| \times |K(w, w)|
\]

and, hence, we have

\[
(10) \quad \text{Conv}(D_z) \times \text{Conv}(D_w) \subset \text{Conv}(D_{z, w}).
\]

In accordance with Nishihara[6], we have

\[
(11) \quad \tilde{D}_T \subset \text{Conv}(D_z)
\]
and, hence, we have

(12) \[ \tilde{D}_T \times \tilde{D}_T \subset \text{Conv}(D_{z,w}). \]

According to Nishihara[6] and the above preparations, the domain of convergence \( K(z,z) \) is a bounded complete logarithmically convex Reinhardt domain in the Hilbert space \( \ell^2 \).

In order to prove the inequality reverse to (12) by the method of reduction to absurd, we assume that there were a point \((z^{(0)}, w^{(0)}) \in \text{Conv}(D_{z,w}) \) with \((z^0, w^{(0)}) \notin \tilde{D}_T \times \tilde{D}_T \). We may assume that \( z^{(0)} \notin \tilde{D}_T \). Since there holds \((z^{(0)}, w^{(0)}) \in \text{Conv}(D_{z,w}) \), there exist neighborhood \( U \) and \( V \), respectively of \( z^{(0)} \) and \( w^{(0)} \) in \( \ell^2 \), such that there holds

(13) \[ (U \times V) \cap (\ell^2 \times \ell^2) \subset \text{Conv}(D_{z,w}). \]

Since there holds \( z^{(0)} \notin \tilde{D}_T \), by the assumption (6), the point \( z^{(0)} \) belongs to the closure of the union \( \bigcup_{n=1}^{\infty} p_n^{-1}(\text{complement of } \tilde{D}_n) \). Hence, there exists a positive integer \( n \) with \( U \cap p_n^{-1}(\text{complement of } \tilde{D}_n) \neq \emptyset \), a point of which is denoted by \( z^{(1)} \). By the theory of convex sets of finite dimension, there would be a continuous real valued linear functional \( s_n \) on \( R^n \) such that we would have \( s_n \leq 0 \) on \( \tilde{D}_n \) and \( s_n(z^{(1)}) > 0 \). Without loss of generality, we may assume that all coefficients of \( s_n \) are non negative integers. There would be a complex valued continuous complex linear functional \( h_n(z) \) with coefficients non negative integers on the complex linear space \( C^n \) such that \( \text{Real}(h_n) = s_n \). We may assume that the imaginary part of \( h_n(z^{(1)}) = 0 \). Then we have

(14) \[ |h_n(z^{(1)})| > 1, \sup\{|h_n(z)|; z \in \tilde{D}_n\} \leq 1. \]

Since the holomorphic function \( f(z) \) on \( D \) defined by

(15) \[ f(z) := \frac{1}{e^{h_n(z)} - e^{h_n(z^{(1)})}} \]
Continuation of kernel functions

is $D$-bounded on $D$ and, therefore, belongs to the Hilbert space $A_2(D, d\mu)$. Since $K(z, w)$ is the reproducing kernel of the function space $A_2(D, d\mu)$, according to [3] there holds the integral representation

(16) \[ f(z) = \int_D K(z, w)f(w)d\mu_T(w), \]

and the function $f(z)$ is holomorphically continued to the point $z^{(1)} \in C^n \subset T(\ell^2)$, what conflicts with the above construction of the holomorphic function $f(z)$, which has the point $z^{(1)}$ as a singularity.

References


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