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q–ANALOGUES OF p–ADIC FOURIER TRANSFORMS

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0. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the ring of integers and the set of all positive integers respectively and let $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the field of rational numbers, the field of real numbers, the field of complex numbers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$ respectively. $\mathbb{R}_{\geq 0}$ is denoted by the set consisting of all positive real numbers and 0 and $\mathbf{J}$ denote the set $\mathbb{N}$ with the p-adic valuation $|\ |$, which is normalized as $|p| = p^{-1}$; hence $\mathbf{J}$ is regarded as a dense subset of the p-adic integer ring $\mathbb{Z}_p$.

The purpose of this paper is to give generalized Fourier transforms on test functions which is important to the study of p-adic quantum mechanics. Our goals are to construct p-adic q-Fourier transform on X, which is defined in section 2. In section 1, we will introduce already known results to obtain our results in section 2. In section 2, we study the q-analogues of p-adic Fourier transforms on the space of bounded functions.

1. p-adic Fourier transforms of test functions on $\mathbb{Q}_p$

Recently new models of quantum physics were proposed on the basis of p-adic number field $\mathbb{Q}_p$. The p-adic Fourier transforms are important to the study of p-adic quantum mechanics. Integral of the form

$$\int_{\mathbb{Q}_p} \chi_p(\xi x)\varphi(x)dx, \qquad \xi, x \in \mathbb{Q}_p$$

is called *p-adic Forier transforms* of test function $\varphi(x)$(see [7]).

Any p-adic number $x \neq 0$ is uniquely represented in the canonical form $x = p^{\gamma}(x_0 + x_1 p + x_2 p^2 + \cdots)$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers such that $0 \leq x_j \leq p-1$, $x_0 > 0$, $j = 0, 1, 2, \cdots$.

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The field $\mathbb{Q}_p$ is a commutative and associative group with respect to addition. $\mathbb{Q}_p^* = \mathbb{Q}_p - \{0\}$ is a commutative and associative group with respect to multiplication.

From the representation of p-adic number $x \neq 0$, the fractional part $\{x\}_p$ of a number $x \in \mathbb{Q}_p$ is given by

$$\{x\}_p = \begin{cases} 0 & \text{if } \gamma \geq 0 \text{ or } x = 0, \\ p^\gamma(x_0 + x_1 p + x_2 p^2 + \cdots + x_{|\gamma|-1} p^{|\gamma|-1}) & \text{if } \gamma < 0. \end{cases}$$

The space $\mathbb{Q}_p^n = \mathbb{Q}_p \times \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ consists of points $x = (x_1, \cdots, x_n), x_j \in \mathbb{Q}_p, j = 1, 2, \cdots, n$. The norm on $\mathbb{Q}_p^n$ is $|x|_p = \max_{1 \leq j \leq n} |x_j|_p$, $x \in \mathbb{Q}_p^n$. This is a non-Archimedean norm since $|x+y|_p \leq \max(|x|_p, |y|_p)$, $x, y \in \mathbb{Q}_p^n$. The space $\mathbb{Q}_p^n$ is a clearly complete metric locally-compact and totally disconnected space. We introduce the inner product by $< x, y >= x_1 y_1 + \cdots + x_n y_n;\ x, y \in \mathbb{Q}_p^n$. From this, the following Schwartz inequality is valid: $| < x, y > |_p \leq |x|_p |y|_p,\ x, y \in \mathbb{Q}_p^n$.

Denote by $B_\gamma(a)$ the ball of radius p^γ with center at the point $a \in \mathbb{Q}_p^n$ and by $S_\gamma(a)$ its boundary (sphere). For the notational convenience let $B_\gamma(0) = B_\gamma$ and $S_\gamma(0) = S_\gamma$, $\gamma \in \mathbb{Z}$. If $a = (a_1, \cdots, a_n) \in \mathbb{Q}_p^n$ then $B_\gamma(a) = B_\gamma(a_1) \times \cdots \times B_\gamma(a_n)$ in $\mathbb{Q}_p^n$. $B_\gamma(a)$ and $S_\gamma(a)$ are clearly closed-open sets.

An additive character of an additive group $\mathbb{Q}_p$ is a continuous complex valued function

$$\chi : \mathbb{Q}_p \longrightarrow \mathbb{C}$$

satisfying the conditions; (i) $|\chi(x)| = 1$ and (ii) $\chi(x + y) = \chi(x)\chi(y)$, $x, y \in \mathbb{Q}_p$. It is clear that every additive character of the field $\mathbb{Q}_p$ is a character of any group $B_\gamma, \gamma \in \mathbb{Z}$.

The function $\chi_p(\xi x) = \exp(2\pi i \{\xi x\}_p)$ for every fixed $\xi \in \mathbb{Q}_p$ is an additive character on the field $\mathbb{Q}_p$ and the group B_γ(see [7]). From the definition of the fractional part we have $\{x+y\}_p = \{x\}_p + \{y\}_p - N, N = 0, 1$.

The Haar measure dx is the (essentially) unique invariant measure on the additive group $\mathbb{Q}_p \to \mathbb{C}$: for any $a \in \mathbb{Q}_p$, $d(x + a) = dx$. Its normalization is fixed by taking the measure of $\mathbb{Z}_p$, the set of p-adic integers, as equal to 1: $\mu(\mathbb{Z}_p) = \int_{\mathbb{Z}_p} dx = \int_{|x|_p \leq 1} dx = 1$.

Let us now take the set of numbers with a given p-adic norm p^γ. Cleary

$$\mu(\{\, |x|_p = p^\gamma\}) = \mu(p^{-\gamma}\mathbb{Z}_p) - \mu(p^{-\gamma+1}\mathbb{Z}_p) = p^\gamma(1 - p^{-1}).$$

That is

$$\int_{S_\gamma} dx = \int_{B_\gamma} dx - \int_{B_{\gamma-1}} dx = p^\gamma\left(1-\frac{1}{p}\right). \tag{1.1}$$

The formula (1.1) is essentially all that is needed for integration over $\mathbb{Q}_p$ or any of its subsets.

PROPOSITION 1.1 [7]. *For* $\gamma \in \mathbb{Z}$,

(a) $\int_{S_\gamma, x_0=k} dx = p^{\gamma-1}, \quad k = 1, 2, \cdots, p-1.$

(b) $\int_{S_\gamma, x_0\neq k} dx = p^\gamma\left(1-\frac{2}{p}\right), \quad k = 1, 2, \cdots, p-1.$

(c) $\int_{S_\gamma, x_l=k} dx = p^{(\gamma-1)}\left(1-\frac{1}{p}\right), \quad l = 1, 2, \cdots, \; k = 0, 1, 2, \cdots, p-1.$

(d) *For* $l = 0, 1, 2, \cdots, 0 \leq k_j < p-1, k_0 \neq 0,$

$$\int_{S_\gamma, x_0=k_0, x_1=k_1, \cdots, x_l=k_l} dx = p^{(\gamma-l-1)}.$$

(e) *For* $\gamma \in \mathbb{Z}$, *let* χ_p *be the additive character of the field* $\mathbb{Q}_p$. *Then*

$$\int_{B_\gamma} \chi_p(\xi x)dx = p^\gamma \Omega(|\xi p^{-\gamma}|_p),$$

where $\Omega(\alpha)$ *is 1 if* $0 \leq \alpha \leq 1$ *and 0 if* $\alpha > 1$.

(f) *Let* χ_p *be the additive character of field* $\mathbb{Q}_p$. *Then we have*

$$\int_{S_\gamma} \chi_p(\xi x)dx = \begin{cases} p^\gamma\left(1-\frac{1}{p}\right), & |\xi|_p \leq p^{-\gamma} \\ -p^{(\gamma-1)}, & |\xi|_p = p^{-\gamma+1} \\ 0, & |\xi|_p \geq p^{-\gamma+2}. \end{cases}$$

A complex-valued function $f(x)$ defined on $\mathbb{Q}_p$ is called *locally- constant* if for any point $x \in \mathbb{Q}_p$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$f(x+x') = f(x), \quad |x'|_p \leq p^{l(x)}.$$

The set of locally-constant functions on $\mathbb{Q}_p$ denotes as $\mathcal{E} = \mathcal{E}(\mathbb{Q}_p)$. We call it a *test function* if every function from $\mathcal{E}$ with compact support. When the set of test function is linear, we denote it by $\mathcal{D} = \mathcal{D}(\mathbb{Q}_p)$. Let $\varphi \in \mathcal{D}$. Then there exists $l \in \mathbb{Z}$, such that

$$\varphi(x + x') = \varphi(x), \quad x' \in B_l, \ x \in \mathbb{Q}_p.$$

Such largest number l we call *the parameter of constancy* of a function φ, $l = l(\varphi)$. Let $\mathcal{D}^l_N = \mathcal{D}^l_N(\mathbb{Q}_p)$ be denoted by the set of test function with support in the disc B_N and with parameter of constancy $\geq l$. Let $\varphi \in \mathcal{D}$. Its *Fourier-transform* $F[\varphi] = \widetilde{\varphi}$ is defined by the formula

$$\widetilde{\varphi}(\xi) = \int_{\mathbb{Q}_p} \chi_p(\xi x)\varphi(x)dx, \quad \xi \in \mathbb{Q}_p. \tag{1.2}$$

PROPOSITION 1.2 [7]. *The Fourier-transform $\varphi \longrightarrow \widetilde{\varphi}$ is the linear isomorphism $\mathcal{D}$ onto $\mathcal{D}$, and also have the inversion Fourier-transform formula*

$$\varphi(x) = \int_{\mathbb{Q}_p} \chi_p(-x\xi)\widetilde{\varphi}(\xi)d\xi, \quad \widetilde{\varphi}, \varphi \in \mathcal{D}.$$

Thus the Parseval-Steklov equalities are valid:

$$\int_{\mathbb{Q}_p} \varphi(x)\overline{\psi(x)}dx = \int_{\mathbb{Q}_p} \widetilde{\varphi}(\xi)\overline{\widetilde{\psi}(\xi)}d\xi, \quad \varphi, \psi \in \mathcal{D},$$

$$\int_{\mathbb{Q}_p} \varphi(x)\widetilde{\psi}(x)dx = \int_{\mathbb{Q}_p} \widetilde{\varphi}(\xi)\psi(\xi)d\xi, \quad \varphi, \psi \in \mathcal{D}.$$

2. p-adic q-analogue Fourier transforms $F_q[f^\rho] = \widehat{f^\rho}_q$

Now let us consider a bounded function $f(x)$ defined on $\mathbf{J}$ and taking its values in $\mathbb{C}_p$, namely $f(x) \in \mathbb{C}_p$ and there exists a constant L depending on f such that $|f(x)| \leq L$ for any $x \in \mathbf{J}$. The set $B(\mathbf{J}, \mathbb{C}_p)$ of all bounded functions makes an algebra over $\mathbb{C}_p$ under the pointwise addition and multiplication.

Let $\mathbb{Z}/p^N\mathbb{Z}$ be the residue class ring of the rational integer ring $\mathbb{Z}$ module p^N ($N \in \mathbb{N}$), and ζ is a primitive p^N-th root of unity in $\mathbb{C}_p$.

Then functions ζ^{mx} of $x \in \mathbb{Z}$ $(m = 0, 1, \cdots, p^N - 1)$ are all the characters of the additive group of $\mathbb{Z}/p^N\mathbb{Z}$. We identify any of the induced function on $\mathbb{Z}/p^N\mathbb{Z}$. For any given bounded function $f \in B(\mathbf{J}, \mathbb{C}_p)$ we make an induced function $f^N(x)$ on $\mathbb{Z}/p^N\mathbb{Z}$ by

$$f^N(x) = f\left(x - p^N\left[\frac{x}{p^N}\right]_g\right) \qquad (x \in \mathbb{Z}), \tag{2.1}$$

where $[z]_g$ denotes the greatest integer not exceeding the real number z, namely $[\ ]_g$ means the Gauss' symbol.

If $q \in \mathbb{C}$, and assume again that $|q| < 1$. If $q = 1 + t \in \mathbb{C}_p$, we normally assume $|t|_p < 1$. We shall further suppose that $ord_p t > \frac{1}{1-p}$, so that $q^x = \exp(x \log_p q)$ for $|x|_p \leq 1$.

We use the notation

$$[x] = [x; q] = \frac{1 - q^x}{1 - q}. \tag{2.2}$$

Thus, we obtain $\lim_{q\to 1}[x; q] = x$ for any x with $|x|_p \leq 1$.

For any fixed positive integer d we easily see that

$$\frac{1}{[p; q^{dp^N}]} \sum_{i=0}^{p-1} q^{idp^N} = 1. \tag{2.3}$$

Let d be a fixed positive integer, and let p be a fixed prime number. Let

$$X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}),$$

where the map from $\mathbb{Z}/dp^M\mathbb{Z}$ to $\mathbb{Z}/dp^N\mathbb{Z}$ for $M \geq N$ is a reduction mod dp^N. Let $a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\}$. Without loss of generality, we may always choose a so that $0 \leq a < dp^N$. Also,

$$a + dp^N\mathbb{Z}_p = \bigcup_{0 \leq b < p} (a + bdp^N) + dp^{N+1}\mathbb{Z}_p \qquad \text{(disjointed union)}. \tag{2.4}$$

We write $a + dp^N\,\mathbb{Z}_p = a + (dp^N)$.

PROPOSITION 2.1 [3]. *Let μ_q be given by*

$$\mu_q(a+(dp^N)) := \frac{q^a}{[dp^N]} = \frac{q^a}{[dp^N;q]}. \tag{2.5}$$

Then μ_q extends to a distribution on the compact open sets $U \subset X$.

REMARK 2.2. For the ordinary p-adic distribution μ_0 defined by (see [4])

$$\mu_0(a+(dp^N)) = \frac{1}{dp^N},$$

we see

$$\lim_{q\to 1}\mu_q = \mu_0.$$

We can evaluate $\int_X f d\mu_q$ as the limit

$$\begin{aligned}\int_X f d\mu_q &= \lim_{N\to\infty}\sum_{0\le a<dp^N} f(a)\mu_q(a+(dp^N)) \\ &= \frac{1}{[dp^N]}\sum_{a=0}^{dp^N-1} f(a)q^a.\end{aligned} \tag{2.6}$$

Also, we have(see [3])

$$\int_X f(x)\,d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x).$$

Thus we easily see that

$$\int_X d\mu_q(x) = \int_{\mathbb{Z}_p} d\mu_q(x),$$

because we set $f=1$.

Now, for $f\in B(\mathbf{J},\mathbb{C}_p)$ we will construct the p-adic q-Fourier trans-from $F_q[f^\rho] = \widehat{f^\rho}_q$ on $\mathbb{C}_p$. If $\lim_{q\to 1}\widehat{f^\rho}_q = \widehat{f^\rho}_1 = \widehat{f^\rho}$, $\widehat{f^\rho}$ is similar to Fourier transform on the complex number field $\mathbb{C}$.

We consider the $F_q[f^\rho] = \widehat{f^\rho}_q$ as

(2.7)
$$\begin{aligned}\widehat{f^\rho}_q(y) &= \int_X \zeta^{yx} f^\rho(x) d\mu_q(x)\\ &= \lim_{\rho\to\infty} \frac{1}{[dp^\rho]} \sum_{x=0}^{dp^\rho - 1} \zeta^{yx} f^\rho(x) q^x, \qquad y \in \mathbb{Z}.\end{aligned}$$

Thus this function $\widehat{f^\rho}_q$ is q-analogue form of the Fourier transform. Also its inverse q-analogue Fourier transform is

(2.8)
$$\begin{aligned}f^\rho(x) &= \int_X \zeta^{-xy} \widehat{f^\rho}_q(y) q^{-y} d\mu_q(y)\\ &= \lim_{\rho\to\infty} \frac{1}{[dp^\rho]} \sum_{y=0}^{dp^\rho - 1} \zeta^{-xy} \widehat{f^\rho}_q(y), \qquad x \in \mathbb{Z}.\end{aligned}$$

since

$$\begin{aligned}&\int_X \zeta^{-xy} \widehat{f^\rho}_q(y) q^{-y} d\mu_q(y)\\ &\quad= \int_X \zeta^{-xy} \int_X \zeta^{yx'} f^\rho(x') d\mu_q(x') d\mu_q(y)\\ &\quad= \int_X f^\rho(x') \int_X \zeta^{-xy+yx'} q^{-y} d\mu_q(y) d\mu_q(x')\\ &\quad= f^\rho(x) + \int_{X, x\neq x'} f^\rho(x') \int_X \zeta^{(x'-x)y} q^{-y} d\mu_q(y) d\mu_q(x')\\ &\quad= f^\rho(x)\end{aligned}$$

Hence we get:

PROPOSITION 2.3. *Let $f \in B(\mathbf{J}, \mathbb{C}_p)$ and assume that $\widehat{f^\rho}_q \in B(\mathbf{J}, \mathbb{C}_p)$. Then for all $x \in \mathbb{Z}$*

$$f^\rho(x) = \int_X \zeta^{-xy} \widehat{f^\rho}_q(y) q^{-y} d\mu_q(y).$$

In this case, we calls $\widehat{f^\rho}_q$ is p-adic q-Fourier transform of f^ρ.

PROPOSITION 2.4. *If* $\lim_{q \to 1} \mu_q = \mu_o$, *then:*

- $\lim_{q \to 1} \int_X \zeta^{-xy} \widehat{f^\rho}_q(y) q^{-y} d\mu_q(y) = \int_X \zeta^{-xy} \widehat{f^\rho}_1(y) d\mu_0(x)$.
- $\widehat{f^\rho}_1(y) = \lim_{\rho \to \infty} \frac{1}{dp^\rho} \sum_{x=0}^{dp^\rho - 1} \zeta^{yx} f_1^\rho(x), \quad y \in \mathbb{Z}$.
- $f_1^\rho(x) = \lim_{\rho \to \infty} \frac{1}{dp^\rho} \sum_{y=0}^{dp^\rho - 1} \zeta^{-xy} \widehat{f_1^\rho}(y), \quad x \in \mathbb{Z}$.

This formula resembles the formula of a finite Fourier transform argument in [2],[5],and [8].

COROLLARY 2.5. *If we consider a sequence of partitions of* $\mathbb{Z}_p$, *then*

$$\mathcal{P}(\mathbb{Z}_p) = \bigcup_{j=0}^{dp^N - 1} B_{-N}(j)$$

and computes $\int_{\mathbb{Z}_p} f(x) d\mu_0(x)$ *as the limit*

$$\int_{\mathbb{Z}_p} f(x) d\mu_0 = \lim_{N \to \infty} \sum_{x=0}^{dp^N - 1} f(x) \frac{1}{dp^N}, \qquad x \in B_{-N}(j).$$

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