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A NOTE ON SEQUENTIAL CONVERGENCE STRUCTURES

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1. Introduction and preliminaries

Recall that a topological space X is called *a Fréchet-Urysohn space* [1] (also simply called *a Fréchet space*) if it satisfies the following property(called *the Fréchet-Urysohn property*[10]): *every point in the closure of a subset A of X is a limit point of a sequence of points in A*. Indeed, topological spaces that satisfy the first axiom of countability form a special group in the class of Fréchet-Urysohn spaces and the metric spaces are distinguished in the former. Many authors have studied several properties of Fréchet-Urysohn spaces and related topics (See [1-10]). Recently, in [8], the author introduced sequential convergence structures and showed that Fréchet-Urysohn spaces are determined by these structures.

In order to construct our goal of this paper, we first introduce some results of [8]. Let X be a non-empty set, $P(X)$ the power set of X, and let $S(X)$ be the set of all sequences in X. Sequences in X will be denoted by small Greek letters α, β, γ etc. The k-th term of the sequence α is denoted by $\alpha(k)$. The small Latin letters s, t etc. denote monotone increasing functions of the natural number set $\mathbb{N}$ into itself. The composition $\alpha \circ s$ is the subsequence of α which has $\alpha(s(k))$ as the k-th term.

A non-empty subfamily L of the cartesian product $S(X) \times X$ is called a *sequential convergence structure on* X[8] if it satisfies the following three conditions:

(SC 1): For each $x \in X$, $((x), x) \in L$, where (x) is the constant sequence whose n-th term is x for all indices $n \in N$.

(SC 2): If $(\alpha, x) \in L$, then $(\beta, x) \in L$ for each subsequence β of α.

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(SC 3): Let $x \in X$ and $A \subset X$. If $(\alpha, x) \notin L$ for each $\alpha \in S(A)$, then $(\beta, x) \notin L$ for each $\beta \in S(\{y \in X : (\gamma, y) \in L$ for some $\gamma \in S(A)\})$.

Let $SC[X]$ denote the set of all sequential convergence structures on X.

THEOREM 1.1 [8, THEOREM 1 AND THEOREM 6]. *For $L \in SC[X]$, define a function $C_L : P(X) \to P(X)$ as follows: for each subset A of X, $C_L(A) = \{x \in X : (\alpha, x) \in L$ for some $\alpha \in S(A)\}$. Then, (X, C_L) is a Fréchet-Urysohn space endowed with the topological closure operator C_L.*

Let $\mathcal{L}(C_L)$ denote the set of all pairs $(\alpha, x) \in S(X) \times X$ such that α converges to x in the space (X, C_L).

THEOREM 1.2 [8, THEOREM 3 AND COROLLARY 4]. *For each $L \in SC[X]$, we have*

(1) $L \subset \mathcal{L}(C_L) \in SC[X]$,

(2) $C_L = C_{\mathcal{L}(C_L)}$, and

(3) $\bigcup\{L' \in SC[X] : C_L = C_{L'}\} = \mathcal{L}(C_L)$.

EXAMPLE 1.3 [8, EXAMPLE 5]. In general, $L \neq \mathcal{L}(C_L)$. Let Q be the rational number set with the usual topology and let $L_Q = \{(\alpha, x) \in S(Q) \times Q : \alpha$ converges to x in $Q\}$ and $L = \{((x), x) : x \in Q\} \cup \{(\alpha, x) \in S(Q) \times Q : \alpha$ converges to x in Q and α is either increasing or decreasing$\}$. Then we have

$$L \subsetneq L_Q = \mathcal{L}(C_{L_Q}) = \mathcal{L}(C_L).$$

Hence a question, which is concerned with a sequential convergence structure L on a set X, arises naturally: Is there a sufficient condition for $L = \mathcal{L}(C_L)$?

The purpose of this paper is to give sufficient conditions for $L = \mathcal{L}(C_L) \in SC[X]$.

2. Results

To prove our main theorem(Theorem 2.3 below), we begin with the following lemmas.

LEMMA 2.1. *Let $L \in SC[X]$ and $x, y \in X$. Then, $((x), y) \in L$ if and only if $((x), y) \in \mathcal{L}(C_L)$.*

Proof. By Theorem 1.2(1), it is sufficient to prove that if $((x), y) \in \mathcal{L}(C_L)$, then $((x), y) \in L$. If $((x), y) \in \mathcal{L}(C_L)$, then the constant sequence (x) converges to y in the space (X, C_L) and hence $y \in C_L(\{x\})$. By definition of C_L, $((x), y) \in L$.

LEMMA 2.2. *Let $L \in SC[X]$. Assume that for each $x \in X$, the set $\{z \in X : ((z), x) \in L\}$ is finite. If $(\alpha, x) \in \mathcal{L}(C_L)$, then there is a subsequence β of α such that $(\beta, x) \in L$.*

Proof. If the range $\{\alpha(k) : k \in \mathbb{N}\}$ of α is finite, then $\{k \in \mathbb{N} : \alpha(k) = \alpha(k_0)\}$ is infinite for some $k_0 \in \mathbb{N}$, equivalently, there is an element $\alpha(k_0) \in \{\alpha(k) : k \in \mathbb{N}\}$ such that $(\alpha(k_0))$ is a constant subsequence of α. Since $(\alpha, x) \in \mathcal{L}(C_L)$ and since $\mathcal{L}(C_L) \in SC[X]$ by Theorem 1.2 (1), we have $((\alpha(k_0)), x) \in \mathcal{L}(C_L)$. So, by above Lemma 2.1, $((\alpha(k_0)), x) \in L$ and hence it holds. Hence, it remains to prove the case when $\{\alpha(k) : k \in \mathbb{N}\}$ is infinite. In this case, there is a subsequence α' of α such that $\alpha'(k_1) \neq \alpha'(k_2)$ for each $k_1, k_2 \in \mathbb{N}$ with $k_1 \neq k_2$ Thus, without loss of generality, we now get to assume that $\alpha(k_1) \neq \alpha(k_2)$ for each $k_1, k_2 \in \mathbb{N}$ with $k_1 \neq k_2$ and the set $\{k \in \mathbb{N} : ((\alpha(k)), x) \in L\}$ is empty. Since $(\alpha, x) \in \mathcal{L}(C_L)$, $x \in C_L(\{\alpha(k) : k \in \mathbb{N}\})$, and so we have that $(\gamma, x) \in L$ for some $\gamma \in S(\{\alpha(k) : k \in \mathbb{N}\})$ by the definition of C_L. Either $\{\gamma(k) : k \in \mathbb{N}\})$ is finite or $\{\gamma(k) : k \in \mathbb{N}\})$ is infinite.

Case 1. $\{\gamma(k) : k \in \mathbb{N}\})$ is finite. Since $(\gamma, x) \in L$, it is easy to check that there exists a number $n_0 \in \mathbb{N}$ such that $((\gamma(n_0)), x) \in L$. It is impossible.

Case 2. $\{\gamma(k) . k \in \mathbb{N}\})$ is infinite. Let f be the function from $\mathbb{N}$ into itself with $\gamma(k) = \alpha(f(k))$, for each $k \in \mathbb{N}$. By the definition of C_L, we know that γ need not be a subsequence of α, i.e., f need not be monotone increasing. Since $\{\gamma(k) : k \in \mathbb{N}\}$ is infinite, we get to construct by induction a sequence β as follows: $\beta(1) = \gamma(1) = \alpha(f(1))$ and for each $n \geq 2$, $\beta(n) = \gamma(k_n) = \alpha(f(k_n))$, where $k_1 = 1$ and $k_n = \min\{p \in \mathbb{N} : p > k_{n-1}$ and $f(p) > f(k_{n-1})\}$. Then, it is obvious that $k_1 < k_2 < k_3 < \cdots$ and $f(k_1) < f(k_2) < f(k_3) < \cdots$, and so we see that β is a subsequence of γ and it also a subsequence of α. It follows that $(\beta, x) \in L$ by (SC 2), and therefore it is a desired subsequence of α.

We now obtain our main theorem.

THEOREM 2.3. *Assume the hypothesis of Lemma 2.2. If L satisfies additionally the following two conditions (SC 4) and (SC 5), then $L = \mathcal{L}(C_L)$.*

(SC 4): Let $\alpha \in S(X)$ and $x \in X$. If $(\beta, x) \in L$ for some subsequence β of α, let $\beta = \alpha \circ s$, such that $\mathbb{N} - \{s(k) : k \in \mathbb{N}\}$ is finite, then $(\alpha, x) \in L$.

(SC 5): Let $\alpha \in S(X)$ and $x \in X$. If there is a finite(infinite) sequence (β_n) of subsequences of α, let $\beta_n = \alpha \circ s_n$ for all $n \in \{1, 2, \cdots, p\}$ (for all $n \in \mathbb{N}$), such that $(\beta_n, x) \in L$ for all $n \in \{1, 2, \cdots, p\}$(for all $n \in \mathbb{N}$), $\{s_i(k) : k \in \mathbb{N}\} \cap \{s_j(k) : k \in \mathbb{N}\} = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{p} \{s_i(k) : k \in \mathbb{N}\} = \mathbb{N}$(resp. $\bigcup_{i \in \mathbb{N}} \{s_i(k) : k \in \mathbb{N}\} = \mathbb{N}$), then $(\alpha, x) \in L$.

Proof. By Theorem 1.2 (1), $L \subset \mathcal{L}(C_L)$. It is sufficient to show that $\mathcal{L}(C_L) \subset L$. Let $(\alpha, x) \in \mathcal{L}(C_L)$. Then, by above Lemma 2.2, there is a subsequence β_1 of α such that $(\beta_1, x) \in L$, let $\beta_1 = \alpha \circ s_1$. If $\mathbb{N} - \{s_1(k) : k \in \mathbb{N}\}$ is finite, then by (SC 4) it is obvious that $(\alpha, x) \in L$ and hence it holds. If $\mathbb{N} - \{s_1(k) : k \in \mathbb{N}\}$ is infinite, by (SC 2) and Lemma 2.2, there is again a subsequence β_2 of α, let $\beta_2 = \alpha \circ s_2$, such that $(\beta_2, x) \in L$ and $\{s_1(k) : k \in \mathbb{N}\} \cap \{s_2(k) : k \in \mathbb{N}\} = \emptyset$. If $\mathbb{N} - \bigcup_{i=1}^{2} \{s_i(k) : k \in \mathbb{N}\}$ is finite, then $(\alpha, x) \in L$ by (SC 4) and the finite case of (SC 5). In the case that $\mathbb{N} - \bigcup_{i=1}^{2} \{s_i(k) : k \in \mathbb{N}\}$ is infinite, it recurs to the steps above. Thus, by the recursive arguments as above, we have that either there is a finite family $\{\beta_1, \beta_2, \cdots, \beta_n\}$ of subsequences of α, let $\beta_i = \alpha \circ s_i$ for all $i \in \{1, 2, \cdots, n\}$, such that $(\beta_i, x) \in L$ for all i, $\{s_i(k) : k \in \mathbb{N}\} \cap \{s_j(k) : k \in \mathbb{N}\} = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{n} \{s_i(k) : k \in \mathbb{N}\} = \mathbb{N}$ or there does not exist a finite family of subsequences of α satisfying the above conditions. In any case we have $(\alpha, x) \in L$ by (SC 4) and (SC 5). Note that in the latter(infinite) case we use the inductive method. Therefore, $\mathcal{L}(C_L) \subset L$.

COROLLARY 2.4. *Let $L \in SC[X]$ and assume the hypothesis of Lemma 2.2. If L satisfies additionally the two conditions (SC 4) and (SC 5), then there is a Fréchet-Urysohn topology $\mathcal{T}$ on X such that $L = \{(\alpha, x) \in S(X) \times X : \alpha$ converges to x in the Fréchet-Urysohn space $(X, \mathcal{T})\}$. In fact, $\mathcal{T} = \{X - C_L(A) : A \in P(X)\}$, i.e., C_L is the closure operator on the Fréchet-Urysohn space $(X, \mathcal{T})$.*

Proof. It follows from Theorem 1.1 and Theorem 2.3.

EXAMPLE 2.5. *In general, the converse of Theorem 2.3 is not true and hence the converse of Corollary 2.4 is also not true. Let $\alpha = (x_n)$ be a sequence in the real line $\mathbb{R}$ defined by*

$$x_n = \begin{cases} \frac{k_n}{k_n+1} & \text{if } n = 1+2+\cdots+k_n \text{ for some } k_n \in \mathbb{N} \\ \frac{i}{i+(k'_n+2-i)} & \text{if } 1+\cdots+k'_n < n < 1+\cdots+k'_n+(k'_n+1) \\ & \quad \text{for some } k'_n \in \mathbb{N} \text{ and } n = (1+\cdots+k'_n)+i \\ & \quad \text{for some } i \in \{1,2,\cdots,k'_n\} \end{cases}$$

for all $n \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $\beta_i = \alpha \circ s_i$ be a subsequence of α defined as follows:

$$s_1(k) = \{1+2+\cdots+(k-1)\}+1 \text{ for all } k \in \mathbb{N}$$

and for each $i \geq 2$,

$$s_i(k) = s_{i-1}(k+1)+1 \text{ for all } k \in \mathbb{N}.$$

Then it is easy to check that β_i converges to 0 in $\mathbb{R}$ for each $i \in \mathbb{N}$, $\{s_i(k) : k \in \mathbb{N}\} \cap \{s_j(k) : k \in \mathbb{N}\} = \emptyset$ if $i \neq j$ and $\cup_{i \in \mathbb{N}}\{s_i(k) : k \in \mathbb{N}\} = \mathbb{N}$. Clearly, the sequences α and $\beta_1, \beta_2, \cdots$ above satisfy the hypotheses of (SC 5). But, α does not converge to 0 in $\mathbb{R}$. In fact, the limit of α does not exist. Note that this fact is equivalent to the fact that the double limit of the double sequence (a_{mn}) in $\mathbb{R}$ defined by $a_{mn} = \frac{m}{m+n}$ for all $m, n \in \mathbb{N}$ does not exist. Thus, let $L_{\mathbb{R}}$ be the set of all pairs of a convergent sequence and its the limit in $\mathbb{R}$; $L_{\mathbb{R}} = \{(\alpha, x) \in S(\mathbb{R}) \times \mathbb{R} : \alpha$ converges to x in $\mathbb{R}\}$, we have that $\mathbb{R}$ is surely a Fréchet-Urysohn space and hence $L_{\mathbb{R}} = \mathcal{L}(C_{L_{\mathbb{R}}})$, but $L_{\mathbb{R}}$ does not satisfy the condition (SC 5) as shown above

REMARK. *It is obvious that the condition (SC 4) of Theorem 2.3 is reasonable. But, from Example 2.5, we know that the condition of the infinite case of (SC 5) is strong in Theorem 2.3. Hence, a new question naturally arises in connection with Theorem 2.3 and Example 2.5:*

Question: Is there a necessary and sufficient condition for $L = \mathcal{L}(C_L)$ *?*

Next, in order to give the answer to the above *Question*, we consider a sequential convergence structure determined by a topological space. Let (X, c) be a topological space endowed with a topological closure operator c and let L_c be the set of all pairs of convergent sequence and its limit in the space (X, c); that is, $L_c = \{(\alpha, x) \in S(X) \times X : \alpha$ converges to x in the space $(X, c)\}$. Define a function C_{L_c} of $P(X)$ into $P(X)$ by for each subset A of X, $C_{L_c}(A) = \{x \in X : (\alpha, x) \in L_c$ for some sequence α in $A\}$. We call this closure operator C_{L_c} *the sequential closure operator on the space* (X, c)[3]. It is well known that the sequential closure operator C_{L_c} on a topological space (X, c) is not a topological closure operator; that is, C_{L_c} is not idempotent and if C_{L_c} is a topological closure operator on X(that is, C_{L_c} is idempotent), then the space (X, C_{L_c}) is a Fréchet-Urysohn space. It is also obvious that if a topological space (X, c) is a Fréchet-Urysohn space, then the sequential closure operator C_{L_c} on (X, c) is idempotent and moreover $L_c = \mathcal{L}(C_{L_c})$, where $\mathcal{L}(C_{L_c})$ denotes the set of all pairs of convergent sequence and its the limit in the space (X, C_{L_c}). In [9], the author gave two sufficient conditions that the sequential closure operator C_{L_c} be idempotent. For the sequential closure operator on a topological space and related topics, we refer to the reader [1,3,9,10].

It is easy to verify that for a topological space (X, c), L_c satisfies the conditions (SC 1) and (SC 2), but not the condition (SC 3), in general.

EXAMPLE 2.6. *Let X be the set consisting of pairwise distinct objects of the following three types: points x_{mn} where $m \in \mathbb{N}$ and $n \in \mathbb{N}$, points y_n where $n \in \mathbb{N}$, and a point z. We set $V_k(y_n) = \{y_n\} \cup \{x_{mn} | m \geqq k\}$ and let γ denote the set of subsets $W \subset X$ such that $z \in W$ and there exists a positive integer p such that $V_1(y_n) - W$ is finite and $y_n \in W$ for all $n \geqq p$. The collection $\mathcal{B} = \{\{x_{mn}\} | m \in \mathbb{N}, n \in \mathbb{N}\} \cup \gamma \cup \{V_k(y_n) | n \in \mathbb{N}, k \in \mathbb{N}\}$ is a base of a topology on X. In the space X, for each $n \in \mathbb{N}$, the sequence $(x_{mn} | m \in \mathbb{N})$ converges to the point y_n and the sequence (y_n) converges to the point z. However, for the set $A = \{x_{mn} | m \in \mathbb{N}, n \in \mathbb{N}\}$, we have that there does not exist any sequence in A converging to the point z(see [3], p.13).*

THEOREM 2.7 [9]. *Let (X, c) be a topological space satisfying the following condition*

(*): For each double-sequence $(x_{nm} | n \in \mathbb{N}, m \in \mathbb{N})$ *of points in* X *such that* $((x_{nm} | m \in \mathbb{N}), x_n) \in L_c$ *for each* $n \in \mathbb{N}$ *and* $((x_n), x) \in L_c$, $((y_n), x) \in L_c$ *for some sequence* (y_n) *of points in the set* $\{x_{nm} | n \in \mathbb{N}, m \in \mathbb{N}\}$.
Then, the space (X, C_{L_c}) *is a Fréchet-Urysohn space and* $L_c = \mathcal{L}(C_{L_c})$.

REMARK. *Note that the condition* $(*)$ *is equivalent to the condition (SC 3) of a sequential convergence structure on a set* X. *Since every Fréchet-Urysohn space* (X, c) *satisfies* $L_c = \mathcal{L}(C_{L_c}) \in SC[X]$, *every Fréchet-Urysohn space* (X, c) *satisfies the condition* $(*)$. *Hence, in a topological space, the condition* $(*)$ *of Theorem 2.7 is an affirmative answer to Question. Moreover, we know that in a topological space* (X, c), *a necessary and sufficient condition that the sequential closure operator* C_{L_c} *on the space* (X, c) *be idempotent is surely a necessary and sufficient condition for* $L_c = \mathcal{L}(C_{L_c}) \in SC[X]$.

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