East Asian Math J 13 (1997), No 2, pp 245–251

A NOTE ON SIMPLE PARAMEDIAL GROUPOIDS

JUNG R. CHO

1. Introduction

Let G be a groupoid, that is, a non-empty set equipped with a binary operation. For a groupoid G, we define a transformation o_G of G by $o_G(x) = xx$ for $x \in G$. An element of G is called *idempotent* if $xx = x$ and G is called *idempotent* if every element of G is idempotent. We denote the set of all idempotent elements of G by $\mathrm{Id}(G)$. A mapping $f : G \to G$ is said to be an *antihomomorphism* if $f(xy) = f(y)f(x)$ for all $x, y, \in G$. For each $x \in G$, we define transformations L_x and R_x of G by $L_x(y) = xy$ and $R_x(y) = yx$ for every $y \in G$.

A groupoid G is said to be

- *cancellative* if both L_x and R_x are injective for all $x \in G$,
- a *quasigroup* if both L_x and R_x are bijective for all $x \in G$,
- a *LZ-semigroup* if $xy = x$ for all $x, y \in G$,
- a *RZ-semigroup* if $xy = y$ for all $x, y \in G$,
- a *band* if G is an idempotent semigroup;
- a *rectangular band* if G is a band and $xyx = x$ for all $x, y \in G$,
- a *semilattice* if it is commutative and idempotent.
- *unipotent* if $xx = yy$ for all $x, y = G$,
- *zeropotent* if $xx \cdot y = y \cdot xx = xx$ for all $x, y \in G$,
- *medial* if $xa \cdot by = xb \cdot ay$ for all $a, b, x, y \in G$,
- *paramedial* if $ax \cdot yb = bx \cdot ya$ for all $a, b, x, y \in G$,
- *simple* if G has no nontrivial congruences.

Finally, let Δ denote the identical congruence $\{(x, x) | x \in G\}$ of a groupoid G. For other terminology, we refer to [1].

Received Sep. 10, 1997

The present study was supported in part by the Matching Fund Program of Research Institute for Basic Sciences, Pusan National University, Project No RIBS-PNU-96-103

2. Basic properties of simple paramedial groupoids

THEOREM 2.1. *Let G be a nontrivial simple paramedial groupoid. Then exactly one of the following three cases takes place:*

(1) *o_G is an injective transformation of G.*
(2) *G is a finite unipotent medial quasigroup.*
(3) *G is zeropotent.*

Proof. By [2, 2.1(iii)], $r = \ker(o_G)$ is a congruence of G, and hence either $r = \Delta$ or $r = G \times G$. If $r = \Delta$ then o_G is injective, and so we assume $r = G \times G$. Then, $xx = yy$ for all $x, y \in G$, and so G is unipotent. Note that $e = xx$ is the unique idempotent element of G. By [2, 2.9(i)], $s = \ker(L_e R_e)$ is a congruence of G. Again, we have either $s = \Delta$ or $s = G \times G$. If $s = \Delta$ then $L_e R_e = R_e L_e$ is an injective homomorphism of G, and hence both L_e and R_e are injective transformations of G. Then, by [2, 2.9(ii)], G is a cancellative medial groupoid. However, it is proved in [4] that every simple cancellative medial groupoid is a finite quasigroup. Now suppose $s = G \times G$. Then $e \cdot xe = e \cdot ye$ for all $x, y \in G$ and it follows that $e \cdot xe = e \cdot ee = e$. It also follows that $ex \cdot e = ey \cdot e$ for all $x, y \in G$ and $ex \cdot e = e$. By [2, 2.6(i)], $t = \ker(L_e^2) = \ker(R_e^2)$ is a congruence of G. If $t = \Delta$ then both L_e and R_e are injective transformations of G, and so G is cancellative by [2, 2.6(ii)] and [2, 2.9(ii)]. However, then, $e \cdot xe = e \cdot ye$ implies $x = y$ for all $x, y \in G$, which is a contradiction since G is nontrivial. Therefore, $t = G \times G$ and we have

$$e \cdot ex = L_e^2(x) = L_e^2(e) = e \cdot ee = e = ee \cdot e = R_e^2(e) = R_e^2(x) = xe \cdot e$$

for all $x \in G$. Put $I = \{a \in G \mid ae = e = ea\}$. Clearly, $e \in I$ and if $a \in I$ and $x \in G$, then $e \cdot ax = ee \cdot ax = xe \cdot ae = xe \cdot e = e$ and similarly we have $ax \cdot e = e$,, $e \cdot xa = e$ and $xa \cdot e = e$. Thus ax, $xa \in I$, and so I is an ideal of G. But then $w = (I \times I) \cup \Delta$ is a congruence of G. Suppose $w = \Delta$ then $I = \{e\}$. For all $x \in G$, since $xe \cdot e = e = e \cdot xe$, we have $xe \in I$ and, since $ex \cdot e = e = e \cdot ex$, we have $ex \in I$. That is, $xe = e = ex$ for all $x \in G$, which means $G = I$, a contradiction. Thus, $w = G \times G$ and so $I = G$, which implies G is zeropotent.

LEMMA 2.2. *Let G be a nontrivial finite idempotent medial groupoid and let f be an antiautomorphism of G such that Δ and $G \times G$ are the only congruences of G which are invariant under f. Then exactly one of the following three cases takes place:*

(1) *G is a quasigroup.*
(2) *G is a semilattice.*
(3) *G is a rectangular band.*

Proof. Let r denote the intersection of all cancellative congruences of G. Then r is the smallest cancellative congruence of G. If we define a relation r_1 on G by $(a, b) \in r_1$ if and only if $(f(a), f(b)) \in r$, then r_1 is a cancellative congruence of G so that $r \subseteq r_1$. This shows that r is invariant under f. If $r = \Delta$ then G is cancellative and, since G is finite, it is a quasigroup. Now assume that $r = G \times G$, that is, no proper homomorphic image is cancellative. Let s be the smallest congruence of G such that corresponding factor is a semilattice. Again, define a relation s_1 on G by $(a, b) \in s_1$ if and only if $(f(a), f(b)) \in s$. Then it can be checked that s_1 is a congruence of G and $(x, xx) \in s_1$, $(xy, yx) \in s_1$ and $(x \cdot yz, xy \cdot z) \in s_1$ for all $x, y, z \in G$. Thus, G/s_1 is a semilattice, $s \subseteq s_1$ and s is invariant under f. So, $s = \Delta$ or $s = G \times G$. If $s = \Delta$ then $G \cong G/s$ is a semilattice. Therefore, we assume that $s = G \times G$. Now, since G is nontrivial and finite, G has a proper nontrivial simple factor groupoid H. By our assumptions that $r = G \times G$ and $s = G \times G$, H is neither cancellative nor a semilattice. Using the description of simple idempotent medial groupoids as is given in [4], we conclude that H is either an LZ-semigroup or an RZ-semigroup. In both cases, $t \neq G \times G$, where t is the smallest congruence such that G/t is a rectangular band. As above, t is invariant under f and, therefore, $t = \Delta$. In other words, G is a rectangular band.

The next proposition classifies the simple paramedial groupoids of Theorem 2.1.(1) more in detail.

PROPOSITION 2.3. *Let G be a nontrivial finite simple paramedial groupoid such that o_G is injective. Then exactly one of the following three cases takes place:*

(1) *G is a quasigroup.*
(2) *G is commutative (hence medial) but not cancellative.*

(3) *There exist a rectangular band $G(*)$ and an antiautomorphism f of $G(*)$ such that $xy = f(x) * f(y)$ for all $x, y \in G$.*

Proof. Clearly, o_G is bijective and, by [2, 2.4], there exist an idempotent medial groupoid $G(*)$ and an antiautomorphism f of $G(*)$ such that $xy = f(x) * f(y)$ for all $x, y \in G$. Since G is simple, Δ and $G \times G$ are the only congruences of $G(*)$ which are invariant under f. Now, we apply Lemma 2.2. If $G(*)$ is a quasigroup then G is also a quasigroup. If $G(*)$ is a semilattice then G is commutative and not commutative. If $G(*)$ is a rectangular band then G is also a rectacgular band.

LEMMA 2.4. *Let G be a simple paramedial groupoid containing at most three elements then G is medial.*

Proof. Easy to check.

EXAMPLE 2.5. Consider the following four element groupoids G_1, G_2, G_3 ~~and G_4~~.

G_1	0	1	2	3
0	0	0	0	0
1	0	0	1	0
2	0	0	0	3
3	0	2	0	0

G_2	0	1	2	3
0	0	0	0	0
1	0	0	1	0
2	0	0	0	3
3	0	2	0	0

G_3	0	1	2	3
0	0	0	0	0
1	0	0	1	0
2	0	0	0	3
3	0	2	0	0

G_4	0	1	2	3
0	0	0	0	0
1	0	0	1	0
2	0	0	0	3
3	0	2	0	0

(1) G_1 is a simple zeropotent non-medial paramedial groupoid.
(2) G_2 is a simple non-medial paramedial groupoid which is not cancellative. o_{G_2} is a permutation and $\mathrm{Id}(G_2)$ is not a subgroupoid of G_2.
(3) G_3 is a simple non-medial paramedial quasigroup and $\mathrm{Id}(G_3)$ is not a subgroupoid of G_3.
(4) G_4 is a simple medial paramedial groupoid. Notice that G_2 and G_4 are not isomorphic.

Let G be a nontrivial paramedial groupoid. We shall say G is

- of type (I) if G is cancellative;

- of type of type (II) if G is zeropotent;
- of type of type (III) if G is commutative but neither cancellative nor zeropotent;
- of type of type (IV) there exist a rectangular band $G(*)$ and an antiautomorphism f of $G(*)$ such that $xy = f(x) * f(y)$ for all $x, y \in G$.
- of type of type (V) if G is not of one of the above types.

Clearly, every simple paramedial groupoid is of just one of the above types. Further, by Theorem 2.1 and Proposition 2.3, every simple paramedial groupoid of type (V) is infinite.

3. Simple paramedial groupoids of type (I) - linear representations

We recall that a nontrivial cancellative groupoid G is called *c-simple* if Δ and $G \times G$ are the only cancellative congruences of G. Further, let A be the group generated by two elements α, β with a relation $\alpha^2 = \beta^2$, and let $R = \mathbb{Z}A$ be the group ring of A over the ring $\mathbf{Z}$ of integers.

PROPOSITION 3 1. *The following conditions are equivalent for a quasigroup Q.*

(1) *Q is a c-simple paramedial quasigroup.*
(2) *There exist a simple R-module structure $(Q, +, R)$ defined on Q and an element $w \in Q$ such that $ab = \alpha a + \beta b + w$ for all $a, b \in Q$.*

Proof. The result is an easy consequence of [2, 6.2].

PROPOSITION 3.2. *Let G be a c-simple paramedial cancellative groupoid. Then the smallest quasigroup containing G as a subgroupoid is a c-simple paramedial quasigroup.*

Proof. See [2, 4.11] and [3, 5.1, 5.3].

4. Simple paramedial groupoids of type (II) - linear representations

PROPOSITION 4.1. *Let G be a simple paramedial groupoid of type (II) and put $0 = aa$ for $a \in G$. Then there exist a commutative semigroup $S(+)$ and automorphisms f, g of $S(+)$ such that the following conditions are satisfied:*

(1) $G \subseteq S$ *and* $ab = f(a) + g(b)$ *for all* $a, b \in G$.
(2) *0 is an absorbing element of S and* $f(x) + g(x) = 0$ *for all* $x \in S$.
(3) $f^2 = g^2$.
(4) $S(+)$ *is either zeropotent or idempotent.*
(5) *The algebra* $(S, +, f, g, f^{-1}, g^{-1})$ *is simple and generated by* G.

Proof. By [5], there exist a commutative semigroup $S(+)$ and automorphisms f, g of $S(+)$ such that conditions (1), (2) and (3) are satisfied; obviously, we can assume that the algebra $\tilde{S} = (S, +, f, g, f^{-1}, g^{-1})$ is generated by G. Considering the factor algebra $\tilde{S}/s$, where s is a congruence of $\tilde{S}$ which is maximal with respect to the property that $s \cap (G \times G) = \Delta$, we can assume that $r \cap (G \times G) \neq \Delta$ for every non-identical congruence r of $\tilde{S}$. Now, if r is such a congruence and $t = r \cap (G \times G)$, then t is a non-identical congruence of G, and hence $t = G \times G \subseteq r$. Consequently, $G \subseteq A = \{x \in S \mid (0, x) \in r\}$ and $A = S$, since A is evidently a subalgebra of $\tilde{S}$. Thus $r = S \times S$ and this proves (5). Now it remains to show (4), and we assume that $S(+)$ is not zeropotent. Then the mapping $x \mapsto x + x$ of $\tilde{S}$ is not constant and, because S is simple, it is an injective endomorphism of $\tilde{S}$. Using an obvious and standard construction we embed $\tilde{S}$ into a (simple) algebra $\tilde{S}_1 = (S_1, +, f, g, f^{-1}, g^{-1})$ for which the mapping $x \mapsto x + x$ is an automorphism. Now, proceed as in [6] and we can show that $S_1(+)$ is idempotent. Then, $S(+)$ is idempotent as a subsemigroup of $S_1(+)$.

REMARK 4.2. Let G be a simple paramedial groupoid of type (II). Proceeding as in the above proof, we can show that there exist a commutative semigroup $S(+)$ and endomorphisms f, g of $S(+)$ such that, in addition to the conditions (1) - (4) of Proposition 4.2, the following holds;

(5$'$) The algebra $\hat{S} = (S, +, f, g)$ is simple and generated by G.

In this case, $f^2 = g^2$ is an endomorphism of $\hat{S}$, and hence either

$\ker(f^2) = G \times G$ or $\ker(f^2) = \Delta$. In the former case, we have $|G| = 2$, and so $\ker(f^2) = \Delta$ provided that $|G| \geq 3$. However, then f and g are injective endomorphisms of $S(+)$.

5. Simple paramedial groupoids of type (III)

Let $n \geq 1$, let $Y_n = \{a_0, a_1, \cdots, a_n\}$ and let a multiplication be defined on Y_n by $a_i a_j = a_0$ for $i \neq j$, $a_0 a_0 = a_0$, and $a_i a_i = a_{i+1}$ for $i \neq 0$, subscript modulo n. Then, it can be checked that Y_n is a simple paramedial groupoid of type (III). Further, Y_1 is a two-element semilattice and, for $n \geq 2$, Y_n contains only one idempotent element, namely a_0; in both cases, a_0 is an absorbing element of Y_n. Notice that Y_n contains no proper subgroupoids except $\{a_0\}$ for $n \geq 2$.

PROPOSITION 5 1. (1) *The groupoids Y_n, $n \geq 1$ are pairwise non-isomorphic simple paramedial groupoids of type (III).*

(2) *Every nontrivial simple paramedial groupoid of type (III) is finite and isomorphic to one of the groupoids Y_n*

Proof. Every commutative paramedial groupoid is medial, and hence our result follows from the classification of simple commutative medial groupoids given in [3].

For a complete classification of simple paramedial groupoids, we need to determine the groupoids of type (V). For the groupoids of type (IV), we also need more concrete detailed study

References

1 R Bruck, *A survey of binary systems*, Springer-Verlag, New York - Heidelberg - Berlin, 1971

2 J R. Cho, J. Ježek and T Kepka, *Paramedial groupoids*, to appear in Czech. J of Math

3. J Ježek and T Kepka, *Medial groupoids*, Rozpravy ČSAV, Vol 93, No 2, 1983

4 ———, *Simple semimodules over commutative semirings*, Acta Sci Math Szeged 46 (1983), 17–27

5 ———, *Equational theory of paramedial groupoids*, preprint

6 ———, *Linear equational theories of semimodule representations*, preprint

Department of Mathematics
Pusan National University
Pusan 609-735, Korea