ABOUT SOME INFINITE FAMILY OF 3-MANIFOLDS

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1. Introduction

Here we will consider finitely generated groups \( G(2m, n) \), \( m \geq 0 \) and \( n \geq 2 \), with the following cyclic presentation

\[
G(2m, n) = \langle x_1, \ldots, x_n \mid x_i^{-1}(x_{i+1}x_i^{-1})^{m+1}x_{i+1}(x_{i-1}x_i^{-1})^{m+1}x_{i-1} = 1, \quad i = 1, \ldots, n \rangle,
\]

where the indices are taken mod \( n \). We will demonstrate that these cyclically-presented groups are closely connected with the 2-bridge knot \( b(4m + 7, 2m + 3) \), that is the closure of the rational \((\frac{4m+7}{2m+3})\)-tangle.

In section 2 we will describe the fundamental polyhedron for the group \( G(2m, n) \) and demonstrate that this group is the fundamental group of a three-dimensional manifold. In section 3 we will consider the split extension \( \tilde{G}(2m, n) \) of \( G(2m, n) \) by the cyclic automorphism and show that \( \tilde{G}(2m, n) \) is the group of the orbifold \( b(4m + 7, 2m + 3)(n) \). In section 4 we show that the cyclic covering manifolds \( M(2m, n) \) are also obtained as two fold branched coverings over \( \mathbb{K}^{2m}_n \).

2. The manifolds with fundamental groups \( G(2m, n) \)

In this section we construct 3-manifolds by polyhedron description and demonstrate, using the Siefert-Trelfall criterion, that \( G(2m, n) \) arises as a fundamental group of a 3-manifold.

**Theorem 1.** For \( m \geq 0 \) and \( n \geq 2 \), the group \( G(2m, n) \) is a fundamental group of a three-dimensional manifold.

**Proof.** We consider a tessellation on the boundary of 2-ball, which can be regarded as a polyhedron \( P(2m, n) \), consisting of \( n \) quadrilaterals \( F_i \) in the north hemisphere, \( n \) \((2m + 4)\)-gons \( T_i \) in the south...
hemisphere, \( n \) quadrilaterals \( F'_i \) and \( n \) \((2m+4)\)-gons \( T'_i \) in the equatorial zone, where \( i = 1, \ldots, n \) and all indices are taken by \( \mod n \). Then the polyhedron \( P(2m, n) \) has \( 4n \) faces, \( 8n+2nm \) edges and \( 4n+2nm+2 \) vertices. For example, if \( m = 1 \) and \( n = 3 \), we have the polyhedron \( P(2, 3) \) as shown in Figure 1.

![Figure 1. The polyhedron \( P(2, 3) \).](image)

Let us consider the 1-skeleton of \( P_n \) with orientation and labeling of its edges in the following manner.

(i) The oriented edges fall into \( 2n + 1 \) classes: \( x_i, i = 1, \ldots, n \), where each class \( x_i \) consists \( 2nm + 4n \) edges, \( y_i, i = 1, \ldots, n \), where each class \( y_i \) consists \( 3n \) edges and \( z \) consists \( n \) edges. In this case oriented edges from the same class carry the same label.

(ii) For each \( i = 1, \ldots, n \) the boundary cycle of the \((2m+4)\)-gons \( T_i \) and \( T'_i \) is \( x_i y_{i+2} (x_i x_{i+1}^{-1})^{m+1} \) with the indices taken \( \mod n \).

(iii) For each \( i = 1, \ldots, n \) the boundary cycle of the quadrilaterals \( F_i \) and \( F'_i \) is \( y_{i+1} x_i z y_i^{-1} \) with the indices taken \( \mod n \).

Note that the set of all faces splits into pairs of faces with the same sequences of oriented boundary edges. Now we shall identify quadrilat-
erals $F_i$ and $F_i'$, and $(2m+4)$-gons $T_i$ and $T_i'$ such that the corresponding oriented edges on polygons carrying the same label are identified for each $i = 1, \ldots, n$.

The resulting complex has 2 non-equivalent vertices $N$ and $S$, $2n + 1$ non-equivalent edges, $2n$ two-cells and 1 three-cell. Thus the Euler characteristic is 0. Then by the following theorem, due to H. Seifert and W. Threlfall [14, p.216], it forms a 3-dimensional manifold.

**Theorem.** A complex, which is formed by identifying the faces of a polyhedron will be a manifold if and only if its Euler characteristic equals zero.

We select $N$ as the initial point of the closed paths and $z$ as auxiliary path leading to the vertex $S$. Then the generating path classes of the fundamental group of this manifold will be represented by the closed paths: for $i = 1, \ldots, n$,

$$Z = z z^{-1}, \quad X_i = z x_i \text{ and } Y_i = y_i z^{-1},$$

and relations: for $i = 1, \ldots, n$,

$$Z = Z Z^{-1}, \quad X_i = Z X_i, \quad Y_i = Y_i Z^{-1}, \quad X_i Y_{i+2}(X_i X_{i+1}^{-1})^{m+1} = 1 \quad \text{and} \quad Y_{i+1} X_i Z Y_i^{-1} = 1.$$

Thus the fundamental group has the following presentation.

$$\langle X_1, \ldots, X_n, Y_1, \ldots, Y_n \mid X_i Y_{i+2}(X_i X_{i+1}^{-1})^{m+1} = 1, \quad Y_{i+1} X_i Y_i^{-1} = 1, \quad i = 1, \ldots, n \rangle,$$

where the indices are taken mod $n$.

Therefore we see that the fundamental group is isomorphic to the group $G(2m, n)$. □

We remark that the polyhedron $P(2m, n)$ can be considered as the natural generalization of the fundamental polyhedron for the manifolds $M_n$ constructed in [11]. Indeed, throughout this paper, if $m = 0$, then all results coincide with ones in [11].
3. The split extension of the group \(G(2m, n)\)

Let us consider the abstract groups defined by following presentations

\[
\langle x_1, \ldots, x_n, y_1, \ldots, y_n \mid x_1, y_{i+2} (x_i x_{i+1})^{m+1} = 1, y_{i+1} x_i y_i^{-1} = 1, \quad i = 1, \ldots, n \rangle,
\]

where the indices are taken mod \(n\). With \(x_i, y_{i+2} (x_i x_{i+1})^{m+1} = 1\) for all \(i = 1, \ldots, n\), and Tieze transformation, we can easily isolate \(y_i\) and prove that it is another presentation of \(G(2m, n)\). Moreover we see that the group \(G(2m, n)\) has the cyclic automorphism \(\rho : x_i \rightarrow x_{i+1}\) and \(y_i \rightarrow y_{i+1}\) of order \(n\). Now we consider the split extension \(\hat{G}(2m, n)\) of group \(G(2m, n)\) by the cyclic group of automorphisms generated by \(\rho\). The following theorem shows that the group \(\hat{G}(2m, n)\) is interesting from the topological point of view.

For a knot \(K\), we denote by \(K(n)\) the 3-dimensional orbifold with the underlying space \(S^3\) and as a singular set \(K\) with branched index \(n\). Then we have the following relation between \(\hat{G}(2m, n)\) and the knot \(b(4m + 7, 2m + 3)\), that is the closure of the rational \((\frac{4m+7}{2m+3})\)-tangle.

**Theorem 2.** For \(m \geq 0, n \geq 2\), let \(M(2m, n)\) be the \(n\)-fold cyclic branched covering of the knot \(b(4m + 7, 2m + 3)\). Then \(\pi_1(M(2m, n)) \cong G(2m, n)\).

**Proof.** With the presentation (1) and the cyclic automorphism \(\rho\) of order \(n\) such that \(\rho : x_i \rightarrow x_{i+1}\) and \(y_i \rightarrow y_{i+1}\), we get the following presentation of the split extension \(\hat{G}(2m, n)\) of \(G(2m, n)\) by \(\rho\), with notations \(x = x_1\) and \(y = y_1\).

\[
\hat{G}(2m, n) = \langle \rho, x, y \mid x \rho^2(y)(x(\rho(x))^{-1})^{m+1} = 1, \rho(y)xy^{-1} = 1, \rho^n = 1 \rangle
\]

\[
= \langle \rho, x, y \mid x \rho^2(y)(x(\rho(x))^{-1})^{m+1} = 1, \rho = y(\rho x)y^{-1}, \rho^n = 1 \rangle.
\]

Note that \(\rho\) and \(\rho x\) are conjugate. Let \(\mu = \rho x\). Then \(x = \rho^{-1}\mu\) and \(\mu^n = 1\). So

\[
\hat{G}(2m, n) = \langle \rho, \mu, y \mid \rho y = y \mu, y = \rho^2 \mu^{-1} \rho^{-1} \rho_1 \mu^{m+1}, \rho^n = 1, \mu^n = 1 \rangle
\]

\[
= \langle \rho, \mu \mid \rho w = w \mu, \rho^n = 1, \mu^n = 1 \rangle
\]

where \(w = \mu^{-1} \rho^{-1} \mu_1^{m+1}, w_1 = \mu \rho \mu^{-1} \rho^{-1}\).
We recall that the group
\[
\langle \rho, \mu \mid \rho w = w \mu \rangle \quad \text{where} \quad w = \mu^{-1} \rho^{-1} (\mu \rho \mu^{-1} \rho^{-1})^m
\]
is the group of the \( b(4m + 7, 2m + 3) \)-knot, where \( \rho \) and \( \mu \) as shown in the Figure 2 and the index \( 2m + 3 \) in Figure 2 denotes the number of half-twists.

![Figure 2. The knot \( b(4m + 7, 2m + 3) \).](image)

Therefore by [5], the group \( \tilde{G}(2m, n) \) is the fundamental group of the orbifold \( b(4m + 7, 2m + 3)(n) \). Hence \( \pi_1(M(2m, n)) \cong G(2m, n) \).

**Theorem.** (Thurston) Assume \( q > 1 \). Then \( (p/q)(n) \) is hyperbolic for (i) \( p = 5, n \geq 4 \) (ii) \( p \neq 5, n \geq 3 \). Moreover \( (p/q)(2) \) is spherical for all \( p \), and \( (5/3)(3) \) is euclidean.

Thanks to the above theorem (see [4] and [10]) we have that the orbifold \( b(4m + 7, 2m + 3)(n) \) (denoted by \( (\frac{4m+7}{2m+3})(n) \)) is hyperbolic for \( n \geq 3, m \geq 0 \), and it is spherical for \( n = 2, m \geq 0 \).

**Corollary 1.** The manifolds \( M(2m, n) \) is hyperbolic for \( n \geq 3, m \geq 0 \), and \( M(2m, 2) \) is the lens space \( L(4m + 7, 2m + 3) \) for \( m \geq 0 \).

**Corollary 2.** The group \( G(2m, n) \) is infinite for \( n \geq 3, m \geq 0 \), and \( G(2m, 2) \cong \mathbb{Z}_{4m+7} \).
4. The manifolds $M(2m, n)$ as 2-fold coverings

In this section we will study the topological properties of manifolds $M(2m, n)$, that gives a topological approach to the studying of cyclically-presented groups $G(2m, n)$. This studying is analogous to the topological studying of Sieradski groups $S(n)$ and Fibonacci groups $F(2, 2n)$ given in [2], [3], [8] and [12].

![Diagram of the knot $K^2_{4m}$](image)

Figure 3. The knot $K^2_{4m}$.

Firstly we define a series of knots. We recall that any knot can be obtained as the closure of some braid [1]. Let $p$ and $q$ be coprime integers, then by $\sigma_i^{p/q}$ we denote the rational $p/q$-tangle whose incoming arcs are $i$-th and $(i + 1)$-th strings. For an integer $n \geq 1$ we denote by $K^2_{4m}$ the $n$-periodic knot, that is the closure of the rational 3-strings braid $(\sigma_1 \sigma_2^{1/(m+2)})^n$. The diagram of the knot $K^2_{4m}$ is pictured in Figure 3. Note that the knot $K^2_{4m}$ is equivalent, under the Reidemeister moves, to the 2-periodic knot $b(4m + 7, 2m + 3)$.

**Theorem 3.** For $m \geq 0$ and $n \geq 2$, the manifold $M(2m, n)$ is the two-fold covering of the 3-sphere branched over the knot $K^2_{4m}$.

**Proof.** By Theorem 3 the manifold $M(2m, n)$ is the $n$-cyclic branched covering of the 3-sphere $S^3$, branched over the knot $b(4m + 7, 2m + 3)$. To describe $M(2m, n)$ as the 2-cyclic branched covering of $S^3$, branched over an $n$-periodic knot, we will use the following construction which is analogous to [12] and [2] where the Fibonacci groups and the Sieradski groups were topologically studied.
About some infinite family of 3-manifolds

From the Figure 2 for the knot $b(4m+7,2m+3)$ we see that the orbifold $b(4m+7,2m+3)(n)$ has a rotation symmetry of order two denoted by $\tau$ such that the axe of the symmetry is disjoint from the knot $b(4m+7,2m+3)$. It is not difficult to see that this symmetry action produces the orbifold $b(4m+7,2m+3)/\langle \tau \rangle$ with underlying space $S^3$, and as a singular set the 2-component link pictured in Figure 4 with branch indices 2 and $n$.

![Figure 4. The singular set of $b(4m+7,2m+3)/\langle \tau \rangle$.](image)

It is easy to see that the singular set of the quotient orbifold is the two-component link $b(8m+14,2m+3)$ in Figure 5, that is the 2-bridge link obtained as the closure of the rational $(\frac{8m+14}{2m+3})$-tangle.

![Figure 5. The link $b(8m+14,2m+3)$.](image)
We will denote the quotient orbifold \( b(4m+7, 2m+3)/(r) \) by \( b(8m+14, 2m+3)(2, n) \). Then we have the following covering diagram

\[
(2) \ M(2m, n) \to b(4m+7, 2m+3)(n) \to b(8m+14, 2m+3)(2, n)
\]

and a sequence of normal subgroups

\[
G(2m, n) = \pi_1(M(2m, n)) \triangleleft \tilde{G}(2m, n) = \pi_1(b(4m+7, 2m+3)(n)) \triangleleft \Omega(2m, n) = \pi_1(b(8m+14, 2m+3)(2, n)),
\]

where \(|\Omega(2m, n) : \tilde{G}(2m, n)| = 2\) and \(|\tilde{G}(2m, n) : G(2m, n)| = n\).

We describe the orbifold group \( \Omega(2m, n) \) using the Wirtinger representation of the link group \( \pi_1(b(8m+14, 2m+3)) \) in figure 4. The link group has two generators \( \bar{\alpha}, \bar{\beta} \) and one relator of the form \( \bar{\alpha} \bar{w} = \bar{w} \bar{\alpha} \), where a word \( \bar{w} \) is determined as follows:

\[
\bar{w} = \bar{\beta}^{\iota_j} \bar{\alpha}^{8m+13} \ldots \bar{\alpha}^{8m+12} \bar{\beta}^{8m+13},
\]

and \( \iota_j \) is the sign of the number \( (2m+3)j \) by mod \( 2(8m+14) \) on the segment \([- (8m+14), 8m+14]\). For example, if \( m = 0 \), we get a word

\[
\bar{w} = \bar{\beta} \bar{\alpha} \bar{\beta} \bar{\alpha} \bar{\beta}^{-1} \bar{\alpha}^{-1} \bar{\beta}^{-1} \bar{\alpha}^{-1} \bar{\beta} \bar{\alpha} \bar{\beta}.
\]

In this representation the generators \( \bar{\alpha} \) and \( \bar{\beta} \) correspond to the arcs with the same labels on the link diagram of \( b(8m+14, 2m+3) \) as shown in figure 4.

According to [5], we get the following presentation for the group \( \Omega(2m, n) \) of the orbifold \( b(8m+14, 2m+3)(2, n) \):

\[
\Omega(2m, n) = \langle \alpha, \beta \mid \alpha \bar{w} = \bar{w} \alpha, \ \alpha^n = \beta^2 = 1 \rangle,
\]

where the generators \( \alpha \) and \( \beta \) canonically correspond to \( \bar{\alpha} \) and \( \bar{\beta} \) respectively.

Let us consider the group

\[
\mathbb{Z}_n \oplus \mathbb{Z}_2 = \langle a \mid a^n = 1 \rangle \oplus \langle b \mid b^2 = 1 \rangle
\]
and the epimorphism

\[ \theta : \Omega(2, n) \twoheadrightarrow \mathbb{Z}_n \oplus \mathbb{Z}_2 \]

defined by setting \( \theta(\alpha) = a \) and \( \theta(\beta) = b \).

By the construction of the 2-fold covering

\[ b(4m + 7, 2m + 3)(n) \twoheadrightarrow b(8m + 14, 2m + 3)(2, n) \]

the loop \( \beta \in \Omega(2m, n) \) lifts to a trivial loop in \( \hat{G}(2m, n) \), and the loop \( \alpha \in \Omega(2m, n) \) lifts to a loop in \( \hat{G}(2m, n) \) which generates a cyclic subgroup of order \( n \). Thus it follows that

\[ \pi_1(b(4m + 7, 2m + 3)(n)) = \theta^{-1}(\langle a^m | a^n = 1 \rangle) = \theta^{-1}(\mathbb{Z}_n). \]

For the \( 2n \)-fold covering

\[ M(2m, n) \twoheadrightarrow b(8m + 14, 2m + 3)(2, n) \]

both loops \( \alpha \) and \( \beta \) from \( \Omega(2m, n) \) lift to trivial loops in \( G(2m, n) = \pi_1(M(2m, n)) \), hence \( G(2m, n) = \text{Ker } \theta \).

Let \( \Gamma_n \) be the subgroup of \( \Omega(2m, n) \) given by

\[ \Gamma_n = \theta^{-1}(\langle \langle b | b^2 = 1 \rangle \rangle) = \theta^{-1}(\mathbb{Z}_2). \]

Then we get a sequence of normal subgroups

\[ G(2m, n) \triangleleft \Gamma_n \triangleleft \Omega(2m, n), \]

where \( |\Omega(2m, n) : \Gamma_n| = n \) and \( |\Gamma_n : G(2m, n)| = 2 \). We recall, that the orbifold \( b(4m + 7, 2m + 3)(n) \) is spherical for \( n = 2 \), and hyperbolic for \( n \geq 3 \). Hence the group \( \Gamma_n \) acts by isometries on the universal covering \( X_n \), that is the 3-sphere \( S^3 \) for \( n = 2 \), and the hyperbolic space \( \mathbb{H}^3 \) for \( n \geq 3 \). Thus we get the orbifold \( X_n / \Gamma_n \) and the following covering diagram

\[ \begin{array}{ccc}
M(2m, n) & \twoheadrightarrow & X_n / \Gamma_n \\
\downarrow & & \downarrow \\
2 \quad & & 2 \\
\end{array} \quad \begin{array}{ccc}
b(8m + 14, 2m + 3)(2, n) \end{array} \]
In this case the second covering is cyclic and it is branched over the component with index \( n \) of the singular set of \( b(8m + 14, 2m + 3)(2, n) \) in Figure 5. But this component is the knot \( K_1 \) and is trivial. So, underlying space of \( X_n / \Gamma_n \) is the 3-sphere. By the construction of the \( n \)-fold covering

\[
X_n / \Gamma_n \xrightarrow{n} b(8m + 14, 2m + 3)(2, n)
\]

the loop \( \alpha \in \Omega(2m, n) \) lifts to a trivial loop in \( \Gamma_n \), and the loop \( \beta \in \Omega(2m, n) \) lifts to a loop in \( \Gamma_n \) which generates a cyclic group of order 2. Because \( b(8m + 14, 2m + 3) \) are 2-bridge links whose components are equivalent, we can exchange branch indices of components in Figure 5. Therefore, the singular set of \( X_n / \Gamma_n \) is a \( n \)-periodic knot which can be obtained as the closure of the 3-string braid \((\sigma_1 \sigma_2^{1/(m+2)})^n\), that is the knot \( K_{n}^{2m} \). Because the branch index is equal to 2, we denote \( X_n / \Gamma_n = K_{n}^{2m}(2) \).

Comparing (2) and (3), we get that the following covering diagram is commutative:

\[
\begin{array}{ccc}
M(2m, n) & \longrightarrow & M(2m, n) \\
\downarrow n & & \downarrow 2 \\
\quad b(4m + 7, 2m + 3)(n) & \quad & \quad K_{n}^{2m}(2) \\
\downarrow 2 & & \downarrow n \\
\quad b(8m + 14, 2m + 3)(2, n) & \longrightarrow & \quad b(8m + 14, 2m + 3)(2, n)
\end{array}
\]

The diagram of coverings.

In particular, we have that \( M(2m, n) \) is the 2-fold branched covering of the 3-sphere branched over the knot \( K_{n}^{2m} \), and theorem is proved. \( \Box \)

References


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