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ABOUT SOME INFINITE FAMILY OF 3–MANIFOLDS

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1. Introduction

Here we will consider finitely generated groups $G(2m,n)$, $m \geq 0$ and $n \geq 2$, with the following cyclic presentation

$$G(2m,n) = \langle x_1, \ldots, x_n \mid x_i^{-1}(x_{i+1}x_i^{-1})^{m+1}x_{i+1}(x_{i-1}x_i^{-1})^{m+1}x_{i-1} = 1, \quad i = 1, \ldots, n \rangle,$$

where the indices are taken mod n. We will demonstrate that these cyclically presented groups are closely connected with the 2-bridge knot $b(4m+7, 2m+3)$, that is the closure of the rational $(\frac{4m+7}{2m+3})$–tangle.

In section 2 we will describe the fundamental polyhedron for the group $G(2m,n)$ and demonstrate that this group is the fundamental group of a three-dimensional manifold In section 3 we will consider the split extension $\widehat{G}(2m,n)$ of $G(2m,n)$ by the cyclic automorphism and show that $\widehat{G}(2m,n)$ is the group of the orbifold $b(4m+7, 2m+3)(n)$. In section 4 we show that the cyclic covering manifolds $M(2m,n)$ are also obtained as two fold branched coverings over $\mathcal{K}_n^{2m}$.

2. The manifolds with fundamental groups $G(2m,n)$

In this section we construct 3-manifolds by polyhedron description and demonstrate, using the Siefert–Trelfall criterion, that $G(2m,n)$ arises as a fundamental group of a 3-manifold.

THEOREM 1. *For $m \geq 0$ and $n \geq 2$, the group $G(2m,n)$ is a fundamental group of a three-dimensional manifold.*

Proof. We consider a tessellation on the boundary of 2-ball, which can be regarded as a polyhedron $P(2m,n)$, consisting of n quadrilaterals F_i in the north hemisphere, n $(2m+4)$-gons T_i in the south

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hemisphere, n quadrilaterals F_i' and n $(2m+4)$-gons T_i' in the equatorial zone, where $i = 1, \dots, n$ and all indices are taken by mod n. Then the polyhedron $P(2m,n)$ has $4n$ faces, $8n+2nm$ edges and $4n+2nm+2$ vertices. For example, if $m = 1$ and $n = 3$, we have the polyhedron $P(2,3)$ as shown in Figure 1.

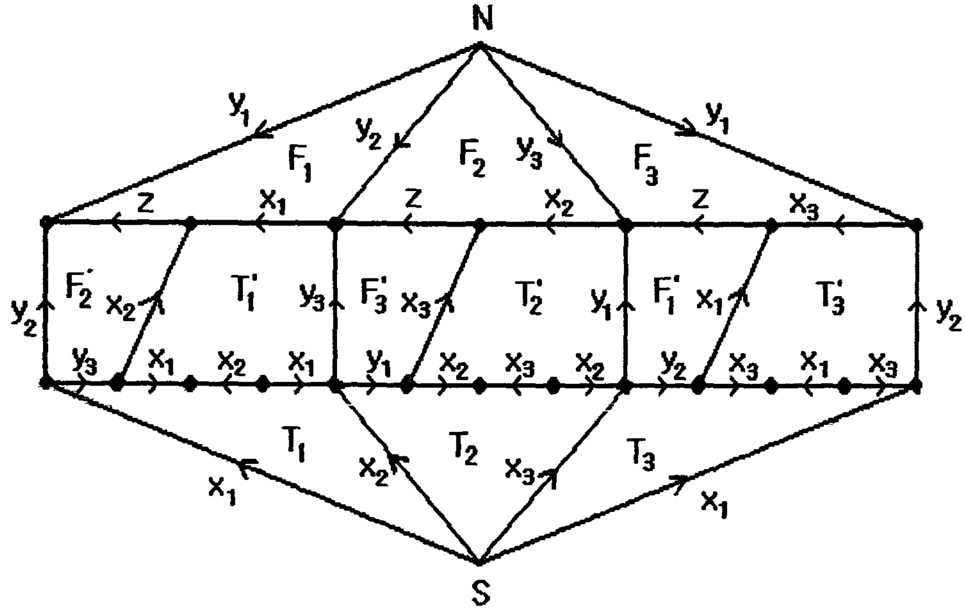

Figure 1. The polyhedron $P(2,3)$.

Let us consider the 1-skeleton of P_n with orientation and labeling of its edges in the following manner.

(i) The oriented edges fall into $2n+1$ classes: x_i, $i = 1, \dots, n$, where each class x_i consists $2nm + 4n$ edges, y_i, $i = 1, \dots, n$, where each class y_i consists $3n$ edges and z consists n edges. In this case oriented edges from the same class carry the same label.

(ii) For each $i = 1, \dots, n$ the boundary cycle of the $(2m+4)$-gons T_i and T_i' is $x_i\, y_{i+2}\, (x_i\, x_{i+1}^{-1})^{m+1}$ with the indices taken mod n.

(iii) For each $i = 1, \dots, n$ the boundary cycle of the quadrilaterals F_i and F_i' is $y_{i+1}\, x_i\, z\, y_i^{-1}$ with the indices taken mod n.

Note that the set of all faces splits into pairs of faces with the same sequences of oriented boundary edges. Now we shall identify quadrilaterals F_i and F'_i, and $(2m+4)$-gons T_i and T'_i such that the corresponding oriented edges on polygons carrying the same label are identified for each $i = 1, \dots, n$.

The resulting complex has 2 non-equivalent vertices N and S, $2n+1$ non-equivalent edges, $2n$ two-cells and 1 three-cell. Thus the Euler characteristic is 0. Then by the following theorem, due to H. Seifert and W. Threlfall [14, p.216], it forms a 3-dimensional manifold.

THEOREM. *A complex, which is formed by identifying the faces of a polyhedron will be a manifold if and only if its Euler characteristic equals zero.*

We select N as the initial point of the closed paths and z as auxiliary path leading to the vertex S. Then the generating path classes of the fundamental group of this manifold will be represented by the closed paths: for $i = 1, ..., n$,

$$Z = z\,z^{-1},\ X_i = z\ x_i\ \ and\ \ Y_i = y_i\,z^{-1},$$

and relations: for $i = 1, ..., n$,

$$Z = Z\,Z^{-1},\ X_i = Z\,X_i,\ \ Y_i = Y_i Z^{-1},\ X_i Y_{i+2}(X_i X_{i+1}^{-1})^{m+1} = 1 \ \ and \quad Y_{i+1} X_i Z\, Y_i^{-1} = 1.$$

Thus the fundamental group has the following presentation.

$$\langle X_1, \dots, X_n, Y_1, \dots, Y_n \mid X_i Y_{i+2}(X_i X_{i+1}^{-1})^{m+1} = 1,\ \ Y_{i+1}\, X_i\, Y_i^{-1} = 1, \quad i = 1, \dots, n\rangle,$$

where the indices are taken mod n.

Therefore we see that the fundamental group is isomorphic to the group $G(2m, n)$. □

We remark that the polyhedron $P(2m, n)$ can be considered as the natural generalization of the fundamental polyhedron for the manifolds M_n constructed in [11]. Indeed, throughout this paper, if $m = 0$, then all results coincide with ones in [11].

3. The split extension of the group $G(2m,n)$

Let us consider the abstract groups defined by following presentations

$$(1)\quad \langle x_1,\dots,x_n,y_1,\dots,y_n \mid x_i\, y_{i+2}\,(x_i\, x_{i+1}^{-1})^{m+1} = 1,\ y_{i+1}x_i y_i^{-1} = 1,\ i = 1,\dots,n\rangle,$$

where the indices are taken mod n. With $x_i\, y_{i+2}\,(x_i\, x_{i+1}^{-1})^{m+1} = 1$ for all $i = 1,\dots n$, and Tieze transformation, we can easily isolate y_i and prove that it is another presentation of $G(2m,n)$. Moreover we see that the group $G(2m,n)$ has the cyclic automorphism $\rho : x_i \to x_{i+1}$ and $y_i \to y_{i+1}$ of order n. Now we consider the split extension $\widehat{G}(2m,n)$ of group $G(2m,n)$ by the cyclic group of automorphisms generated by ρ. The following theorem shows that the group $\widehat{G}(2m,n)$ is interesting from the topological point of views.

For a knot K, we denote by $K(n)$ the 3-dimensional orbifold with the underlying space S^3 and as a singular set K with branched index n. Then we have the following relation between $\widehat{G}(2m,n)$ and the knot $b(4m+7,2m+3)$, that is the closure of the rational $(\frac{4m+7}{2m+3})$-tangle.

THEOREM 2. *For $m \geq 0$, $n \geq 2$, let $M(2m,n)$ be the n-fold cyclic branched covering of the knot $b(4m+7,2m+3)$. Then $\pi_1(M(2m,n)) \cong G(2m,n)$.*

Proof. With the presentation (1) and the cyclic automorphism ρ of order n such that $\rho : x_i \to x_{i+1}$ and $y_i \to y_{i+1}$, we get the following presentation of the split extension $\widehat{G}(2m,n)$ of $G(2m,n)$ by ρ, with notations $x = x_1$ and $y = y_1$.

$$\begin{aligned}\widehat{G}(2m,n) &= \langle \rho,x,y \mid x\rho^2(y)(x(\rho(x))^{-1})^{m+1} = 1, \rho(y)xy^{-1} = 1, \rho^n = 1\ \rangle \\ &= \langle \rho,x,y \mid x\rho^2(y)(x(\rho(x))^{-1})^{m+1} = 1, \rho = y(\rho x)y^{-1}, \rho^n = 1\rangle.\end{aligned}$$

Note that ρ and ρx are conjugate. Let $\mu = \rho x$. Then $x = \rho^{-1}\mu$ and $\mu^n = 1$. So

$$\begin{aligned}\widehat{G}(2m,n) &= \langle \rho,\mu,y \mid \rho y = y\mu, y = \rho^2\mu^{-1}\rho^{-1}w_1^{m+1}, \rho^n = 1,\quad \mu^n = 1\rangle \\ &= \langle \rho,\mu \mid \rho w = w\mu, \rho^n = 1,\quad \mu^n = 1\rangle \\ &\textit{where}\quad w = \mu^{-1}\rho^{-1}w_1^{m+1},\ w_1 = \mu\rho\mu^{-1}\rho^{-1}.\end{aligned}$$

We recall that the group

$$\langle \rho, \mu \mid \rho w = w\mu \rangle \ \ where \ \ w = \mu^{-1}\rho^{-1}(\mu\rho\mu^{-1}\rho^{-1})^m$$

is the group of the $b(4m+7, 2m+3)$-knot, where ρ and μ as shown in the Figure 2 and the index $2m+3$ in Figure 2 denotes the number of half-twists.

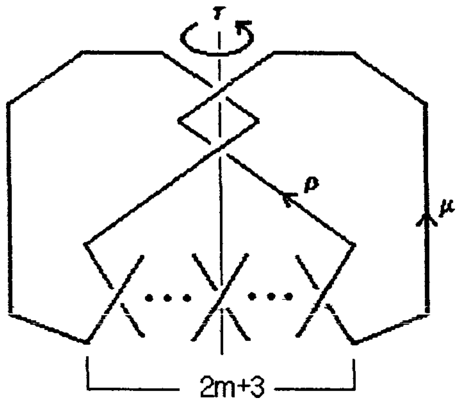

Figure 2. The knot $b(4m+7, 2m+3)$.

Therefore by [5], the group $\widehat{G}(2m, n)$ is the fundamental group of the orbifold $b(4m+7, 2m+3)(n)$. Hence $\pi_1(M(2m, n)) \cong G(2m, n)$. □

THEOREM. *(Thurston) Assume $q > 1$. Then $(p/q)(n)$ is hyperbolic for (i) $p = 5, n \geq 4$ (ii) $p \neq 5, n \geq 3$. Moreover $(p/q)(2)$ is spherical for all p, and $(5/3)(3)$ is euclidean.*

Thanks to the above theorem(see [4] and [10]) we have that the orbifold $b(4m+7, 2m+3)(n)$ (denoted by $(\frac{4m+7}{2m+3})(n)$) is hyperbolic for $n \geq 3, m \geq 0$, and it is spherical for $n = 2, m \geq 0$.

COROLLARY 1. *The manifolds $M(2m, n)$ is hyperbolic for $n \geq 3, m \geq 0$, and $M(2m, 2)$ is the lens space $L(4m+7, 2m+3)$ for $m \geq 0$.*

COROLLARY 2. *The group $G(2m, n)$ is infinite for $n \geq 3$, $m \geq 0$, and $G(2m, 2) \cong \mathbb{Z}_{4m+7}$.*

4. The manifolds $M(2m,n)$ as 2-fold coverings

In this section we will study the topological properties of manifolds $M(2m,n)$, that gives a topological approach to the studying of cyclically-presented groups $G(2m,n)$. This studying is analogous to the topological studying of Sieradski groups $S(n)$ and Fibonacci groups $F(2,2n)$ given in [2], [3], [8] and [12].

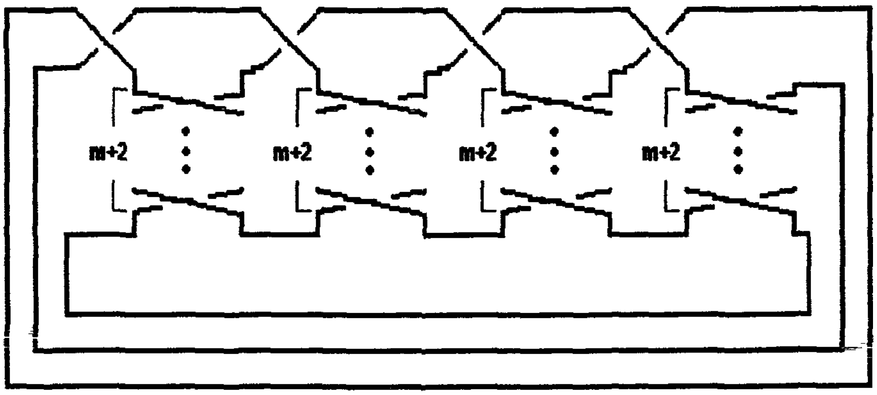

Figure 3. The knot $\mathcal{K}_4^{2m}$.

Firstly we define a series of knots. We recall that any knot can be obtained as the closure of some braid [1]. Let p and q be coprime integers, then by $\sigma_i^{p/q}$ we denote the rational p/q-tangle whose incoming arcs are i-th and $(i+1)$-th strings. For an integer $n \geq 1$ we denote by $\mathcal{K}_n^{2m}$ the n-periodic knot, that is the closure of the rational 3-strings braid $(\sigma_1 \, \sigma_2^{1/(m+2)})^n$. The diagram of the knot $\mathcal{K}_4^{2m}$ is pictured in Figure 3. Note that the knot $\mathcal{K}_2^{2m}$ is equivalent, under the Reidemeister moves, to the 2-periodic knot $b(4m+7, 2m+3)$.

THEOREM 3. *For $m \geq 0$ and $n \geq 2$, the manifold $M(2m,n)$ is the two-fold covering of the 3-sphere branched over the knot $\mathcal{K}_n^{2m}$.*

Proof. By Theorem 3 the manifold $M(2m,n)$ is the n-cyclic branched covering of the 3-sphere S^3, branched over the knot $b(4m+7, 2m+3)$. To describe $M(2m,n)$ as the 2-cyclic branched covering of S^3, branched over an n-periodic knot, we will use the following construction which is analogous to [12] and [2] where the Fibonacci groups and the Sieradski groups were topologically studied.

From the Figure 2 for the knot $b(4m+7, 2m+3)$ we see the that the orbifold $b(4m+7, 2m+3)(n)$ has a rotation symmetry of order two denoted by τ such that the axe of the symmetry is disjoint from the knot $b(4m+7, 2m+3)$. It is not difficult to see that this symmetry action produces the orbifold $b(4m+7, 2m+3)/\langle\tau\rangle$ with underlying space S^3, and as a singular set the 2-component link pictured in Figure 4 with branch indices 2 and n.

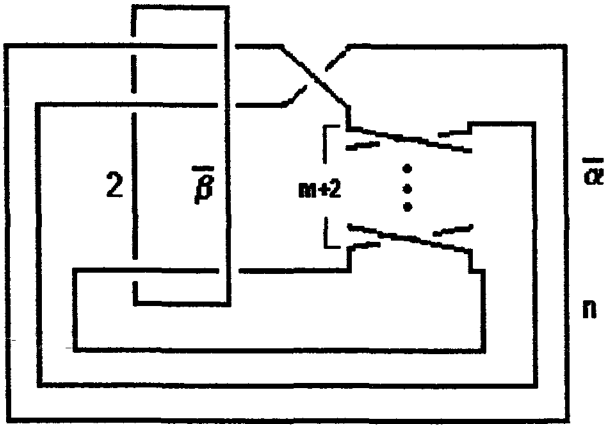

Figure 4. The singular set of $b(4m+7, 2m+3)/\langle\tau\rangle$.

It is easy to see that the singular set of the quotient orbifold is the two-component link $b(8m+14, 2m+3)$ in Figure 5, that is the 2-bridge link obtained as the closure of the rational $(\frac{8m+14}{2m+3})$-tangle.

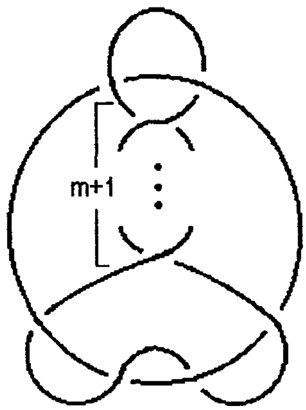

Figure 5. The link $b(8m+14, 2m+3)$.

We will denote the quotient orbifold $b(4m+7, 2m+3)/\langle\tau\rangle$ by $b(8m+14, 2m+3)(2, n)$. Then we have the following covering diagram

$$(2)\ M(2m,n) \xrightarrow{n} b(4m+7,2m+3)(n) \xrightarrow{2} b(8m+14,2m+3)(2,n)$$

and a sequence of normal subgroups

$$G(2m,n) = \pi_1(M(2m,n)) \lhd \widehat{G}(2m,n) = \pi_1(b(4m+7,2m+3)(n)) \lhd \Omega(2m,n) = \pi_1(b(8m+14,2m+3)(2,n)),$$

where $|\Omega(2m,n) : \widehat{G}(2m,n)| = 2$ and $|\widehat{G}(2m,n) : G(2m,n)| = n$.

We describe the orbifold group $\Omega(2m, n)$ using the Wirtinger representation of the link group $\pi_1(b(8m+14, 2m+3))$ in figure 4. The link group has two generators $\overline{\alpha}, \overline{\beta}$ and one relator of the form $\overline{\alpha}\,\overline{w} = \overline{w}\,\overline{\alpha}$, where a word $\overline{w}$ is determined as follows:

$$w = \overline{\beta}^{\imath_1}\,\overline{\alpha}^{\imath_2}\,\overline{\beta}^{\imath_3} \cdots \overline{\alpha}^{\imath(8m+12)}\,\overline{\beta}^{\imath(8m+13)},$$

and $\imath_j$ is the sign of the number $(2m+3)j$ by mod $2(8m+14)$ on the segment $[-(8m+14), 8m+14]$. For example, if $m = 0$, we get a word

$$w = \overline{\beta}\,\overline{\alpha}\,\overline{\beta}\,\overline{\alpha}\,\overline{\beta}^{-1}\,\overline{\alpha}^{-1}\,\overline{\beta}^{-1}\,\overline{\alpha}^{-1}\,\overline{\beta}^{-1}\,\overline{\alpha}\,\overline{\beta}\,\overline{\alpha}\,\overline{\beta}.$$

In this representation the generators $\overline{\alpha}$ and $\overline{\beta}$ correspond to the arcs with the same labels on the link diagram of $b(8m+14, 2m+3)$ as shown in figure 4.

According to [5], we get the following presentation for the group $\Omega(2m, n)$ of the orbifold $b(8m+14, 2m+3)(2, n)$:

$$\Omega(2m,n) = \langle \alpha,\ \beta \mid \alpha\, w = w\, \alpha, \quad \alpha^n = \beta^2 = 1 \rangle,$$

where the generators α and β canonically correspond to $\overline{\alpha}$ and $\overline{\beta}$ respectively.

Let us consider the group

$$\mathbb{Z}_n \oplus \mathbb{Z}_2 = \langle a \mid a^n = 1\rangle \oplus \langle b \mid b^2 = 1\rangle$$

and the epimorphism

$$\theta\ \Omega(2,n) \implies \mathbb{Z}_n \oplus \mathbb{Z}_2$$

defined by setting $\theta(\alpha) = a$ and $\theta(\beta) = b$.

By the construction of the 2-fold covering

$$b(4m+7, 2m+3)(n) \xrightarrow{2} b(8m+14, 2m+3)(2,n)$$

the loop $\beta \in \Omega(2m,n)$ lifts to a trivial loop in $\widehat{G}(2m,n)$, and the loop $\alpha \in \Omega(2m,n)$ lifts to a loop in $\widehat{G}(2m,n)$ which generates a cyclic subgroup of order n. Thus it follows that

$$\pi_1(b(4m+7, 2m+3)(n)) = \theta^{-1}(\langle a \mid a^n = 1\rangle) = \theta^{-1}(\mathbb{Z}_n).$$

For the $2n$-fold covering

$$M(2m,n) \xrightarrow{2n} b(8m+14, 2m+3)(2,n)$$

both loops α and β from $\Omega(2m,n)$ lift to trivial loops in $G(2m,n) = \pi_1(M(2m,n))$, hence $G(2m,n) = \mathrm{Ker}\,\theta$.

Let Γ_n be the subgroup of $\Omega(2m,n)$ given by

$$\Gamma_n = \theta^{-1}(\langle b \mid b^2 = 1\rangle) = \theta^{-1}(\mathbb{Z}_2).$$

Then we get a sequence of normal subgroups

$$G(2m,n) \lhd \Gamma_n \lhd \Omega(2m,n),$$

where $|\Omega(2m,n) : \Gamma_n| = n$ and $|\Gamma_n : G(2m,n)| = 2$. We recall, that the orbifold $b(4m+7, 2m+3)(n)$ is spherical for $n = 2$, and hyperbolic for $n \geq 3$. Hence the group Γ_n acts by isometries on the universal covering X_n, that is the 3-sphere S^3 for $n = 2$, and the hyperbolic space $\mathbb{H}^3$ for $n \geq 3$. Thus we get the orbifold X_n / Γ_n and the following covering diagram

$$\text{(3)} \qquad M(2m,n) \xrightarrow{2} X_n / \Gamma_n \xrightarrow{2} b(8m+14, 2m+3)(2,n).$$

In this case the second covering is cyclic and it is branched over the component with index n of the singular set of $b(8m+14, 2m+3)(2,n)$ in Figure 5. But this component is the knot $\mathcal{K}_1$ and is trivial. So, underlying space of $X_n \,/\, \Gamma_n$ is the 3-sphere. By the construction of the n-fold covering

$$X_n \,/\, \Gamma_n \xrightarrow{\ n\ } b(8m+14, 2m+3)(2,n)$$

the loop $\alpha \in \Omega(2m,n)$ lifts to a trivial loop in Γ_n, and the loop $\beta \in \Omega(2m,n)$ lifts to a loop in Γ_n which generates a cyclic group of order 2. Because $b(8m+14, 2m+3)$ are 2-bridge links whose components are equivalent, we can exchange branch indices of components in Figure 5. Therefore, the singular set of $X_n \,/\, \Gamma_n$ is a n-periodic knot which can be obtained as the closure of the 3-string braid $(\sigma_1\,\sigma_2^{1/(m+2)})^n$, that is the knot $\mathcal{K}_n^{2m}$. Because the branch index is equal to 2, we denote $X_n \,/\, \Gamma_n = \mathcal{K}_n^{2m}(2)$.

Comparing (2) and (3), we get that the following covering diagram is commutative:

$$\begin{array}{ccc}
M(2m,n) & = & M(2m,n) \\
{\scriptstyle n}\big\downarrow & & \big\downarrow{\scriptstyle 2} \\
b(4m+7, 2m+3)(n) & & \mathcal{K}_n^{2m}(2) \\
{\scriptstyle 2}\big\downarrow & & \big\downarrow{\scriptstyle n} \\
b(8m+14, 2m+3)(2,n) & = & b(8m+14, 2m+3)(2,n)
\end{array}$$

The diagram of coverings.

In particular, we have that $M(2m,n)$ is the 2-fold branched covering of the 3-sphere branched over the knot $\mathcal{K}_n^{2m}$, and theorem is proved. □

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