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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A CLASS OF DOUBLY NONLINEAR EVOLUTION EQUATIONS

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1. Introduction

In this paper, we discuss the asymptotic behavior, as $t \to \infty$, of solutions to the nonlinear initial value problem

$$\begin{cases} Au'(t) + Bu(t) \ni f(t), & t \in \mathbb{R}^+ = [0, \infty), \\ u(0) = u_0, \end{cases} \tag{1}$$

in a Hilbert space H. Here A and B are maximal monotone (possibly multivalued) operators in H (or from V to V^*, where V is a reflexive Banach space of dual V^*, with $V \subset H \subset V^*$, densely and continuously), $f : \mathbb{R}^+ \to H$, and $u_0 \in H$. Equations of this type arise in thermodynamics, in the presence of dissipation phenomena (cf., e.g , [5]).

The existence of solutions to (1) on a finite interval has recently been considered in [1, 5, 6], while continuous dependence results appear in [2]. However, we are not aware of any attempt to develop an asymptotic theory for such equations.

The plan of the paper is as follows In Section 2 we extend the existence results of [1] to the case of $\mathbb{R}^+$, and we study the asymptotic properties of solutions to Equation (1) Two theorems are included. Section 3 contains two examples that illustrate the abstract theory.

We assume the familiarity of the reader with maximal monotone operators, Banach space valued functions, and Sobolev spaces. See [3, 4] for background material on these topics.

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2. Main results

Throughout this paper H denotes a real Hilbert space of norm $|\cdot|$ and inner product $(\cdot, \cdot)$. Let $(V, \|\cdot\|)$ be a reflexive Banach space, of dual $(V^*, \|\cdot\|_*)$ such that $V \subset H \subset V^*$, densely and continuously.

The duality pairing between $v^* \in V^*$ and $v \in V$, also denoted (v^*, v), coincides with the scalar product in H whenever $v^* \in H$. Without loss of generality we will assume that both V and V^* are strictly convex.

Let A be a (possibly multivalued) operator from V to V^* with $\mathcal{D}(A) = V$, satisfying

(H1) A is maximal monotone and there exist $0 < c_1 \leq c_2$, and $p \in [2, +\infty)$ such that

$$(y, x) \geq c_1 \|x\|^p, \qquad \|y\|_* \leq c_2 \|x\|^{p-1} \tag{2}$$

for all $x \in V$ and $y \in Ax$.

Next let $B : V \to V^*$ satisfy

(H2) $B = \partial\varphi$ (∂ = subdifferential), where $\varphi : V \to (-\infty, +\infty]$ is proper, convex and lower semicontinuous, and such that for any $r > 0$, the set $\{x \in V : \varphi(x) \leq r\}$ is compact in V,

(H3) $B^{-1}0 \neq \emptyset$.

In what follows, p denotes the constant appearing in (H1), and q is its conjugate, i.e., $p^{-1} + q^{-1} = 1$. We also need the condition

(H4) $f \in L^q(\mathbb{R}^+; V^*)$.

Definition 1. Let $u_0 \in V$ be given. A solution of (1) is a function $u \in W^{1,p}_{loc}(\mathbb{R}^+; V)$ with $u(0) = u_0$, such that there exist measurable $v, w : [0, T] \to V^*$ satisfying

$$\begin{aligned} &v(t) \in Au'(t), \quad w(t) \in Bu(t), \\ &v(t) + w(t) = f(t), \end{aligned} \tag{3}$$

for almost all $t \in \mathbb{R}^+$.

THEOREM 2. *Let (H1)-(H4) hold. Then for each* $u_0 \in \mathcal{D}(\varphi) := \{x \in V : \varphi(x) < +\infty\}$ *the problem* (1) *has a solution* u *with the following properties:*

$$u \in L^\infty(\mathbb{R}^+;\ V) \cap UC(\mathbb{R}^+;\ V), \quad u' \in L^p(\mathbb{R}^+;\ V), \tag{4}$$

where $UC(\mathbb{R}^+;\ V)$ *stands for the set of all uniformly continuous functions on* $\mathbb{R}^+$ *with values in* V,

$$v, w \in L^q(\mathbb{R}^+;\ V^*), \tag{5}$$

where v, w *satisfy* (3),

$$\lim_{t\to\infty} \varphi(u(t)) = \min_{x \in V} \varphi(x), \tag{6}$$

$$\omega(u) \neq \emptyset, \quad \omega(u) \subset B^{-1}0, \tag{7}$$

where $\omega(u) = \{x \in V\ :\ u(t_n) \to x\ \text{ in } V,\ \text{ for some }\ t_n \to \infty\}$.

If also $B^{-1}0$ *is a singleton, then*

$$\lim_{t\to\infty} u(t) = B^{-1}0 \ \text{ in } V. \tag{8}$$

Proof. Let $T \in (0,\ \infty)$ be arbitrarily fixed. By [1, Theorem 2], Eq. (1) has a solution u_1 on $[0,\ T]$ in the sense that $u_1 \in W^{1,p}(0, T; V)$ and there exist $v_1, w_1 \in L^q(0,\ T;\ V^*)$ satisfying (3) (with $u = u_1, v = v_1, w = w_1$) for almost all t in $(0,\ T)$. Also remark (cf., e.g., [5, Lemma 4.1]) that the function $t \to \varphi(u_1(t))$ is absolutely continuous on $[0,\ T]$, so that in particular, $u_1(T) \in \mathcal{D}(\varphi)$. Consider the problem

$$\begin{cases} Ax'(t) + Bx(t) \ni f(t+T), & 0 < t < T, \\ x(0) = u_1(T). \end{cases} \tag{9}$$

Applying Theorem 2 of [1] again, we conclude that (9) has a solution $x : [0,\ T] \to V$ such that $x \in W^{1,p}(0,\ T;\ V)$ and there are functions $y, z \in L^q(0,\ T;\ V^*)$ satisfying (3) on $(0,\ T)$ (with x, y, z in place of u, v, w, respectively). Define $u_2 : [T,\ 2T] \to V$, $\quad v_2, w_2 : [T,\ 2T] \to V^*$ by

$$u_2(t) = x(t-T), \ \ v_2(t) = y(t-T), \ \ w_2(t) = z(t-T) \quad (T \le t \le 2T),$$

and subsequently let $u : [0,\ 2T] \to V$, $v, w : [0,\ 2T] \to V^*$ be given by

$$u(t) = u_n(t),\ \ v(t) = v_n(t),\ \ w(t) = w_n(t)\ \text{ if } (n-1)T \le t \le nT, \tag{10}$$

for $n = 1, 2$. It is easily seen that u, v, w, as given by (10) satisfy Definition 1 on $[0,\ 2T]$. Applying this procedure repeatedly we extend u, v and w on $\mathbb{R}^+$, such that $u \in W^{1,p}_{loc}(\mathbb{R}^+;\ V)$, $v, w \in L^q_{loc}(\mathbb{R}^+;\ V^*)$ and (3) holds.

To obtain the stronger conclusions of Theorem 2, multiply (3_2) by $u'(t)$ and integrate the result over $(0, t)$, $0 < t < \infty$. Using (H1), (H2) and Lemma 4.1 of [5], we obtain

$$\begin{aligned}(c_1/2)\int_0^t \|u'(s)\|^p ds + \varphi(u(t)) \\ \le \varphi(u_0) + 2^{q/p} q^{-1} (c_1 p)^{-q/p} \int_0^t \|f(s)\|_*^q\ ds.\end{aligned} \tag{11}$$

In view of (H3), φ is bounded below on V. This, (H4) and (11) yield $u' \in L^p(\mathbb{R}^+;\ V)$. Moreover it also follows that $\{\varphi(u(t)) : t \in \mathbb{R}^+\}$ is bounded, so that by (H2), $\{u(t)\ :\ t \in \mathbb{R}^+\}$ is compact in V. In particular (4) and (7_1) have been established. Relation (5) is a direct consequence of (3), (4) and (H1). To prove (6), note first that

$$\frac{d}{dt}\varphi(u(\cdot)) \in L^1(0,\ \infty). \tag{12}$$

Indeed, $\frac{d}{dt}\varphi(u(t)) = (w(t),\ u'(t))$ with $u' \in L^p(\mathbb{R}^+;\ V)$ and $w \in L^q(\mathbb{R}^+;\ V)$ (cf. (4), (5)). Since

$$\varphi(u(t)) = \varphi(u_0) + \int_0^t (\varphi(u(s)))' ds$$

we infer, by (12), that

$$\lim_{t\to\infty} \varphi(u(t)) = \varphi_\infty \tag{13}$$

exists. Next, by the definition of a subdifferential and (3),

$$\varphi(x) \ge \varphi(u(t)) + (w(t),\ x - u(t)),\ \text{ a.e. on } \mathbb{R}^+,\ \forall x \in V. \tag{14}$$

Taking into account that $w \in L^q(\mathbb{R}^+;\ V^*)$ and $u \in L^\infty(\mathbb{R}^+;\ V)$ one can find a sequence $t_n \to \infty$, such that $(w(t_n),\ x - u(t_n)) \to 0$.

Letting $t = t_n \to \infty$ in (14) yields by virtue of (13)

$$\varphi(x) \geq \varphi_\infty, \quad \forall x \in V.$$

This in conjunction with (H3) shows that $\varphi_\infty = \min_{x \in V} \varphi(x)$, as desired. Finally, let $x_0 \in \omega(u)$; that is $u(t_n) \to x_0$ in V for some $t_n \to \infty$. The lower semicontinuity of φ, together with (6), implies

$$\varphi(x_0) = \min_{x \in V} \varphi(x),$$

which is equivalent to $x_0 \in B^{-1}0$, so that (7_2) holds. In particular, when $B^{-1}0$ is a singleton, (8) follows from (7). This completes the proof of Theorem 2 □

Our second results complements Theorem 3 in [1]. In place of (H2) we are now using the following conditions :

(H5) $B = \partial\psi$, where $\psi : H \to (-\infty,\ +\infty]$ is proper, convex and lower semicontinuous, and such that the set $\{x \in H\ :\ \psi(x) \leq r\}$ is compact in H for any $r > 0$

(H6) $V \subset \mathcal{D}(B)$ and B^0 maps bounded subsets of V into bounded subsets of H. (Here $B^0 x = Proj_{Bx} 0, \forall x \in \mathcal{D}(B)$, where 'Proj' designates the nearest point projection in H.)

THEOREM 3. *Assume* $(H1)$, $(H3)$-$(H6)$. *Then for every* $u_0 \in V$, *there exists a solution* u *of* (1) *satisfying*

$$u \in L^\infty(\mathbb{R}^+;\ H) \cap UC(\mathbb{R}^+;\ V),\ \ u' \in L^p(\mathbb{R}^+;\ V), \tag{15}$$

$$v \in L^q(\mathbb{R}^+;\ V^*),\ \ w \in L^\infty_{loc}(\mathbb{R}^+;\ H) \cap L^q(\mathbb{R}^+;\ V^*), \tag{16}$$

$$\lim_{t \to \infty} \psi(u(t)) = \psi_\infty \ \text{ exists}, \tag{17}$$

$$\omega_H(u) \neq \emptyset, \tag{18}$$

where $\omega_H(u) = \{x \in H\ :\ u(t_n) \to x \text{ in } H \text{ for some } t_n \to \infty\}$. *If also*

$$u \in L^\infty(\mathbb{R}^+;\ V) \tag{19}$$

then

$$\psi_\infty = \min_{x\in H} \psi(x), \quad \omega_H(u) \subset B^{-1}0. \tag{20}$$

In particular, when $B^{-1}0$ is a singleton, then

$$u(t) \to B^{-1}0 \ \text{ in } H, \ \text{ as } t \to \infty. \tag{21}$$

The proof of Theorem 3 is essentially similar to that of Theorem 2, and is therefore omitted. (One makes use of [1, Theorem 3] to obtain a solution to (1) on a finite interval [0, T], and then extends it to $\mathbb{R}^+$. Relations (15)-(17) and (20), (21) are derived by the technique employed in the proof of Theorem 2.)

An interesting question is when does (19) hold. A partial answer is given below.

REMARK 4. Let $A : V \to V^*$ be a linear bounded self-adjoint operator satisfying

$$(Ax, x) \geq c\|x\|^2, \ \forall x \in V \ (c > 0).$$

This implies (H1) with $p = 2$. In particular, $x \to (Ax, x)$ defines an equivalent norm to $\|x\|$ for $x \in V$.

Let $f \in L^1(0, \infty; V^*) \cap L^2(0, \infty; V^*)$, and $0 \in B0$. If u is a solution to (1), one has, for almost all $t \in \mathbb{R}^+$

$$(Au'(t), u(t)) = (Au(t), u'(t)) = \frac{1}{2}\frac{d}{dt}(Au(t), u(t)).$$

As a consequence, multiplying (3_2) by $u(t)$ and integrating over $(0, t)$, $0 < t < \infty$, yields

$$\|u(t)\| \leq C(\|u_0\| + \int_0^\infty \|f(s)\|_* \, ds)$$

for some $C > 0$, so that $u \in L^\infty(\mathbb{R}^+, V)$, as desired.

It is also worth noting that in this case u is uniquely determined.

3. Examples

Throughout this section, $p \in [2, \infty)$ and $q = p/(p-1)$.

First, let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary Γ, and let $r, s \in [2, \infty)$ satisfy $r^{-1} + s^{-1} = 1$; if $r < N$, we also assume that $p < rN/(N-r)$. Set $V = (L^p(\Omega))^M$, $H = (L^2(\Omega))^M$, $W = (W_0^{1,r}(\Omega))^M$ $(M \geq 1)$, so that $V^* = (L^q(\Omega))^M$, $W^* = (W^{-1,s}(\Omega))^M$ and $W \subset V \subset H \subset V^* \subset W^*$ with dense and continuous injections ; moreover W is compactly imbedded into V.

Let α be a maximal monotone graph in $\mathbb{R}^M \times \mathbb{R}^M$ satisfying

$$\sum_{j=1}^{M} y_j z_j \geq c_1 \sum_{j=1}^{M} |z_j|^p, \quad \sum_{j=1}^{M} |y_j|^q \leq c_2 \sum_{j=1}^{M} |z_j|^p \tag{22}$$

for all $z = (z_1, \cdots, z_M) \in \mathbb{R}^M$ and $y = (y_1, \cdots, y_M) \in \alpha(z)$, and fixed positive constants c_1, c_2.

Define $A : V \to V^*$ by

$$v \in Au \ \text{ if } \ v(x) \in \alpha(u(x)), \ \text{ a.e. on } \ \Omega. \tag{23}$$

In view of (22) it is readily seen that A, as given by (23), satisfies (H1). Next, introduce $\varphi : V \to (-\infty, +\infty]$ by

$$\varphi(u) = \begin{cases} \displaystyle\sum_{j=1}^{M} r^{-1} \int_\Omega \sum_{i=1}^{N} b_{ij}(x) \left| \frac{\partial u_j}{\partial x_i}(x) \right|^r dx, & \text{if } u \in W, \\ +\infty, & \text{if } u \in V \setminus W, \end{cases} \tag{24}$$

where $b_{ij} \in L^\infty(\Omega)$ $(i = 1, \cdots, N;\ j = 1, \cdots, M)$ are such that

$$0 < b_1 \leq b_{ij}(x) \leq b_2, \ \text{ a.e. on } \ \Omega, \tag{25}$$

for some $b_1, b_2 > 0$, and we have set $x = (x_1, \cdots, x_N)$, $u = (u_1, \cdots, u_M)$.

Observe that φ is proper, convex and lower semicontinuous on V, and that its subdifferential (denoted by B) is given by

$$\begin{aligned} &Bu = (\beta_1(u_1), \cdots, \beta_M(u_M)), \\ &\mathcal{D}(B) = \{u \in W \ : \ \beta_j(u_j) \in L^q(\Omega), \ j = 1, \cdots, M\}, \\ &\beta_j(y) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(b_{ij}(x) \left| \frac{\partial y}{\partial x_i} \right|^{r-2} \frac{\partial y}{\partial x_i} \right), \ \forall y \in W_0^{1,r}(\Omega). \end{aligned} \tag{26}$$

On account of (24)-(26) one verifies (H2) and (H3) ; in particular, $B^{-1}0 = 0$. Finally let u_0, f be such that

$$u_0 \in (W_0^{1,r}(\Omega))^M, \quad f \in L^q(\mathbb{R}^+;\ (L^q(\Omega))^M) \tag{27}$$

and consider the initial-boundary value problem ($j = 1, \cdots, M$)

$$(P_1) \qquad \begin{cases} w_j - \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(b_{ij}(x)\left|\frac{\partial u_j}{\partial x_i}\right|^{r-2}\frac{\partial u_j}{\partial x_i}\right) = f_j \text{ on } \mathbb{R}^+ \times \Omega, \\ w \in \alpha\left(\frac{\partial u}{\partial t}\right) \text{ on } \mathbb{R}^+ \times \Omega, \\ u(0,\ x) = u_0(x) \text{ on } \Omega, \quad u = 0 \text{ on } (0,\ T) \times \Gamma, \end{cases}$$

where $u = (u_1, \cdots, u_M)$, $w = (w_1, \cdots, w_M)$, $f = (f_1, \cdots, f_M)$. This can be rewritten in the abstract form (1), with A and B given by (23) and (26) respectively. As remarked earlier conditions (H1)-(H3) are satisfied in this setup, while (H4) and the restriction on u_0 in Theorem 2 are consequences of (27). A direct application of Theorem 2 leads to

THEOREM 5. *Let the assumptions* (22), (25) *and* (27) *be fulfilled. Then the problem* (P_1) *has a solution* u *satisfying*

$$\begin{aligned} &u \in L^\infty(\mathbb{R}^+;(L^p(\Omega))^M) \cap UC(\mathbb{R}^+;(L^p(\Omega))^M), \\ &u_t \in L^p(\mathbb{R}^+;(L^p(\Omega))^M), \\ &v, w \in L^q(\mathbb{R}^+;(L^q(\Omega))^M), \\ &\left(v = -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(b_{ij}(x)\left|\frac{\partial u_j}{\partial x_i}\right|^{r-2}\frac{\partial u_j}{\partial x_i}\right)\right), \\ &\lim_{t\to\infty} u(t,\cdot) = 0 \ \text{ in } \ (W_0^{1,r}(\Omega))^M. \end{aligned}$$

The last conclusion of Theorem 5 follows from (6) and (8), by virtue of (24) and $B^{-1}0 = 0$.

Our second example is related to Theorem 3. We now take $V = W_0^{2,2}(0,1)$, $H = L^2(\Omega)$, $V^* = W^{-2,2}(0,1)$, and define $A : V \to V^*$ by

$$Au = \frac{d^4u}{dx^4}, \quad \forall u \in V \tag{28}$$

Obviously A satisfies the conditions in Remark 4. Next, let $g : \mathbb{R} \to \mathbb{R}$ satisfy

$$g \in C^1(\mathbb{R}), \;\; g(0) = 0, \;\; 0 < m \le g' \le M, \tag{29}$$

for some $m, M > 0$. Define $\psi : H \to (-\infty, +\infty]$ by

$$\psi(u) = \begin{cases} \int_0^1 (\int_0^{u_x} g(s)\, ds)\, dx, & \text{if } u \in W_0^{1,2}(0,1), \\ +\infty, & \text{if } u \in L^2(0,1) \setminus W_0^{1,2}(0,1). \end{cases} \tag{30}$$

Note that ψ is proper, convex and lower semicontinuous on H, with subdifferential $B = \partial\psi$ given by

$$\begin{aligned} Bu &= -(g(u_x))_x, \\ \mathcal{D}(B) &= \{u \in W_0^{1,2}(0,1) \; : \; (g(u_x))_x \in L^2(0,1)\}. \end{aligned} \tag{31}$$

Taking into account (29)-(31), one can easily show that (H5) and (H6) hold ; moreover $B^{-1}0 = 0$.

We are interested in the semilinear problem

$$(P_2) \qquad \begin{cases} u_{txxxx} - (g(u_x))_x = f, \;\; \text{on } \mathbb{R}^+ \times (0,1), \\ u(0,x) = u_0(x), \;\; \text{on } (0,1), \\ \frac{\partial^k u}{\partial x^k}(t,0) = \frac{\partial^k u}{\partial x^k}(t,1), \;\; t \in \mathbb{R}^+ \;\; (k = 0,1), \end{cases}$$

where u_0 and f are such that

$$\begin{aligned} &u_0 \in W_0^{2,2}(0,1), \\ &f \in L^1(\mathbb{R}^+;\, W^{-2,2}(0,1)) \cap L^2(\mathbb{R}^+;\, W^{-2,2}(0,1)). \end{aligned} \tag{32}$$

In view of (28), (31) and (32) it is clear that (P_2) is of the abstract form (1) in $H = L^2(0,\, 1)$. By applying Theorem 3 in conjunction with Remark 4, we obtain

THEOREM 6. *Let conditions* (29) *and* (32) *be satisfied. Then the problem* (P_2) *has a unique solution* u *such that*

$$\begin{aligned} &u \in L^\infty(\mathbb{R}^+;\, W_0^{2,2}(0,1)) \cap UC(\mathbb{R}^+,\, W_0^{2,2}(0,1)), \\ &u_t \in L^2(\mathbb{R}^+,\, W_0^{2,2}(0,1)), \\ &(g(u_x))_x \in L^\infty_{loc}(\mathbb{R}^+;\, L^2(0,1)) \cap L^2(\mathbb{R}^+;\, W^{-2,2}(0,1)), \\ &\lim_{t\to\infty} u(t,\, \cdot) = 0 \;\; in \;\; W_0^{1,2}(0,1). \end{aligned}$$

The convergence of u on $W_0^{1,2}(0,\, 1)$ (as compared to $L^2(0,\, 1)$, only) is a consequence of (20), (21) and (30).

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