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GENERALIZED NONLINEAR QUASIVARIATIONAL INCLUSIONS FOR FUZZY MAPPINGS

NAN-JING HUANG AND YEOL JE CHO

1. Introduction

Variational inequalities, introduced by Hartman and Stampacchia [13] in the early sixties, are a very powerful tool of the current mathematical technology. These have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control theory, nonlinear programming, economics and transportation equilibrium and engineering sciences, etc. Quasivariational inequalities are a generalized form of variational inequalities in which the constraint set depends on the solution These were introduced and studied by Bensoussan, Goursat and Lions [3]. For further details, we refer to [1, 2, 4, 5, 22].

In 1991, Chang and Huang [7, 8] introduced and studied some new class of complementarity problems and variational inequalities for set-valued mappings with compact values in Hilbert spaces. In 1994, Hassouni and Moudafi [12] studied a new class of variational inclusions, which included many variational and quasivariational inequalities considered by Noor [24-26], Isac [19], Siddiqi and Ansari [29, 30] as special cases.

In 1989, Chang and Zhu [11] were first to introduce and study a class of variational inequalities for fuzzy mappings. Recently, several kinds of variational inequalities and complementarity problems for fuzzy mappings were considered by Chang [6], Chang and Huang [9, 10], Huang [15-17], Noor [27] and Lee et al [20, 21]. These works may lead to new and significant results in these areas [28].

The main purpose of this paper is to introduce and study a new class of generalized nonlinear quasivariational inclusions for fuzzy mappings

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which includes some known classes as special cases. We also construct some new iterative algorithms and discuss the existence of solutions for this class of generalized nonlinear quasivariational inclusions and the convergence of iterative sequences generated by these algorithms. Our results extend and improve some known results in this field.

2. Preliminaries

Let H be a real Hilbert space endowed with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{F}(H)$ be a collection of all fuzzy sets in H. A mapping F from H into $\mathcal{F}(H)$ is called a fuzzy mapping on H. If F is a fuzzy mapping on H, then $F(x)$(denote it by F_x in the sequel) is a fuzzy set on H and $F_x(y)$ is the membership function of y in F_x.

Let $M \in \mathcal{F}(H)$ and $q \in [0,1]$. Then the set

$$(M)_q = \{x \in H : \; M(x) \geq q\}$$

is called a $q-$cut set of M.

Let $T, A : H \to \mathcal{F}(H)$ be two fuzzy mappings satisfying the following condition (I):

(I) There exist two mappings $a, b : H \to [0,1]$ such that for all $x \in H$ the sets $(T_x)_{a(x)} \in CB(H)$ and $(A_x)_{b(x)} \in CB(H)$, where $CB(H)$ denotes the family of all nonempty bounded closed subsets of H.

By using the fuzzy mappings T and A, we can define two set-valued mappings $\widetilde{T}$ and $\widetilde{A}$ as follows:

$$\widetilde{T} : H \to CB(H), \;\; x \longmapsto (T_x)_{a(x)},$$
$$\widetilde{A} : H \to CB(H), \;\; x \longmapsto (A_x)_{b(x)},$$

respectively. In the sequel, $\widetilde{T}$ and $\widetilde{A}$ are called the set-valued mappings induced by the fuzzy mappings T and A, respetively.

Let $a, b : H \to [0,1]$ be mappings, $T, A : H \to \mathcal{F}(H)$ be fuzzy mappings and f, p, $g : H \to H$ be single-valued mappings. Suppose $\varphi : H \times H \to R \cup \{+\infty\}$ be a function such that for each fixed $y \in H$, $\varphi(\cdot, y) : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $Img \bigcap dom(\partial\varphi(\cdot, y)) \neq \emptyset$ for each $y \in H$, where

$\partial\varphi(\cdot, y)$ denotes the subdifferential of function $\varphi(\cdot, y)$. We consider the following problem:

Find $u, w, y \in H$ such that

$$
(2.1)\qquad \begin{cases} T_u(w) \geq a(u),\ A_u(y) \geq b(u),\ g(u) \bigcap dom(\partial\varphi(\cdot, u)) \neq \emptyset, \\ \langle f(w) - p(y), v - g(u)\rangle \geq \varphi(g(u), u) - \varphi(v, u) \end{cases}
$$

for all $v \in H$. This problem is called a generalized nonlinear quasivariational inclusion for fuzzy mappings.

If $\varphi(x, y) = \varphi(x)$ for all $y \in H$, then the problem (2.1) is equivalent to finding $u, w, y \in H$ such that

$$
(2.2)\qquad \begin{cases} T_u(w) \geq a(u),\ A_u(y) \geq b(u),\ g(u) \bigcap dom(\partial\varphi) \neq \emptyset, \\ \langle f(w) - p(y), v - g(u)\rangle \geq \varphi(g(u)) - \varphi(v) \end{cases}
$$

for all $v \in H$. This problem is called a generalized nonlinear variational inclusion for fuzzy mappings, which was considered by Huang[17].

If $T, A : H \to 2^H$(where 2^H denotes all the nonempty subsets of H) are classical set-valued mappings, then the problem (2.1) is equivalent to finding $u, w, y \in H$ such that

$$
(2.3)\qquad \begin{cases} w \in Tu,\ y \in Au,\ g(u) \bigcap dom(\partial\varphi(\cdot, u)) \neq \emptyset, \\ \langle f(w) - p(y), v - g(u)\rangle \geq \varphi(g(u), u) - \varphi(v, u) \end{cases}
$$

for all $v \in H$, which is called a generalized nonlinear quasivariational inclusion for set-valued mappings and the problem (2.2) is equivalent to finding $u, w, y \in H$ such that

$$
(2.4)\qquad \begin{cases} w \in Tu,\ y \in Au,\ g(u) \bigcap dom(\partial\varphi) \neq \emptyset, \\ \langle f(w) - p(y), v - g(u)\rangle \geq \varphi(g(u)) - \varphi(v) \end{cases}
$$

for all $v \in H$, which is called a generalized nonlinear variational inclusion for set-valued mappings, which was considered by Huang [14].

REMARK 2.1 For an appropriate and suitable choice of the mappings f, p, g, T, A and the functions a, b, φ, a number of known classes of variational inequalities and quasi-variational inequalities can be obtained as special cases studied previously by many authors in [5, 7-10, 12, 14-19, 22, 24-26, 29-31].

3. Iterative algorithms

First, let us give the following lemma.

LEMMA 3.1. *u, w and y are a solution of the problem (2.1) if and only if $w \in \widetilde{T}u, y \in \widetilde{A}u$ such that*

$$g(u) = J_\alpha^{\varphi(\cdot,u)}(g(u) - \alpha(f(w) - p(y))), \tag{3.1}$$

where $\alpha > 0$ is a constant and $J_\alpha^{\varphi(\cdot,u)} = (I + \alpha\partial\varphi(\cdot,u))^{-1}$ is the so-called proximal mapping on H.

Proof. From the definition of $J_\alpha^{\varphi(\cdot,u)}$, we have

$$g(u) - \alpha(f(w) - p(y)) \in g(u) + \alpha\partial\varphi(\cdot,u)(g(u))$$

and hence

$$p(y) - f(w) \in \partial\varphi(\cdot,u)(g(u)).$$

From the definition of $\partial\varphi(\cdot,u)$, it follows that

$$\varphi(v,u) \geq \varphi(g(u),u) + \langle p(y) - f(w), v - g(u)\rangle$$

for all $v \in H$. Thus u, w and y are a solution of (2.1). This completes the proof.

To obtain an approximate solution of (2.1), we can apply a successive approximation method to the problem of solving

$$u \in F(u) \tag{3.2}$$

where

$$F(u) = u - g(u) + J_\alpha^{\varphi(\cdot,u)}(g(u) - \alpha(f(Tu) - p(Au))).$$

Based on (3.1) and (3.2), we proceed our algorithms.

Suppose that $T, A : H \to \mathcal{F}(H)$ satisfy the conditon (I). Let $\widetilde{T}, \widetilde{A} : H \to CB(H)$ be set-valued mappings induced by T, A, respectively. For given $u_0 \in H$, let $w_0 \in \widetilde{T}u_0$, $y_0 \in \widetilde{A}u_0$ and

$$u_1 = u_0 - g(u_0) + J_\alpha^{\varphi(\cdot,u_0)}(g(u_0) - \alpha(f(w_0) - p(y_0))).$$

By Nadler [23] there exists $w_1 \in \widetilde{T}u_1$ and $y_1 \in \widetilde{A}u_1$ such that

$$\|w_1 - w_0\| \leq (1+1)\widehat{\mathbf{H}}(\widetilde{T}u_1, \widetilde{T}u_0),$$

$$\|y_1 - y_0\| \leq (1+1)\widehat{\mathbf{H}}(\widetilde{A}u_1, \widetilde{A}u_0),$$

where $\widehat{\mathbf{H}}$ is the Hausdorff metric on $CB(H)$. By induction, we can obtain our algorithm as follows:

ALGORITHM 3.1. *Suppose that $T, A : H \to \mathcal{F}(H)$ satisfy the conditon* (I). *Let $\widetilde{T}, \widetilde{A} : H \to CB(H)$ be set-valued mappings induced by T, A, respectively, and f, p, $g : H \to H$ be mappings. For given $u_0 \in H$, we can get an algorithm for* (2.1) *as follows:*

$$
(3.3) \qquad \begin{cases} u_{n+1} = u_n - g(u_n) + J_\alpha^{\varphi(\cdot, u_n)}(g(u_n) - \alpha(f(w_n) - p(y_n))), \\ \|w_{n+1} - w_n\| \le (1 + (1+n)^{-1})\widehat{\mathbf{H}}(\widetilde{T}u_{n+1}, \widetilde{T}u_n), \\ \|y_{n+1} - y_n\| \le (1 + (1+n)^{-1})\widehat{\mathbf{H}}(\widetilde{A}u_{n+1}, \widetilde{A}u_n) \\ w_n \in \widetilde{T}u_n, \quad y_n \in \widetilde{A}u_n \end{cases}
$$

for $n = 0, 1, 2, \cdots$.

Similarly, we have the following algorithms:

ALGORITHM 3.2. *Suppose that $T, A : H \to \mathcal{F}(H)$ satisfy the conditon* (I). *Let $\widetilde{T}, \widetilde{A} : H \to CB(H)$ be set-valued mappings induced by T, A, respectively, and f, p, $g : H \to H$ be mappings. For given $u_0 \in H$, we can get an algorithm for* (2.2) *as follows:*

$$
(3.4) \qquad \begin{cases} u_{n+1} = u_n - g(u_n) + J_\alpha^{\varphi}(g(u_n) - \alpha(f(w_n) - p(y_n))), \\ \|w_{n+1} - w_n\| \le (1 + (1+n)^{-1})\widehat{\mathbf{H}}(\widetilde{T}u_{n+1}, \widetilde{T}u_n), \\ \|y_{n+1} - y_n\| \le (1 + (1+n)^{-1})\widehat{\mathbf{H}}(\widetilde{A}u_{n+1}, \widetilde{A}u_n) \\ w_n \in \widetilde{T}u_n, \quad y_n \in \widetilde{A}u_n \end{cases}
$$

for $n = 0, 1, 2, \cdots$.

ALGORITHM 3.3. *Let $T, A : H \to CB(H)$ be set-valued mappings and f, p, $g : H \to H$ be mappings. For given $u_0 \in H$, we can get an algorithm for* (2.3) *as follows:*

$$
(3.5) \qquad \begin{cases} u_{n+1} = u_n - g(u_n) + J_\alpha^{\varphi(\cdot, u_n)}(g(u_n) - \alpha(f(w_n) - p(y_n))), \\ \|w_{n+1} - w_n\| \le (1 + (1+n)^{-1})\widehat{\mathbf{H}}(Tu_{n+1}, Tu_n), \\ \|y_{n+1} - y_n\| \le (1 + (1+n)^{-1})\widehat{\mathbf{H}}(Au_{n+1}, Au_n) \\ w_n \in Tu_n, \quad y_n \in Au_n \end{cases}
$$

for $n = 0, 1, 2, \cdots$.

ALGORITHM 3.4. *Let $T, A : H \to CB(H)$ be set-valued mappings and f, p, $g : H \to H$ be mappings. For given $u_0 \in H$, we can get an algorithm for (2.4) as follows:*

$$\tag{3.6} \begin{cases} u_{n+1} = u_n - g(u_n) + J_\alpha^\varphi(g(u_n) - \alpha(f(w_n) - p(y_n))), \\ \|w_{n+1} - w_n\| \le (1 + (1+n)^{-1})\,\widehat{\mathbf{H}}(Tu_{n+1}, Tu_n), \\ \|y_{n+1} - y_n\| \le (1 + (1+n)^{-1})\,\widehat{\mathbf{H}}(Au_{n+1}, Au_n) \\ w_n \in Tu_n, \quad y_n \in Au_n \end{cases}$$

for $n = 0, 1, 2, \cdots$.

REMARK 3.1. Algorithms 3.1~3.4 include several known algorithms of [5, 7, 8, 10, 12, 14, 16-19, 24-26, 29-31] as special cases.

4. Existence and convergence

DEFINITION 4.1. A mapping $g : H \to H$ is said to be

(1) *strongly monotone* if there exists some $\delta > 0$ such that

$$\langle g(u_1) - g(u_2), u_1 - u_2 \rangle \ge \delta \|u_1 - u_2\|^2$$

for $u_i \in H$, $i = 1, 2$,

(2) *Lipschitz continuous* if there exists some $\sigma > 0$ such that

$$\|g(u_1) - g(u_2)\| \le \sigma \|u_1 - u_2\|$$

for $u_i \in H$, $i = 1, 2$.

DEFINITION 4.2. A set-valued mapping $T : H \to 2^H$ is said to be

(1) *strongly monotone* with respect to a mapping $f : H \to H$ if there exists some $\beta > 0$ such that

$$\langle f(w_1) - f(w_2), u_1 - u_2 \rangle \ge \beta \|u_1 - u_2\|^2$$

for $u_i \in H$ and $w_i \in Tu_i$, $i = 1, 2$,

(2) $\widehat{\mathbf{H}}$-*Lipschitz continuous* if there exists some $\gamma > 0$ such that

$$\widehat{\mathbf{H}}(Tu_1, Tu_2) \le \gamma \|u_1 - u_2\|$$

for $u_i \in H$, $i = 1, 2$.

THEOREM 4.1. *Let $g : H \to H$ be strongly monotone and Lipschitz continuous, $f, p : H \to H$ be Lipschitz continuous, $T, A : H \to \mathcal{F}(H)$ be fuzzy mappings satisfying the condition (I). Let $\widetilde{T}, \widetilde{A} : H \to CB(H)$ be set-valued mappings induced by T, A, respectively, and $\widetilde{T}, \widetilde{A}$ be $\widehat{\mathbf{H}}$-Lipschitz continuous and $\widetilde{T}$ be strongly monotone with respect to f. Suppose there exists a constant $\xi > 0$ such that for each $x, y, z \in H$,*

$$\|J_\alpha^{\partial\varphi(\cdot,x)}(z) - J_\alpha^{\partial\varphi(\cdot,y)}(z)\| \le \xi\|x - y\|.$$

If the following conditions hold:

$$(4.1) \qquad \left|\alpha - \frac{\beta + \epsilon\mu(k-1)}{\eta^2\gamma^2 - \epsilon^2\mu^2}\right| < \frac{\sqrt{(\beta + (k-1)\epsilon\mu)^2 - (\gamma^2 - \epsilon^2\mu^2)k(2-k)}}{\eta^2\gamma^2 - \epsilon^2\mu^2},$$

$$(4.2) \qquad \beta > (1-k)\epsilon\mu + \sqrt{(\eta^2\gamma^2 - \epsilon^2\mu^2)k(2-k)}, \quad \eta\gamma > \epsilon\mu,$$

$$(4.3) \qquad \alpha\mu\epsilon < 1 - k, \quad k = \xi + 2\sqrt{1 - 2\delta + \sigma^2}, \quad k < 1,$$

where β and δ are constants of strong monotonicity of $\widetilde{T}$ and g, respectively, γ and μ are $\widehat{\mathbf{H}}$-Lipschitz constants of $\widetilde{T}$ and $\widetilde{A}$, respectively, σ, η and ϵ are Lipschitz constants of g, f and p, respectively, then there exist u, w, $y \in H$ such that (2.1) is satisfied. Moreover, as $n \to \infty$,

$$u_n \to u, \quad w_n \to w, \quad y_n \to y,$$

where $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are defined in Algorithm 3.1.

Proof. From (3.3), it follows that

$$\begin{aligned}\|u_{n+1} - u_n\| = \|&u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \\ &+ J_\alpha^{\varphi(\cdot,u_n)}(h(u_n)) - J_\alpha^{\varphi(\cdot,u_{n-1})}(h(u_{n-1}))\|,\end{aligned}$$

where

$$h(u_n) = g(u_n) - \alpha(f(w_n) - p(y_n))$$

Also we have

$$\begin{aligned}
&\|J_\alpha^{\varphi(\cdot,u_n)}(h(u_n)) - J_\alpha^{\varphi(\cdot,u_{n-1})}(h(u_{n-1}))\| \\
&\le \|J_\alpha^{\varphi(\cdot,u_n)}(h(u_{n-1})) - J_\alpha^{\varphi(\cdot,u_{n-1})}(h(u_{n-1}))\| \\
&\quad + \|J_\alpha^{\varphi(\cdot,u_n)}(h(u_n)) - J_\alpha^{\varphi(\cdot,u_n)}(h(u_{n-1}))\| \\
&\le \xi\|u_n - u_{n-1}\| + \|h(u_n) - h(u_{n-1})\| \\
&\le \xi\|u_n - u_{n-1}\| + \|u_n - u_{n-1} - \alpha(f(w_n) - f(w_{n-1}))\| \\
&\quad + \|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| + \alpha\|p(y_n) - p(y_{n-1})\|.
\end{aligned}$$

That is

$$\begin{aligned}
\|u_{n+1} - u_n\| \le \xi&\|u_n - u_{n-1}\| \\
&+ 2\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| \\
&+ \|u_n - u_{n-1} - \alpha(f(w_n) - f(w_{n-1}))\| \\
&+ \alpha\|p(y_n) - p(y_{n-1})\|.
\end{aligned} \tag{4.4}$$

By Lipschitz continuity and strong monotonicity of g, we obtain

$$\begin{aligned}
&\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\|^2 \\
&\qquad \le (1 - 2\delta + \sigma^2)\|u_n - u_{n-1}\|^2.
\end{aligned} \tag{4.5}$$

Also from $\widehat{\mathbf{H}}$-Lipschitz continuity and strong monotonicity of $\widetilde{T}$ and Lipschitz continuity of f, we have

$$\begin{aligned}
&\|u_n - u_{n-1} - \alpha(f(w_n) - f(w_{n-1}))\|^2 \\
&\le (1 - 2\beta\alpha + \alpha^2\eta^2(1+n^{-1})^2\gamma^2)\|u_n - u_{n-1}\|^2.
\end{aligned} \tag{4.6}$$

By $\widehat{\mathbf{H}}$-Lipschitz continuity of $\widetilde{A}$, Lipschitz continuity of p and (3.3), it follows that

$$\alpha\|p(y_n) - p(y_{n-1})\| \le \alpha\epsilon(1+n^{-1})\mu\|u_n - u_{n-1}\|. \tag{4.7}$$

Thus, by combining (4.4)~(4.7), we have

$$\|u_{n+1} - u_n\| \le \theta_n\|u_n - u_{n-1}\|,$$

where

$$\theta_n := \xi + 2\sqrt{1-2\delta+\sigma^2} + \sqrt{1-2\beta\alpha+\alpha^2\eta^2(1+n^{-1})^2\gamma^2} + \alpha\epsilon(1+n^{-1})\mu.$$

Letting

$$\theta := \xi + 2\sqrt{1-2\delta+\sigma^2} + \sqrt{1-2\beta\alpha+\alpha^2\eta^2\gamma^2} + \alpha\epsilon\mu,$$

we know that $\theta_n \searrow \theta$. It follows from (4.1)~(4.3) that $\theta < 1$. Hence $\theta_n < 1$ for n sufficiently large. Therefore, $\{u_n\}$ is a Cauchy sequence in H and we can suppose that $u_n \to u \in H$.

Now we prove that $w_n \to w \in \widetilde{T}u$ and $y_n \to y \in \widetilde{A}u$ as $n \to \infty$. In fact, it follows from Algorithm 3.1 that

$$\begin{aligned} \|w_n - w_{n-1}\| &\le (1+n^{-1})\gamma\|u_n - u_{n-1}\|, \\ \|y_n - y_{n-1}\| &\le (1+n^{-1})\mu\|u_n - u_{n-1}\|, \end{aligned}$$

i.e., $\{w_n\}$ and $\{y_n\}$ are Cauchy sequences in H. Let $w_n \to w$ and $y_n \to y$ as $n \to \infty$. Further, we have

$$\begin{aligned} \varrho(w,\widetilde{T}u) &= \inf\{\|w - z\| : z \in \widetilde{T}u\} \\ &\le \|w - w_n\| + \varrho(w_n, \widetilde{T}u) \\ &\le \|w - w_n\| + \widehat{\mathbf{H}}(\widetilde{T}u_n, \widetilde{T}u) \\ &\le \|w - w_n\| + \gamma\|u_n - u\| \to 0. \end{aligned}$$

Hence, $w \in \widetilde{T}u$. Similarly, $y \in \widetilde{A}u$. This completes the proof.

From Theorem 4.1, we have the following results:

THEOREM 4.2. *Let $g : H \to H$ be strongly monotone and Lipschitz continuous, $f, p : H \to H$ be Lipschitz continuous, $T, A : H \to \mathcal{F}(H)$ be fuzzy mappings satisfying the condition* (I). *Let $\widetilde{T}, \widetilde{A} : H \to CB(H)$ be set-valued mappings induced by T, A, respectively, and $\widetilde{T}, \widetilde{A}$ be $\widehat{\mathbf{H}}$-Lipschitz continuous and $\widetilde{T}$ be strongly monotone with respect to f. If* *the following conditions hold:*

$$\begin{aligned}
(4.8)\qquad & \left|\alpha - \frac{\beta + \epsilon\mu(k-1)}{\eta^2\gamma^2 - \epsilon^2\mu^2}\right| \\
& < \frac{\sqrt{(\beta + (k-1)\epsilon\mu)^2 - (\gamma^2 - \epsilon^2\mu^2)k(2-k)}}{\eta^2\gamma^2 - \epsilon^2\mu^2}, \\
(4.9)\qquad & \beta > (1-k)\epsilon\mu + \sqrt{(\eta^2\gamma^2 - \epsilon^2\mu^2)k(2-k)}, \\
& \eta\gamma > \epsilon\mu, \\
(4.10)\qquad & \alpha\mu\epsilon < 1-k, \quad k = 2\sqrt{1 - 2\delta + \sigma^2}, \quad k < 1,
\end{aligned}$$

where β and δ are constants of strong monotonicity of $\widetilde{T}$ and g, respectively, γ and μ are $\widehat{\mathbf{H}}$-Lipschitz constants of $\widetilde{T}$ and $\widetilde{A}$, respectively, σ, η and ϵ are Lipschitz constants of g, f and p, respectively, then there exist u, w, $y \in H$ such that (2.2) is satisfied. Moreover, as $n \to \infty$,

$$u_n \to u, \quad w_n \to w, \quad y_n \to y,$$

where $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are defined in Algorithm 3.2.

THEOREM 4.3. *Let $g : H \to H$ be strongly monotone and Lipschitz continuous, $f, p : H \to H$ be Lipschitz continuous, $T, A : H \to CB(H)$ be $\widehat{\mathbf{H}}$-Lipschitz continuous and T be strongly monotone with respect to f. Suppose there exists a constant $\xi > 0$ such that for each $x, y, z \in H$,*

$$\|J_\alpha^{\partial\varphi(\ ,x)}(z) - J_\alpha^{\partial\varphi(\ ,y)}(z)\| \le \xi\|x - y\|.$$

If the conditions (4.1)~(4.3) of Theorem 4.1 hold, where β and δ are constants of strong monotonicity of T and g respectively, γ and μ are $\widehat{\mathbf{H}}$-Lipschitz constants of T and A, respectively, σ, η and ϵ are Lipschitz constants of g, f and p, respectively, then there exist u, w, $y \in H$ such that (2.3) is satisfied. Moreover, as $n \to \infty$,

$$u_n \to u, \quad w_n \to w, \quad y_n \to y,$$

where $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are defined in Algorithm 3.3.

THEOREM 4.4. *Let $g : H \to H$ be strongly monotone and Lipschitz continuous, $f, p : H \to H$ be Lipschitz continuous, $T, A : H \to CB(H)$ be $\widehat{\mathbf{H}}$-Lipschitz continuous and T be strongly monotone with respect to f. If the conditions (4.8)~(4.10) of Theorem 4.2 hold, where β and δ are constants of strong monotonicity of T and g respectively, γ and μ are $\widehat{\mathbf{H}}$-Lipschitz constants of T and A, respectively, σ, η and ϵ are Lipschitz constants of g, f and p, respectively, then there exist u, w, $y \in H$ such that (2.4) is satisfied. Moreover, as $n \to \infty$,*

$$u_n \to u, \quad w_n \to w, \quad y_n \to y,$$

where $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are defined in Algorithm 3.4.

REMARK 4.1 For a suitable choice of the operators g, T, A, f, p and the function φ, we can obtain several known results [5, 7, 8, 10, 14, 16-19, 24-26, 29-31] as special cases of our main results.

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Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064
People's Republic of China

Department of Mathematics
Gyeongsang National University
Chinju 660-701, Korea