LINEAR OPERATORS THAT STRONGLY PRESERVES THE SIGN-CENTRAL MATRICES

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1. Introduction

Let $M_{m,n}$ be the set of all $m \times n$ real matrices. For a matrix $A = [a_{ij}] \in M_{m,n}$, the sign of $a_{ij}$ is defined by

$$
\text{sgn } a_{ij} = \begin{cases} 
0 & \text{if } a_{ij} = 0, \\
+1 & \text{if } a_{ij} > 0, \\
-1 & \text{if } a_{ij} < 0.
\end{cases}
$$

The sign pattern of $A$, $A$ is the $m \times n \{0, 1, -1\}$-matrix

$$
A = [\text{sgn } a_{ij}] = \text{sgn } A
$$

obtained from $A$ by replacing each entry with its sign. If $A$ and $B$ are sign pattern matrices with same size, then $A + B$ exists, that is, $A + B$ is qualitatively defined if $a_{ij}b_{ij} \neq -1$ for all $i$ and $j$, $1 \leq i \leq m$, $1 \leq j \leq n$. If $a_{ij}b_{ij} = -1$, then $a_{ij} + b_{ij}$ is $0$, $-1$ or $+1$. So, we cannot determine the sign of the entry $a_{ij} + b_{ij}$. That is, $A + B$ is undefined.

Let $Q(B)$ be the qualitative class of $B$ such that the sign pattern of any matrix in $Q(B)$ equals to the sign pattern of $B = [b_{ij}]$, i.e.,

$$
Q(B) = \{ A = [a_{ij}] \in M_{m,n} \mid b_{ij} = \text{sgn } a_{ij} \text{ for all } i, j \}.
$$

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The column vectors $a^{(1)}, a^{(2)}, \ldots, a^{(n)}$ of a matrix $A$ in $Q(B)$ determine a convex polytope

$$\mathcal{CP}(A) = \left\{ \sum_{i=1}^{n} c_i a^{(i)} \mid \sum_{i=1}^{n} c_i = 1, \ c_i \geq 0 \ (1 \leq i \leq n) \right\}.$$ 

We define the matrix $A$ to be central provided that the origin $(0, \ldots, 0)^T$ is contained in the polytope $\mathcal{CP}(A)$. The matrix $A \in Q(B)$ is called sign-central provided that each matrix in $Q(B)$ is central. That is, a matrix $A \in Q(B)$ is a sign-central matrix if and only if each matrix in $Q(B)$ is sign-central. For example, the $m \times (m+1)$ matrix with exactly one 1 and exactly one $-1$ in each row defined by

$$F_m = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$$

is easily seen to be a sign-central matrix. The matrix

$$E_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

is also a sign-central matrix. More generally, for each positive integer $m$, the $m \times 2^m$ matrix $E_m$ such that each $m$-tuple of 1's and $-1$'s is a column of $E_m$ is a sign-central matrix.

A diagonal matrix $D \neq 0$ each of whose diagonal entries equals 0, 1, or $-1$ is called a signing. A signing with no 0's on its main diagonal is called a strict signing. Let $A$ be an $m \times n$ matrix, and let $P$ and $Q$ be permutation matrices of order $m$ and $n$, respectively. Let $D$ be a strict signing. Then it follows from the definition that $A$ is a sign-central matrix if and only if $PDAQ$ is a sign-central matrix. That is, a sign-central matrix is permutation invariant.

In [1], the sign-central matrix was characterized as following;
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**Theorem 1.1.** [Ando and Brualdi, 1, Theorem 2.1]. Let $A$ be an $m \times n \{0, 1, -1\}$-matrix. Then the following are equivalent:

(i) $A$ is a sign-central matrix.

(ii) For every strict signing $D$ of order $m$, the matrix $DA$ has a nonnegative column vector.

(iii) For every strict signing $D$ of order $m$, the matrix $DA$ has a nonpositive column vector.

(iv) Each set of the blocker $b(A)$ contains as a subset at least one of the sets $\{1, \bar{1}\}, \ldots, \{m, \bar{m}\}$.

(v) There do not exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} A_1 & X_1 \\ X_2 & A_2 \end{bmatrix}$$

where $A_1$ is a possibly vacuous matrix with at least one 1 in each column and $A_2$ is a possibly vacuous matrix with at least one $-1$ in each column.

In the above theorem, (ii) and (iii) are clearly equivalent. By the above theorem, if a matrix $A$ has a zero column, then $A$ is a sign-central matrix. And, if a matrix $A$ is a sign-central matrix with no zero column vector, then the matrix $DA$ have both a nonnegative column vector and a nonpositive column vector for every strict signing $D$ of order $m$.

Let $T : M_{m,n} \rightarrow M_{m,n}$ be a linear operator. We say $T$ preserves the subset $\mathcal{K}$ of $M_{m,n}$ if $T$ maps each matrix in the set $\mathcal{K}$ to a matrix in $\mathcal{K}$. We say $T$ strongly preserves the subset $\mathcal{K}$ of $M_{m,n}$ if $T$ preserves both $\mathcal{K}$ and $M_{m,n} \setminus \mathcal{K}$, the complement of $\mathcal{K}$ in $M_{m,n}$.

Let $E_{ij}$ denote the $(0,1)$-matrix whose only nonzero entry is in the $(i, j)$ position. A cell is a scalar multiple of $E_{ij}$ for some $(i, j)$, so that the set of cells is the set

$$\{\alpha_{ij}E_{ij} | \alpha_{ij} \in \mathbb{R}, \text{ the reals, } 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}.$$ 

Let $\mathcal{R}_i = \sum_{j=1}^{n} E_{ij}$ and $\mathcal{C}_j = \sum_{i=1}^{m} E_{ij}$. That is, $\mathcal{R}_i$ is the matrix whose $i$th row is all ones and all other entries are zero. Let $J$ be an
$m \times n$ matrix whose entries are all ones and let $I_m$ be the identity matrix of order $m$. Clearly, $\mathcal{R}_i$, $J$ and $I_m$ are not sign-central matrices.

We denote the Hadamard product of $A = [a_{ij}]$ and $B = [b_{ij}]$ in $M_{m,n}$ by $A \odot B$, i.e., $A \odot B = [a_{ij}b_{ij}]$.

The term rank is the minimum number, $t(A)$, of lines(columns or rows) which contain all non-zero entries of $A$.

In [3], Beasley and Pullman characterized the linear operators that preserve term rank 1 as following:

**Theorem 1.2.** [Beasley and Pullman, 3, Corollary 3.1.2]. Suppose that $T$ is a nonsingular linear operator on $M_{m,n}$. The linear operator $T$ preserves the set of matrices of term rank 1 if and only if $T$ is one of or a composition of some of the following operators:

(i) $X \rightarrow X^T$ if $m = n$.

(ii) $X \rightarrow PXQ$ for some fixed but arbitrary permutation matrices $P$ and $Q$ of order $m$ and $n$, respectively.

(iii) $X \rightarrow X \odot M$ for some fixed but arbitrary matrix $M$ in $M_{m,n}$ with no zero entries.

In this paper, we characterize linear operators $T$ preserve the set of sign-central matrices using the above theorem.

**2. STRONG PRESERVERS OF SIGN-CENTRAL MATRICES**

In this section we will investigate the linear operators that strongly preserve sign-central matrices. We will prove that if $T$ is a linear operator that strongly preserves the sign-central matrices then

$$T(X) = PDXQ \quad \text{for all} \quad X \in M_{m,n},$$

or

$$T'(X) = PDX^TQ \quad \text{for} \quad m = n \quad \text{and} \quad X := X^T,$$

where $P$ and $Q$ are permutation matrices of order $m$ and $n$, respectively, and $D$ is a strict signing of order $m$. 

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Throughout this section, let $T$ be a linear operator that strongly preserves sign-central matrices.

**Lemma 2.1.** Let $X = [x^{(1)} \cdots x^{(n)}]$ be a nonzero sign-central matrix. Then there is a sign-central matrix $Y = [y^{(1)} \cdots y^{(n)}]$ such that $X + Y$ is not a sign-central matrix.

**Proof.** First, suppose that the matrix $X$ has no zero column vector. Then, for any strict signing $D$, $DX$ have both a nonnegative column vector and a nonpositive column vector. For some fixed $D$, without loss of generality, let $DX^{(1)}, \ldots, DX^{(i)}$ be nonnegative vectors and let $DX^{(i+1)}, \ldots, DX^{(i+j)}$ be nonpositive vectors, $i \geq 1$, $j \geq 1$, $i + j \leq n$. For some $p \geq 2$, let

$$
y^{(1)} = -px^{(1)}, \ldots, y^{(i)} = -px^{(i)},$$

$$y^{(i+1)} = \ldots = y^{(i+j)} = 0,$$

$$y^{(i+j+1)} = x^{(i+j+1)}, \ldots, y^{(n)} = x^{(n)}.$$

Then the matrix $Y$ is a sign-central and the matrix $X + Y$ has no zero column. Since $D(X + Y)$ does not have a nonnegative column vector, $X + Y$ is not a sign-central matrix.

Now, suppose that the matrix $X$ have zero columns. Without loss of generality, let $x^{(1)} = \ldots = x^{(i)} = 0$ and $x^{(i+1)}, \ldots, x^{(n)}$ are nonzero vectors, $1 \leq i \leq n - 1$. First, assume that $DX^{(i+1)}, \ldots, DX^{(n)}$ are not nonpositive (respectively, nonnegative) vectors for some strict signing $D$. Then, let

$$y^{(1)} = \ldots = y^{(i)} = x^{(i+1)}, y^{(i+1)} = \ldots = y^{(n)} = 0.$$

Then the matrix $Y$ is a sign-central matrix and the matrix $X + Y$ has no zero column. Since $D(X + Y)$ does not have a nonpositive (respectively, nonnegative) vector, $X + Y$ is not a sign-central matrix. Next, assume that there are nonpositive (respectively, nonnegative) vectors and there is no nonnegative (respectively, nonpositive) vector among the vectors $DX^{(i+1)}, \ldots, DX^{(n)}$. Without loss of generality, we may assume
that \( Dx^{(i+1)}, \ldots, Dx^{(i+j)} \) are nonpositive (respectively, nonnegative) vectors, \( 1 \leq j \leq n - i \). Let
\[
y^{(1)} = \ldots = y^{(i)} = x^{(i+1)}, y^{(i+1)} = \ldots = y^{(n)} = 0.
\]

Then the matrix \( Y \) is a sign-central and \( X + Y \) has no zero column vector. Since \( D(X + Y) \) does not have a nonnegative (respectively, nonpositive) vector, the matrix \( X + Y \) is not a sign-central matrix. Finally, assume that the \( \{Dx^{(i+1)}, \ldots, Dx^{(n)}\} \) have both a nonnegative vector and a nonpositive vector. Without loss of generality, we may assume that \( Dx^{(i+1)}, \ldots, Dx^{(i+j)} \) are nonnegative vectors and \( Dx^{(i+j+1)}, \ldots, Dx^{(i+k)} \) are nonpositive vectors. Then, for some \( p \geq 2 \), let
\[
y^{(1)} = \ldots = y^{(i)} = x^{(i+j+1)}
y^{(i+1)} = -px^{(i+1)}, \ldots, y^{(i+j)} = -px^{(i+j)}
y^{(i+j+1)} = \ldots = y^{(i+k)} = y^{(i+k+1)} = \ldots = y^{(n)} = 0.
\]

Then, the matrix \( Y \) is a sign-central matrix and the matrix \( X + Y \) has no zero column vector. Since \( D(X + Y) \) does not have a nonnegative vector, the matrix \( X + Y \) is not a sign-central matrix.

Therefore, if \( X \) is a sign-central matrix then there is a sign-central matrix \( Y \) such that \( X + Y \) is not a sign-central matrix. ■

Lemma 2.2. \( T \) is a nonsingular linear operator.

Proof. Suppose \( T(X) = 0 \) for some \( X \neq 0 \). Since \( T \) is a strongly preserver, \( X \) is a sign-central matrix. So, there is a sign-central matrix \( Y \) such that \( X + Y \) is not a sign-central. Then
\[
T(X + Y) = T(X) + T(Y) = T(Y).
\]

This is a contradiction. Therefore, \( T \) is nonsingular ■

By above lemma, since \( T \) is a nonsingular and dimension of domain of \( T \) equals dimension of image of \( T \), the linear operator \( T \) is bijective on \( M_{m,n} \). And, an immediate consequence of the above lemmas is the following:
Strongly preserves the sign-central matrices

**Theorem 2.3.** The mapping $T$ is bijective on the set of cells.

For matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same order, write $A \leq B$ if $a_{ij} \leq b_{ij}$ for all $i$ and $j$.

**Lemma 2.4.** If $T(\mathcal{R}_i) = X$ for each $i$, then $X \geq 0$ or $X \leq 0$.

**Proof.** Suppose that $T(\mathcal{R}_i) = X_1 - X_2$ for $X_1, X_2 \geq 0$. Let

$$A = E_{i1} + \cdots + E_{ik} \text{ and } B = E_{ik+1} + \cdots + E_{in}$$

for some $k, 1 \leq k \leq n$. Since $T$ is nonsingular and bijective on the cells, without loss of generality, let $T(A) = X_1$ and $T(B) = -X_2$. Since $\mathcal{R}_i$ is not a sign-central matrix, $T(\mathcal{R}_i) = X_1 - X_2$ is not a sign-central matrix. So, the matrix $X_1 - X_2$ does not have a zero column vector and hence $X_1 + X_2$ does not have a zero column vector. Now, we consider a sign-central matrix $A - B$. Then, $T(A - B) = X_1 + X_2$. So, $X_1 + X_2$ is a sign-central matrix. But, the matrix $X_1 + X_2$ is not a sign-central matrix, since $X_1 + X_2$ does not have a zero column vector and $X_1 + X_2 \geq 0$. \[\square\]

We now show that $T$ preserves the term rank of any matrix. We say that a matrix $A$ is a row matrix if $\mathcal{R}_i \geq A$ for some $i$. Also, we say that a matrix $A$ is a column matrix if $\mathcal{C}_j \geq A$ for some $j$.

**Theorem 2.5.** If $T$ strongly preserves the set of sign-central matrices, then $T$ preserves the set of matrices of term rank $1$.

**Proof.** If $m = 1$ or $n = 1$, then, clearly, the mapping $T$ preserves the set of matrices of term rank $1$. If $n = 2$, then, for any real numbers $a, b, \ldots, c$, the matrices

$$\begin{bmatrix} a & 0 \\ b & 0 \\ \vdots & \vdots \\ c & 0 \end{bmatrix} \text{ and } \begin{bmatrix} a & -a \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

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are only sign-central matrices. Since $T$ strongly preserve the sign-central matrices and $T$ is nonsingular, $T$ is a term rank 1 preserving operator. Thus, we consider the case $m \geq 2$ and $n \geq 3$.

Suppose that $m \geq 2$ and $n \geq 3$. If $T$ preserve row matrices and column matrices, respectively, then $T$ is a term rank 1 preserver.

First, assume that $T(\mathcal{R}_i)$ is not a row matrix for some $i$. Let $A_{ip} = \sum_{j=1}^{p} E_{ij}$ and $B_{ip} = \sum_{j=p+1}^{n} E_{ij}$ for some $i, p$. Since $T$ is bijection on the set of cells, we may assume that $T(A_{ip}) = A_{kp}$ and $T(B_{ip}) = B_{rp}$ for $k \neq r$. Then, by lemma 2.4,

$$T(A_{ip} - B_{ip}) = A_{kp} - B_{rp}.$$  

Since the matrix $A_{ip} - B_{ip}$ is a sign-central matrix, the matrix $A_{kp} - B_{rp}$ is a sign-central matrix. Let $D = \text{diag}\{d_1, \ldots, d_m\}$ be a strict signing of order $m$ with $d_k = 1$ and $d_r = -1$. Then, the matrix $D(A_{kp} - B_{rp})$ does not have a nonpositive column vector, i.e., the matrix $A_{kp} - B_{rp}$ is not a sign-central matrix. Thus, $T$ preserves row matrices.

Now, suppose that $T(\mathcal{C}_j)$ is not a column matrix for some $j$. Let $G_{kp} = \sum_{i=1}^{k} E_{ip}$ and $H_{kp} = \sum_{i=k+1}^{m} E_{ip}$ for some $k, p$. Without loss of generality, we may assume that $T(G_{kp}) = G_{kl}$ and $T(H_{kp}) = H_{ks}$ for $l \neq s$. Then, there exist cells $C_1, \ldots, C_k$ such that $T(C_1 + \cdots + C_k) = G_{ks}$. Since $C_i \neq E_{ip}$ for $i = 1, \ldots, k$,

$$J \setminus (C_1 + \cdots + C_k + H_{kp})$$

is not a sign-central matrix. But,

$$T(J \setminus (C_1 + \cdots + C_k + H_{kp}))$$

have a zero column vector. That is, $T(J \setminus (C_1 + \cdots + C_k + H_{kp}))$ is a sign-central matrix. Thus, $T$ preserves column matrices.

Therefore, $T$ is a term rank 1 preserver. ■

We note that the sign-central matrices can be varied by the transpose, in general. That is, there is a sign-central matrix $A$ such that $A^T$ is not a sign-central matrix. For example, if

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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then the matrix $A$ is a sign-central matrix. But the matrix $A^T$ is not a sign-central matrix, since the identity matrix $I_3$ is a strict signing and the matrix $I_3 A^T$ does not have a nonpositive column vector. If

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

then $B$ is not a sign-central matrix. But $B^T$ is a sign-central matrix.

Note that if $m = n$ and $A$ is a symmetric sign-central matrix, then $A^T$ is a sign-central matrix. We have thus established the following lemma;

**Lemma 2.6.** Let $A$ does not have a zero vector in rows and columns. The transpose operator preserves a sign-central matrix $A$ if and only if $A$ is a symmetric sign-central matrix.

**Lemma 2.7.** Let $X \in M_{m,n}$. If $T$ strongly preserves sign-central matrices and if $T(X) = X \circ M$, then there exists a strict signing $D$ of order $m$ such that $M = DJ$; thus $T(X) = DX$.

**Proof.** Let $T(X) = X \circ M$. A real matrix $X$ is a sign-central matrix if and only if each matrix in $Q(X)$ is sign-central. So, without loss of generality, let $X$ be a $(0,1,-1)$-matrix and $M = [m_{ij}]$ be a $(1,-1)$-matrix. Since $T$ preserves term rank 1, $T(E_{ij} + E_{ik}) = \alpha E_{pq} + \beta E_{ps}$, for $j \neq k$ and $q \neq s$. Suppose that $\text{sgn} \alpha \neq \text{sgn} \beta$. Then, without loss of generality, let $T(E_{ij} + E_{ik}) = E_{pq} - E_{ps}$. So, $\mathcal{R}_i$ and $T(\mathcal{R}_i)$ are not sign-central matrices and $T(\mathcal{R}_i)$ has exactly one nonzero entry in each column. Since

$$T(\mathcal{R}_i) = T(E_{ij} + E_{ik} + \mathcal{R}_i \setminus (E_{ij} + E_{ik})) = E_{pq} - E_{ps} + T(\mathcal{R}_i \setminus (E_{ij} + E_{ik})),$$

for every strict signing $D$ of order $m$,

$$DT(\mathcal{R}_i) = D(E_{pq} - E_{ps} + T(\mathcal{R}_i \setminus (E_{ij} + E_{ik})))$$

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have both a nonnegative column vector and a nonpositive column vector. That is, \( T(\mathbf{A}_i) \) is a sign-central matrix. This is a contradiction. Thus \( \text{sgn} \alpha = \text{sgn} \beta \). Since, for a sign-central matrix \( X \), \( T(X) = X \circ M \),

\[
sgn m_{i1} = \cdots = sgn m_{in} \quad \text{for} \quad i = 1, \ldots, m.
\]

Let \( D = \text{diag}\{sgn m_{11}, sgn m_{21}, \ldots, sgn m_{m1}\} \). Then, \( X \circ M = DX \) and hence \( T(X) = DX \) for strict signing \( D \). \( \blacksquare \)

An immediate consequence of the above lemmas and theorems is the following;

**Theorem 2.8.** Let a linear operator \( T \) strongly preserves sign-central matrices. Then,

\[
T(X) = PDXQ \quad \text{for any} \quad X \in M_{m,n},
\]

or

\[
T(X) = PDX^TQ \quad \text{for} \quad m = n \quad \text{and} \quad X = X^T,
\]

where \( P \) and \( Q \) are permutation matrices of order \( m \) and \( n \), respectively, and \( D \) is a strict signing of order \( m \).

**Proof.** Since a real matrix \( X \) is a sign-central matrix if and only if each matrix in \( Q(X) \) is sign-central matrix, without loss of generality, let \( X \) be a sign-pattern matrix. Then, we have an immediate consequence by the above lemmas and theorems. \( \blacksquare \)

**References**

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