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A METRIC INDUCED BY A NORM ON NORMED ALMOST LINEAR SPACES

SUNG MO IM AND SANG HAN LEE

In [3,4,5], G. Godini introduced a normed almost linear space(*nals*), generalizing the concept of a normed linear space. In contrast with the case of a normed linear space, the norm of a *nals* $(X, |||\cdot|||)$ does not generate a metric on X (for $x \in X \setminus V_X$ we have $|||x-x||| \neq 0$). G. Godini [5] proved that for a *nals* X there exists a semi-metric which satisfy some properties. In this paper, we prove that the above semi-metric is a metric if a *nals* X has a basis. Also, we construct a new metric for such a space in a simpler way and, prove that in the case when a *nals* X has a basis and splits as $X = W_X + V_X$, then X is complete if and only if V_X and W_X are complete.

We recall some definitions and results used in this paper. All spaces involved in this paper are over the real field $\mathbb{R}$. Let us denote by $\mathbb{R}_+$ the set $\{\lambda \in \mathbb{R} : \lambda \geq 0\}$.

An *almost linear space* (*als*) is a set X together with two mappings $s : X \times X \to X$ and $m : \mathbb{R} \times X \to X$ satisfying the conditions $(L_1)-(L_8)$ given below. For $x, y \in X$ and $\lambda \in \mathbb{R}$ we denote $s(x,y)$ by $x+y$ and $m(\lambda, x)$ by λx, when these will not lead to misunderstandings. Let $x, y, z \in X$ and $\lambda, \mu \in \mathbb{R}$. (L_1) $x+(y+z) = (x+y)+z$; (L_2) $x+y = y+x$; (L_3) There exists an element $0 \in X$ such that $x+0 = x$ for each $x \in X$; (L_4) $1x = x$; (L_5) $\lambda(x+y) = \lambda x + \lambda y$; (L_6) $0x = 0$; (L_7) $\lambda(\mu x) = (\lambda\mu)x$; (L_8) $(\lambda+\mu)x = \lambda x + \mu x$ for $\lambda \geq 0$, $\mu \geq 0$. We denote $-1x$ by $-x$, and $x - y$ means $x + (-y)$. For an *als* X we introduce the following two sets:

$$V_X = \{x \in X : x - x = 0\}$$

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$$W_X = \{x \in X : x = -x\}.$$

V_X and W_X are almost linear subspaces of X(i.e., closed under addition and multiplication by scalars) and, in fact, V_X is a linear space. Clearly an *als* X is a linear space iff $V_X = X$ iff $W_X = \{0\}$. Note that $V_X \cap W_X = \{0\}$ and $W_X = \{x - x : x \in X\}$.

A *norm* on an *als* X is a functional $|||\cdot||| : X \to \mathbb{R}$ satisfying the conditions $(N_1) - (N_3)$ below. Let $x, y, z \in X$ and $\lambda \in \mathbb{R}$. (N_1) $|||x - z||| \le |||x-y|||+|||y-z|||$; (N_2) $|||\lambda x||| = |\lambda|\, |||x|||$; (N_3) $|||x||| = 0$ iff $x = 0$. An *als* X together with $|||\cdot||| : X \to \mathbb{R}$ satisfying $(N_1)-(N_3)$ is called a *normed almost linear space* (*nals*). Using (N_1) we get $|||x + y||| \le |||x|||+|||y|||$ and $|||x-y||| \ge |\, |||x|||-|||y|||\, |$ for $x, y \in X$. By the above axioms it follows that $|||x||| \ge 0$ for each $x \in X$.

A subset B of an *als* X is called a *basis* for X if for each $x \in X \setminus \{0\}$ there exist unique sets $\{b_1, b_2, ..., b_n\} \subset B$, $\{\lambda_1, \lambda_2, ..., \lambda_n\} \subset \mathbb{R} \setminus \{0\}$ (n depending on x) such that $x = \sum_{i=1}^{n} \lambda_i b_i$, where $\lambda_i > 0$ for $b_i \notin V_X$. Clearly, if B is a basis for X then $0 \notin B$.

Now, we give some propositions needed in the sequel.

PROPOSITION 1 ([3]). *Let $(X, |||\cdot|||)$ be a nals. Then,*

(a) *For $x \in X$, $w \in W_X$,* $\max\{|||x|||, |||w|||\} \le |||x + w|||$.
(b) *For $x, x_n \in X$, $n \in N$, if $\lim_{n\to\infty} |||x_n + x||| = 0$ then $x \in V_X$.*

PROPOSITION 2 ([3]). *Let X be an als with a basis B. Then,*

(a) *The relations $x + y = x + z$, $x, y, z \in X$ imply that $y = z$.*
(b) *For each $x \in X \setminus V_X$, there exist unique $b_1, b_2, ..., b_n \in B \setminus V_X$, $v \in V_X$, and $\lambda_1, \lambda_2, ..., \lambda_n > 0$ such that $x = \sum_{i=1}^{n} \lambda_i b_i + v$.*
(c) *There exists a basis B' of X with the property that for each $b' \in B' \setminus V_X$ we have $-b' \in B' \setminus V_X$. Moreover* $\text{card}(B \setminus V_X) = \text{card}(B' \setminus V_X)$. *We shall call such a basis B' a* **symmetric** *basis.*
(d) *There exists a basis B'' of $W_X + V_X$ with the property that $B'' = B_1 \cup B_2$, where B_1 is a basis for W_X and B_2 is a basis for V_X.*

We say that a commutative semigroup X with zero [i.e. satisfying (L_1) - (L_3)] is an *abstract convex cone* if there is also given a mapping

$(\lambda, x) \to \lambda x$ of $\mathbb{R}_+ \times X$ into X such that (L_4), (L_5), (L_7) and (L_8) hold for $x, y \in X$ and $\lambda, \mu \in \mathbb{R}_+$. X satisfies the *law of cancellation* if the relations $x, y, z \in X$, $x+y = x+z$ imply $y = z$. A map $T : X \to L$ from an abstract convex cone to a linear space is *positively homogeneous* if $T(\alpha x) = \alpha T(x)$ for $\alpha \in \mathbb{R}_+$.

PROPOSITION 3 ([5,7]). *Let X be an abstract convex cone satisfying the law of cancellation. Then there exist a linear space L and a one-to-one additive and positively homogeneous mapping $T : X \to L$ such that $L = T(X) - T(X) = \{T(x) - T(y) : x, y \in X\}$.*

Note that such an additive and positively homogeneous mapping T is linear on the subspace V_X.

PROPOSITION 4. *Let $(X, |||\cdot|||)$ be a nals and let $x \in X$. Then for each $\alpha, \beta \in \mathbb{R}$, we have*

$$\alpha x + \beta x = (\alpha + \beta)x + w$$

for some $w \in W_X$. Furthermore, for $\alpha_i, \beta_i \in \mathbb{R}$, $x_i \in X$, $(i = 1, 2, ..., n)$, we have

$$|||\sum_{i=1}^{n}(\alpha_i + \beta_i)x_i||| \leq |||\sum_{i=1}^{n}\alpha_i x_i + \sum_{i=1}^{n}\beta_i x_i|||.$$

Proof. If $\alpha\beta \geq 0$, then $(\alpha + \beta)x = \alpha x + \beta x$. Let $\alpha\beta < 0$. We may assume that $\alpha < 0 < \beta$, without loss of generality. If $|\alpha| \leq \beta$, we have $\alpha x + \beta x = \alpha x + (-\alpha + \alpha + \beta)x = \alpha x + (-\alpha)x + (\alpha + \beta)x = w + (\alpha + \beta)x$, where $w = \alpha x + (-\alpha)x$. If $|\alpha| > \beta$, we have $\alpha x + \beta x = (\alpha + \beta - \beta)x + \beta x = (\alpha + \beta)x + (-\beta)x + \beta x = (\alpha + \beta)x + w$, where $w = \beta x + (-\beta)x$. Therefore, $\alpha x + \beta x = (\alpha + \beta)x + w$ for some $w \in W_X$. From Proposition 1(a), we have

$$|||\alpha x + \beta x||| = |||(\alpha + \beta)x + w||| \geq |||(\alpha + \beta)x|||.$$

For the second statement, let $\alpha_i, \beta_i \in \mathbb{R},\ x_i \in X,\ i = 1, 2, ..., n$. Then

$$\begin{aligned} |||\sum_{i=1}^{n} \alpha_i x_i + \sum_{i=1}^{n} \beta_i x_i||| &= |||\sum_{i=1}^{n}(\alpha_i x_i + \beta_i x_i)||| \\ &= |||\sum_{i=1}^{n}((\alpha_i + \beta_i)x_i + w_i)||| \\ &\geq |||\sum_{i=1}^{n}(\alpha_i + \beta_i)x_i|||. \quad \square \end{aligned}$$

On every *nals* $(X, |||\cdot|||)$, Godini introduced a semi-metric ρ. In the case when X has a basis, the construction can be simplified as follows: A *nals* X with a basis is an abstract convex cone satisfying the law of cancellation. By Proposition 3, there exist a linear space L and a one-to-one additive and positively homogeneous mapping $T : X \to L$ such that $L = T(X) - T(X)$. For $l \in L$ define

$$||l|| = \inf\{|||x||| + |||y||| : l = T(x) - T(y),\ x, y \in X\}.$$

Then $||\cdot||$ is a semi-norm on L and $||T(x)|| = |||x|||$ for each $x \in X$([5; Theorem 3.2]). The semi-metric ρ on X is given by $\rho(x, y) = ||T(x) - T(y)||$ for $x, y \in X$([5; Corollary 3.3]) and satisfies the following properties:

(1) $$\rho(x, v) = |||x - v||| \quad (x \in X,\ v \in V_X),$$

(2) $$\rho(x + z, y + z) = \rho(x, y) \quad (x, y, z \in X),$$

(3) $$\rho(\lambda x, \lambda y) = |\lambda|\rho(x, y) \quad (x, y \in X,\ \lambda \in \mathbb{R}),$$

(4) $$|\ |||x||| - |||y|||\ | \leq \rho(x, y) \leq |||x - y||| \quad (x, y \in X),$$

(5) $$\lim_{\lambda \to \lambda_0} \rho(\lambda x, x) = \rho(\lambda_0 x, x) \quad (x \in X,\ \lambda_0 > 0).$$

We prove that the semi-metric is a metric in the case when X has a basis.

THEOREM 5. *If a nals $(X, |||\cdot|||)$ has a basis, then G. Godini's semi-metric is a metric on X.*

Proof. Let B be a basis for a *nals* X. We shall show that $||\cdot||$ is a norm on L.

Let $l \in L$. Choose $x_0, y_0 \in X$ so that $l = T(x_0) - T(y_0)$. Using our basis B, we can write x_0 and y_0 as

$$x_0 = \sum \alpha_i b_i + \sum \beta_j b_j + \sum \gamma_k b_k + \sum \delta_l b_l,$$

$$y_0 = \sum \alpha'_i b_i + \sum \beta'_j b_j + \sum \gamma'_k b_k + \sum \delta'_l b_l$$

where

$$b_i, b_j \in V_X \cap B,\ b_i \neq b_j,\ \alpha_i \geq \alpha'_i,\ \beta_j < \beta'_j$$

and

$$b_k, b_l \in B \setminus V_X,\ b_k \neq b_l,\ \gamma_k \geq \gamma'_k \geq 0,\ 0 \leq \delta_l < \delta'_l.$$

Since T is additive and positively homogeneous (so T is linear on V_X), we have

$$\begin{aligned} T(x_0) - T(y_0) &= \sum(\alpha_i - \alpha'_i)T(b_i) + \sum(\gamma_k - \gamma'_k)T(b_k) \\ &\quad - \left(\sum(\beta'_j - \beta_j)T(b_j) + \sum(\delta'_l - \delta_l)T(b_l)\right) \\ &= T\left(\sum(\alpha_i - \alpha'_i)b_i + \sum(\gamma_k - \gamma'_k)b_k\right) \\ &\quad - T\left(\sum(\beta'_j - \beta_j)b_j + \sum(\delta'_l - \delta_l)b_l\right). \end{aligned}$$

Let

$$\widetilde{x}_0 = \sum(\alpha_i - \alpha'_i)b_i + \sum(\gamma_k - \gamma'_k)b_k = \sum_p \xi_p b_p$$

$$\widetilde{y}_0 = \sum(\beta'_j - \beta_j)b_j + \sum(\delta'_l - \delta_l)b_l = \sum_q \zeta_q b_q.$$

Note that all the coefficients are positive; that is, $\xi_p, \zeta_q > 0$ for all p, q. Moreover, the subsets $\{b_p\}$ and $\{b_q\}$ of B, appearing in $\widetilde{x}_0$ and $\widetilde{y}_0$, are mutually disjoint. The above equalities also show that $T(x_0) - T(y_0) = T(\widetilde{x}_0) - T(\widetilde{y}_0)$. We shall try to denote general elements in terms of these particular elements $\widetilde{x}_0, \widetilde{y}_0$.

Let $x, y \in X$ be general elements such that $l = T(x) - T(y)$. Then $T(x) - T(y) = T(\widetilde{x}_0) - T(\widetilde{y}_0)$. Since L is a linear space, $T(x) + T(\widetilde{y}_0) = T(y) + T(\widetilde{x}_0)$. By additivity of T, $T(x + \widetilde{y}_0) = T(y + \widetilde{x}_0)$. Since T is one-to-one, $x + \widetilde{y}_0 = y + \widetilde{x}_0$; i.e., $x + \sum \zeta_q b_q = y + \sum \xi_p b_p$. Recall that $\xi_p, \zeta_q > 0$ for all p, q, and the subsets $\{b_p\}$ and $\{b_q\}$ of B, appearing in $\widetilde{x}_0$ and $\widetilde{y}_0$, are mutually disjoint. We conclude there exists $u \in X$ such that

$$x = \widetilde{x}_0 + u, \quad y = \widetilde{y}_0 + u.$$

Suppose

$$0 = ||l|| = \inf\{|||x||| + |||y||| : l = T(x) - T(y),\ x, y \in X\}.$$

There exist sequences $(x_n), (y_n) \in X$ such that

$$l = T(x_n) - T(y_n)$$

for each n, and

$$\lim_{n\to\infty} (|||x_n||| + |||y_n|||) = 0.$$

However, as we observed earlier, for each n, there exists $u_n \in X$ so that

$$x_n = \widetilde{x}_0 + u_n, \quad y_n = \widetilde{y}_0 + u_n.$$

Moreover

$$\lim_{n\to\infty} |||\widetilde{x}_0 + u_n||| = 0 \quad \text{and} \quad \lim_{n\to\infty} |||\widetilde{y}_0 + u_n||| = 0.$$

By Proposition 1(b), we have $\widetilde{x}_0, \widetilde{y}_0 \in V_X$. Note that $T|_{V_X}$ is a linear operator. Thus $l = T(\widetilde{x}_0) - T(\widetilde{y}_0) = T(\widetilde{x}_0 - \widetilde{y}_0) \in T(X)$, so we have

$$0 = ||l|| = ||T(\widetilde{x}_0 - \widetilde{y}_0)|| = |||\widetilde{x}_0 - \widetilde{y}_0|||,$$

where the last equality holds since $\widetilde{x}_0 - \widetilde{y}_0 \in X$. Therefore $\widetilde{x}_0 - \widetilde{y}_0 = 0$. Thus $l = T(\widetilde{x}_0 - \widetilde{y}_0) = T(0) = 0 \in L$. We have shown that $|| \cdot ||$ is a norm on L. Thus ρ is a metric. □

We construct a new metric on X which is simpler than G. Godini's. This does not involve construction of L.

THEOREM 6. *For a nals $(X, |||\cdot|||)$ with a basis, there exists a metric d on X satisfying the properties of ρ in* (1) – (5).

Proof. Let B be a symmetric basis for X. For $x, y, z \in X$, we may assume that $x = v_x + \sum_{i=1}^{n} \alpha_i b_i$, $y = v_y + \sum_{i=1}^{n} \beta_i b_i$, $z = v_z + \sum_{i=1}^{n} \gamma_i b_i$, where $v_x, v_y, v_z \in V_X$, $b_i \in B \setminus V_X$, $\alpha_i, \beta_i, \gamma_i \geq 0$, $i = 1, 2, \ldots, n$.

Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = |||v_x - v_y + \sum_{i=1}^{n} (\alpha_i - \beta_i) b_i||| \quad (x, y \in X). \tag{6}$$

From Proposition 4 and the triangle inequality, we have

$$\begin{aligned} d(x, y) + d(y, z) &= |||v_x - v_y + \sum_{i=1}^{n} (\alpha_i - \beta_i) b_i||| \\ &\quad + |||v_y - v_z + \sum_{i=1}^{n} (\beta_i - \gamma_i) b_i||| \\ &\geq |||v_x - v_z + \sum_{i=1}^{n} (\alpha_i - \beta_i) b_i + \sum_{i=1}^{n} (\beta_i - \gamma_i) b_i||| \\ &\geq |||v_x - v_z + \sum_{i=1}^{n} (\alpha_i - \gamma_i) b_i||| \\ &= d(x, z). \end{aligned}$$

Thus $d(x, y) + d(y, z) \geq d(x, z)$ for $x, y, z \in X$. Clearly, we have $d(x, y) = d(y, x)$ and $d(x, y) = 0$ if and only if $x = y$ for each $x, y \in X$. Therefore d is a metric on X.

It is easy to show that d satisfies (1) - (5) except for (3) when $\lambda < 0$. To show this, let $\lambda < 0$. Then $\lambda x = \lambda v_x + \sum_{i=1}^{n} (-\lambda \alpha_i)(-b_i)$, $\lambda y =$

$\lambda v_y + \sum_{i=1}^{n}(-\lambda\beta_i)(-b_i)$. Hence we have

$$\begin{aligned} d(\lambda x, \lambda y) &= |||\lambda v_x - \lambda v_y + \sum_{i=1}^{n}(-\lambda\alpha_i - (-\lambda\beta_i))(-b_i)||| \\ &= |||\lambda(v_x - v_y) + \sum_{i=1}^{n}\lambda(\alpha_i - \beta_i)b_i||| \\ &= |\lambda|\ |||v_x - v_y + \sum_{i=1}^{n}(\alpha_i - \beta_i)b_i||| \\ &= |\lambda| d(x, y). \end{aligned}$$

Therefore $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for $x, y \in X$, $\lambda < 0$. The proof of the theorem is complete. □

A metric induced by the norm on a normed linear space X satisfies (2) and (3). Also, the metric d defined by (6) satisfies (2), (3), and $d(x, y) = |||x - y|||$ if $X = V_X$. This shows that (6) generalizes the notion of the metric induced by the norm on a normed linear space X.

EXAMPLE 7. Let $\mathbb{R}^2$ be endowed with the Euclidean norm $||\cdot||$ and let $e_1 = (1, 0)$, $e_2 = (0, 1)$. Let $X = \{\alpha e_1 + \beta e_2 : \alpha \geq 0,\ \beta \geq 0\}$. Define $s(x, y) = x + y$, $m(\lambda, x) = |\lambda| x$ for $x, y \in X$, $\lambda \in \mathbb{R}$. Then $V_X = \{0\}$ and $W_X = X$. Define $|||x||| = ||x||$ for $x \in X$. Then $(X, |||\cdot|||)$ is a *nals*. And $\{e_1, e_2\}$ is a basis for X. Let ρ be a metric on X defined by G. Godini and d a metric on X defined by (6). We have

$$d(e_1, e_2) = |||e_1 - e_2||| = |||e_1 + e_2||| = \sqrt{2}.$$

But

$$\rho(e_1, e_2) = |||e_1||| + |||e_2||| = 2.$$

Indeed, if $T : X \to L = T(X) - T(X)$ is the positively homogeneous map used in defining the semi-metric ρ, and if $T(e_1) - T(e_2) = T(x) - T(y)$ for some $x, y \in X$, then $x = e_1 + u$, $y = e_2 + u$ for some $u \in X$. Hence $|||e_1||| \leq |||x|||$ and $|||e_2||| \leq |||y|||$ since $u \in X = W_X$. Therefore $||T(e_1) - T(e_2)|| = |||e_1||| + |||e_2||| = 2$.

From now on, every *nals* X with a basis is assumed to have a metric d given by Theorem 6.

THEOREM 8. *If a nals X has a basis, then V_X and W_X are closed in X.*

Proof. Let (v_n) be a sequence in V_X such that

$$\lim_{n\to\infty} d(v_n, x) = 0$$

for some $x \in X$. Since

$$\begin{aligned} d(0, x-x) &= d(v_n - v_n, x - x) \\ &\leq d(v_n - v_n, x - v_n) + d(x - v_n, x - x) \\ &= d(v_n, x) + d(-v_n, -x) \\ &= d(v_n, x) + |-1| d(v_n, x) \\ &= 2d(v_n, x) \end{aligned}$$

for each $n \in N$, we have $d(0, x-x) = 0$. Hence $x - x = 0$. Therefore $x \in V_X$.

Let (w_n) be a sequence in W_X such that

$$\lim_{n\to\infty} d(w_n, x) = 0$$

for some $x \in X$. Since

$$\begin{aligned} d(x, -x) &\leq d(x, w_n) + d(w_n, -x) \\ &= d(x, w_n) + d(-w_n, -x) \\ &= 2d(x, w_n) \end{aligned}$$

for each $n \in N$, we have $d(x, -x) = 0$. Hence $x = -x$. Therefore $x \in W_X$. □

Thus, if a *nals* X with a basis is complete, then V_X and W_X are complete. However, if a *nals* X is not split, then the converse does not hold as shown in the following Example:

EXAMPLE 9. Let $\mathbb{R}^2$ be endowed with the Euclidean norm $||\cdot||$ and let $e_1 = (1,0)$, $e_2 = (0,1)$. Let $X = \{\alpha e_1 + \beta e_2 : \alpha, \beta \in \mathbb{R},\ \beta \geq 0\}$. Define $s(x,y) = x + y$ for $x, y \in X$ and $m(\lambda, x) = (\lambda\alpha)e_1 + (|\lambda|\beta)e_2$ for $x = \alpha e_1 + \beta e_2 \in X$ and $\lambda \in \mathbb{R}$. And define $|||x||| = ||x||$ for $x \in X$. Then $(X, |||\cdot|||)$ is a *nals* with a basis $B = \{e_1, e_2\}$. Also, $V_X = \{\alpha e_1 : \alpha \in \mathbb{R}\}$ and $W_X = \{\beta e_2 : \beta \geq 0\}$. Hence $X = W_X + V_X$. Let $x = \alpha_1 e_1 + \beta_1 e_2$ and $y = \alpha_2 e_1 + \beta_2 e_2$. Then $d(x,y) = |||\alpha_1 e_1 - \alpha_2 e_1 + (\beta_1 - \beta_2)e_2||| = ||\alpha_1 e_1 - \alpha_2 e_1 + (\beta_1 - \beta_2)e_2|| = ||(\alpha_1 e_1 + \beta_1 e_2) - (\alpha_2 e_1 + \beta_2 e_2)|| = ||x - y||$. Hence X is complete.

Let $Y = \{\alpha e_1 + \beta e_2 \in X : \alpha, \beta \in \mathbb{R},\ \beta > 0\} \cup \{(0,0)\}$. Then Y is an almost linear subspace of X which has no basis. $V_Y = \{0\}$, $W_Y = \{\beta e_2 : \beta \geq 0\}$, and $Y \neq W_Y + V_Y$. Clearly, W_Y and V_Y are complete but Y is not complete.

When X splits as $X = W_X + V_X$, Theorem 8 yields the following

THEOREM 10. *If a nals X has a basis and splits as $X = W_X + V_X$, then X is complete if and only if V_X and W_X are complete.*

Proof. By Proposition 2(d), there exists a basis B for W_X. Suppose that V_X and W_X are complete. Let (x_n) be a Cauchy sequence in X. Let $x_n = v_n + \sum_{i=1} \alpha_{n_i} b_i$, where $v_n \in V_X$, $b_i \in B$, $\alpha_{n_i} \geq 0$, and $\sum_i$ denote a finite sum. By Proposition 1(a), we have

$$|||v_n - v_m||| \leq |||v_n - v_m + \sum_i (\alpha_{n_i} - \alpha_{m_i})b_i||| = d(x_n, x_m)$$

and

$$|||\sum_i (\alpha_{n_i} - \alpha_{m_i})b_i||| \leq |||v_n - v_m + \sum_i (\alpha_{n_i} - \alpha_{m_i})b_i||| = d(x_n, x_m).$$

Thus (v_n) is a Cauchy sequence in V_X, and $(\sum_i \alpha_{n_i} b_i)$ is a Cauchy sequence in W_X. Since V_X, W_X are complete, there exist $v \in V_X$, $w = \sum_i \alpha_i b_i \in W_X$ such that

$$\lim_{n\to\infty} d(v_n, v) = 0 \text{ and } \lim_{n\to\infty} d\left(\sum_i \alpha_{n_i} b_i, \sum_i \alpha_i b_i\right) = 0.$$

Put $x = v + \sum_i \alpha_i b_i \in X$. Then, since

$$\begin{aligned} d(x_n, x) &= d\left(v_n + \sum_i \alpha_{n_i} b_i, v + \sum_i \alpha_i b_i\right) \\ &\leq d\left(v_n + \sum_i \alpha_{n_i} b_i, \sum_i \alpha_{n_i} b_i + v\right) \\ &\quad + d\left(\sum_i \alpha_{n_i} b_i + v, v + \sum_i \alpha_i b_i\right) \\ &= d(v_n, v) + d\left(\sum_i \alpha_{n_i} b_i, \sum_i \alpha_i b_i\right), \end{aligned}$$

we have

$$\lim_{n\to\infty} d(x_n, x) \leq \lim_{n\to\infty} d(v_n, v) + \lim_{n\to\infty} d\left(\sum_i \alpha_{n_i} b_i, \sum_i \alpha_i b_i\right) = 0.$$

Hence (x_n) converges to $x \in X$. Therefore X is complete. □

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Department of Mathematics, College of Natural Science, Chungbuk National University, Cheongju 360-763, Korea