Bull. Korean Math. Soc. **34** (1997), No. 2, pp. 155–171

A CLASS OF COMPACT SUBMANIFOLDS WITH CONSTANT MEAN CURVATURE

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0. Introduction

Let M^n be a connected submanifold of a Euclidean space E^m, equipped with the induced metric. Denote by Δ the Laplacian operator of M^n and by x the position vector. A well-known T. Takahashi's theorem [13] says that $\Delta x = \lambda x$ for some constant λ if and only if M^n is either minimal submanifold of E^m or minimal submanifold in a hypersphere of E^m. In [9], O. Garay studied the hypersurfaces M^n in E^{n+1} satisfying $\Delta x = Dx$, where D is a diagonal matrix, and he classified such hypersurfaces. Garay's condition can be seen as a generalization of T. Takahashi's condition. But one can observe that Garay's condition is not coordinate-invariant. F. Dillen, J. Pas and L. Verstraelen generalized T. Takahashi's condition, following Garay's idea, in a way which is independent of the choice of coordinates. They considered submanifolds in E^m satisfying

$$(*) \qquad \Delta x = Ax + b,$$

where A is a $m \times m$ constant matrix and b is a constant vector in E^m, and proved that the only surfaces in E^3 satisfying $(*)$ are minimal surfaces, spheres and circular cylinders [4]. Submanifolds satisfying three authors' condition were studied by serveral authors [1,2,3,5,6,7,8,11]. Recently, Th. Hasanis and Th. Vlachos classified submanifold of codimension 2 with constant mean curvature satisfying $(*)$ for a diagonal

Received June 25, 1996.
1991 Mathematics Subject Classification: 53C40, 53A05.
Key words and phrases: Laplacian, submanifolds satisfying $\Delta x = Ax + b$.

matrix A and $b = 0$ [12]. In this paper, we are going to deal with a part of the following problem:

Classify submanifolds of codimension 2 with constant mean curvature satisfying (*) for a constant matrix A (not neccesarily diagonal or symmetric) and a constant vector b.

Our main result is given by the following.

THEOREM. *Let M^n be a compact submanifold of E^{n+2} with constant mean curvature. Then M^n satisfies (*) for a constant matrix A and a constant vector b in E^{n+2} if and only if M^n is one of the following:*

(1) *an ordinary sphere in E^{n+1},*
(2) *a minimal hypersurface of a hypersphere S^{n+1},*
(3) *a Riemann product $S^m(r_1) \times S^{n-m}(r_2)$, where $1 \leq m \leq n-1$ and $\frac{r_1}{r_2} \neq \sqrt{\frac{m}{n-m}}$.*

1. Preliminaries

Let M^n be an n-dimensional submanifold of E^m. We denote by $\overline{\nabla}, \nabla, h, H$ and x, the Euclidean connection of E^m, the induced connection, the second fundamental form, the mean curvature vector and the position vector of M^n, respectively. Then the Gauss formula is obtained by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1.1}$$

for vector fields X, Y tangent to M^n. Let $e_1, e_2, \cdots, e_m$ be an adapted local orthonormal frame in E^m such that $e_1, \cdots, e_n$ tangent to M^n and $e_{n+1}, e_{n+2}, \cdots, e_m$ normal to M^n. Then we have

$$\Delta x = \sum_{i=1}^{n} h(e_i, e_i) = H. \tag{1.2}$$

The connection form ω_A^B are defined by

$$de_A = \sum \omega_A^B e_B, \; A, B, C, \cdots = 1, \cdots, m. \tag{1.3}$$

Then we have $\omega_B^A = -\omega_A^B$. We will use the same notation $\langle \; , \; \rangle$ for the induced metric of M^n and that of E^m. We need the following lemmas.

LEMMA 1.1. [4] *Let M^n be a compact submanifold of E^m and let M^n satisfy $(*)$. Then we may assume $b = 0$.*

Proof. Now assume that M^n is compact and satisfies $(*)$. Integrating $(*)$ over M^n and using divergence theorem, implies that

$$Ax_0 + b = 0$$

for a constant vector x_0 in E^{n+2}. Using this, then $(*)$ further implies that

$$\Delta(x - x_0) = A(x - x_0).$$

□

LEMMA 1.2. *Let M^n be a compact submanifold of E^{n+2} satisfying $(*)$ for a singular matrix A. Then M^n is an ordinary sphere in a hyperplane of E^{n+2}.*

Proof. By Lemma 1.1, we may assume $b = 0$. Since A is singular, we may assume that $A_{n+2} = \alpha_1 A_1 + \cdots + \alpha_{n+1} A_{n+1}$, where A_i are the ith row vector of A and α_i are suitable constants. Then $(*)$ implies that $\Delta x_{n+2} = \langle A_{n+2}, x\rangle = \langle \sum_{i=1}^{n+1} \alpha_i A_i, x\rangle = \sum_{i=1}^{n+1} \alpha_i \Delta x_i$, where x_i are the ith coordinate funtion of M^n. So we have

$$\Delta(x_{n+2} - \sum \alpha_i x_i) = 0.$$

This and Hopf's lemma imply that

$$x_{n+2} - \sum \alpha_i x_i = \text{constant}.$$

Thus M^n is contained in a hyperplane of E^{n+2}. The conclusion follows from the classification theorem of hypersurfaces satisfying $(*)$ [1,5,11]. □

LEMMA 1.3. *Let M^n be a submanifold of E^m satisfying $(*)$. Then every vector field CX, where C is the skew symmetric matrix $\frac{1}{2}(A - {}^tA)$ and X is a local tangent vector of M^n, is normal to M^n.*

Proof. Since $Ax+b$ is normal to M^n, we have

$$\langle Ax+b, X\rangle = 0,\ \langle Ax+b, Y\rangle = 0$$

for local tangent vectors X, Y of M^n. Differentiating these in Y, X respectively, by (1.1) we have

$$\langle AY, X\rangle + \langle Ax+b, h(Y,X)\rangle = 0$$
$$\langle AX, Y\rangle + \langle Ax+b, h(X,Y)\rangle = 0.$$

From these and the symmetry of h, we can conclude that CX is normal to M^n for a tangent vector X. □

LEMMA 1.4. *Under the same hypothesis of Lemma 1.3, we have the following formula*

$$\sum_{i=1}^{n}\langle Ae_i, e_i\rangle = -\langle Ax+b, Ax+b\rangle,$$

where $e_1, \cdots, e_n$ are local orthnormal tangent frame of M^n.

Proof. By similar method to that in the proof of Lemma 1.3, we have

$$\langle Ae_i, e_i\rangle = -\langle Ax+b, h(e_i, e_i)\rangle$$

for $i = 1, \cdots, n$. By summation we obtain

$$\sum_{i=1}^{n}\langle Ae_i, e_i\rangle = -\langle Ax+b, \sum_{i=1}^{n} h(e_i, e_i)\rangle.$$

From this and (1.2), the conclusion follows. □

LEMMA 1.5. *Let M^n be a compact submanifold of E^{n+2} satisfying $(*)$. Then the rank of the skew symmetric matrix $C = \frac{1}{2}(A - {}^tA)$ is at most 2.*

Proof. Suppose $b = 0$ (Lemma 1.1). Since M^n is compact, the function $\langle Ax, x\rangle$ has minimum and maximum. So we may assume that the function $\langle Ax, x\rangle$ has an extremal value different from zero at a point $p(\in M^n)$. In a local orthonormal tangent frame $e_1, e_2, \cdots, e_n$ on a neighborhood of p, differentiating $\langle Ax, x\rangle$, we have

$$e_i\langle Ax, x\rangle = 0 \quad \text{at} \quad p.$$

This imply that

$$\langle Ae_i, x\rangle = \langle e_i, {}^tAx\rangle = 0 \quad \text{at} \quad p.$$

Thus $\langle Ce_i, x\rangle = 0$ at p. Decomposing x as $x^N + x^T$, where x^N is normal to M^n and x^T is tangent to M^n, we have

$$\langle Ce_i, x^N\rangle = \langle Cx^N, x^N\rangle = 0 \quad \text{at} \quad p.$$

From these, considering $Ce_1, Ce_2, \cdots, Ce_n, Cx^N$ are normal to M^n and $x^N \neq 0$ at p, we can see that the rank of C is at most 2. □

2. Proof of theorem

Throughout this section we will assume that M^n is a compact submanifold of E^{n+2} with constant mean curvature.

LEMMA 2.1. *If M^n satisfies $\Delta x = Ax$ for an $(n+2) \times (n+2)$ nonsymmetric matrix A of the following form*

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & \\ & 0 & \beta I \end{pmatrix}, \tag{2.1}$$

where βI is an $n \times n$ diagonal matrix with all diagonal entries β and $a_{ij}(i, j = 1, 2)$ are constants, then M^n is an ordinary sphere.

Proof. Since A is not symmetric, the rank of the skew symmetric matrix $C = \frac{1}{2}(A - {}^tA)$ is 2. Let $E_1, E_2, E_3, \cdots, E_{n+2}$ be the standard basis of E^{n+2}. Then *ker* C is spanned by $E_3, \cdots, E_{n+2}$. If the dimension of the normal space $\{CX_p | X_p \in T_pM\}$ at $p(p \in M^n)$ is 2, then $E_3, \cdots, E_{n+2}$ span the tangent space T_pM. This is impossible, since $CE_i = 0$ for $i = 3, \cdots, n+2$. Hence the dimension of $\{CX_p | X_p \in T_pM\}$ is at most 1 at every point p in M^n. Now suppose that the dimension of $\{CX | X \in TM\}$ is 0 at every point of an open subset U of M^n. Then $E_3, \cdots, E_{n+2}$ span the tangent space of U. This means that M^n is a minimal submanifold of E^{n+2}, since the mean curvature of M^n is constant. This is impossibe. Thus we may assume that $\dim\{CX | X \in TM^n\}$ is 1 at every point of a dense open subset W of M^n. We will work in W. Since $\dim\{CX | X \in TW\}$ is 1 at every point of W, we can choose a local orthonormal frame $e_1, \cdots, e_n$ of W such that

$$(2.2) \qquad Ce_1 \neq 0, \quad Ce_i = 0 \quad \text{for } i = 2, \cdots, n.$$

The second equation in (2.2) and (2.1) imply that

$$(2.3) \qquad Ae_i = \beta e_i, \quad i = 2, \cdots, n.$$

Since the mean curvature of M^n is constant, (1.2) implies

$$\langle Ax, Ax \rangle = \langle H, H \rangle = \alpha^2$$

for a positive constant α, where H is the mean curvature vector field of M^n. By Lemma 1.4 we have $\sum_{i=1}^n \langle Ae_i, e_i \rangle = -\langle Ax, Ax \rangle = -\alpha^2$. By this and (2.3) we obtain $\langle Ae_1, e_1 \rangle + (n-1)\beta = -\alpha^2$. This implies that $\langle Ae_1, e_1 \rangle = -\alpha^2 - (n-1)\beta$. We will let $-\alpha^2 - (n-1)\beta = \gamma$. Let e_{n+1}, e_{n+2} be orthonrmal normal frame of M^n such that $e_{n+1} = \frac{1}{\alpha} Ax$. Then we have

$$(2.4) \qquad Ae_1 = \gamma e_1 + \delta e_{n+2},$$

where $\delta = \alpha \omega_{n+1}^{n+2}(e_1)$, since $\langle Ae_1, Ax \rangle = 0$. Now we will show that the dimension of $\text{Im}h = \{h(X, Y) | X$ and Y are local tangent vectors$\}$ is

1. Suppose that the dimension of Imh is 2, locally. Differentiating the second equation in (2.2) in the direction e_j $(j = 1, \cdots, n)$, we have

$$Ch(e_i, e_j) = -C\nabla_{e_j} e_i, \quad i = 2, \cdots, n. \tag{2.5}$$

(2.5) and Lemma 1.3 imply

$$\langle Ch(e_i, e_j), e_1 \rangle = -\langle h(e_i, e_j), Ce_1 \rangle = 0.$$

for $i = 2, \cdots, n$ and $j = 1, \cdots, n$. Hence $h(e_i, e_j)$ are parallel each other for $i = 2, \cdots, n$ and $j = 1, \cdots, n$. (2.3) and (2.4) imply that

$$\langle Ax, h(e_i, e_i) \rangle = -\langle Ae_i, e_i \rangle = \beta \tag{2.6}$$

$$\text{for} \quad i = 2, \cdots, n \quad \text{and}$$

$$\langle Ax, h(e_i, e_j) \rangle = -\langle Ae_i, e_j \rangle = 0 \tag{2.7}$$

$$\text{for} \quad i \neq j, \quad i, j = 1, \cdots, n.$$

Since $h(e_i, e_j)(i = 1, \cdots, n,\ j = 2, \cdots, n)$ are parallel each other, (2.6) and (2.7) imply

$$h(e_2, e_2) = \cdots = h(e_n, e_n), \tag{2.8}$$

$$h(e_i, e_j) = 0 \quad \text{for} \quad i \neq j. \tag{2.9}$$

(2.5) and (2.9) imply

$$\langle \nabla_{e_j} e_i, e_1 \rangle = 0 \tag{2.10}$$

for $i \neq j$, $\quad j = 1, \cdots, n$, $\quad i = 2, \cdots, n$. From (2.3),(2.4),(2.8) and the equation $Ax = \alpha e_{n+1} = \sum h(e_i, e_i)$, we have

$$h(e_1, e_1) = -\frac{\gamma}{\alpha} e_{n+1} + k e_{n+2}, \tag{2.11}$$

$$h(e_2, e_2) = -\frac{\beta}{\alpha} e_{n+1} - \frac{k}{n-1} e_{n+2} \tag{2.12}$$

for some function k. The assumption that $\dim Im\ h = 2$ implies that $k \neq 0$. And we have

$$\overline{\nabla}_{e_1} e_{n+2} = -k e_1 - \frac{\delta}{\alpha} e_{n+1} \tag{2.13}$$

Differentiating (2.4) in the direction e_1, we get

$$Ah(e_1, e_1) = \gamma h(e_1, e_1) + (e_1\delta)e_{n+2} + \delta\overline{\nabla}_{e_1} e_{n+2},$$

since $\nabla_{e_1} e_1 = 0$ by (2.10). Substituting (2.11) and (2.13) into the above equation, we have
(2.14)

$$-\frac{\gamma}{\alpha}Ae_{n+1} + kAe_{n+2} = (-\delta k)e_1 + (-\frac{\gamma^2+\delta^2}{\alpha})e_{n+1} + (\gamma k + e_1\delta)e_{n+2}.$$

Differentiating (2.3) in the direction e_2 and using (2.4), we have

$$Ah(e_2, e_2) = \beta h(e_2, e_2) + (\beta - \gamma)\omega_2^1(e_2)e_1 - \omega_2^1(e_2)\delta e_{n+2}.$$

Substituting (2.12) into the above equation, we get

$$\begin{aligned} (2.15) \qquad & -\frac{\beta}{\alpha}Ae_{n+1} - \frac{k}{n-1}Ae_{n+2} \\ & = (\gamma - \beta)\omega_2^1(e_2)e_1 - \frac{\beta^2}{\alpha}e_{n+1} + (-\frac{\beta k}{n-1} - \omega_2^1(e_2)\delta)e_{n+2}. \end{aligned}$$

Differentiating $\langle Ae_1, e_2\rangle = 0$ in the direction e_2, we have

$$\langle A\overline{\nabla}_{e_2} e_1, e_2\rangle + \langle Ae_1, h(e_2, e_2)\rangle + \langle Ae_1, \nabla_{e_2} e_2\rangle = 0.$$

From (2.3), (2.4), (2.9) and this it follows that

$$\langle Ae_1, h(e_2, e_2)\rangle = \omega_2^1(e_2)(\beta - \gamma).$$

Thus

$$(2.16) \qquad -\frac{\delta k}{n-1} = \omega_2^1(e_2)(\beta - \gamma).$$

If $\beta = \gamma$, then (2.16) implies that $\delta = 0$. Then we can deduce that $Ae_{n+1} = \beta e_{n+1}$ and $Ae_{n+2} = \beta e_{n+2}$ from (2.14) and (2.15). This contradicts to the assumption A is not symmetric. Thus we have

$$\omega_2^1(e_2) = \frac{1}{\beta - \gamma}(-\frac{\delta k}{n-1}).$$

Substituting this into (2.15), we have

$$(2.17)\qquad -\frac{\beta}{\alpha}Ae_{n+1}-\frac{k}{n-1}Ae_{n+2}$$
$$=(\gamma-\beta)\omega_2^1(e_2)e_1-\frac{\beta^2}{\alpha}e_{n+1}+(-\frac{\beta k}{n-1}+\frac{\delta^2 k}{(\beta-\gamma)(n-1)})e_{n+2}.$$

From (2.14),(2.17) and $\alpha^2=-\gamma-(n-1)\beta$, we have

$$(2.18)\qquad Ae_{n+1}$$
$$=-\frac{2\delta k}{\alpha}e_1-\frac{\gamma^2+(n-1)\beta^2+\delta^2}{\alpha^2}e_{n+1}+\frac{1}{\alpha}\{e_1\delta+(\gamma-\beta)k+\frac{\delta^2 k}{\beta-\gamma}\}e_{n+2},$$

$$(2.19)$$
$$Ae_{n+2}=\{\frac{2(n-1)\beta\delta}{\alpha^2}+\delta\}e_1+\frac{(n-1)\beta(\gamma^2-\beta\gamma+\delta^2)}{k\alpha^3}e_{n+1}$$
$$+\{\beta-\frac{\delta^2}{\beta-\gamma}-\frac{(n-1)\beta}{k\alpha^2}(e_1\delta+(\gamma-\beta)k+\frac{\delta^2 k}{\beta-\gamma})\}e_{n+2}.$$

Now we will derive the following equality

$$(2.20)\qquad \langle Ae_{n+2},e_{n+1}\rangle=\frac{1}{\alpha}\{e_1\delta+(\gamma-\beta)k-\frac{\delta^2 k}{\beta-\gamma}\}.$$

Using $e_{n+1}=\frac{1}{\alpha}\{h(e_1,e_1)+(n-1)h(e_2,e_2)\}$ and $e_{n+2}=\frac{n-1}{k\alpha^2}\{\gamma h(e_2,e_2)-\beta h(e_1,e_1)\}$, we have

$$(2.21)\quad \langle Ce_{n+1},e_{n+2}\rangle$$
$$=\frac{n-1}{k\alpha^3}\langle Ch(e_1,e_1)+(n-1)Ch(e_2,e_2),\gamma h(e_2,e_2)-\beta h(e_1,e_1)\rangle$$
$$=\frac{n-1}{k\alpha^3}\{\gamma\langle Ch(e_1,e_1),h(e_2,e_2)\rangle-(n-1)\beta\langle Ch(e_2,e_2),h(e_1,e_1)\rangle\}$$
$$=\frac{n-1}{k\alpha^3}\{-(n-1)\beta-\gamma\}\langle Ch(e_2,e_2),h(e_1,e_1)\rangle$$
$$=\frac{n-1}{k\alpha}\langle Ch(e_2,e_2),h(e_1,e_1)\rangle.$$

Differentiating $Ce_2 = 0$ in the direction e_2, we have

$$Ch(e_2, e_2) = -\omega_2^1(e_2)Ce_1 = \frac{\delta k}{(\beta - \gamma)(n-1)}Ce_1.$$

From this and (2.21) we obtain

$$\langle Ce_{n+1}, e_{n+2}\rangle = \frac{\delta}{\alpha(\beta - \gamma)}\langle Ce_1, h(e_1, e_1)\rangle. \tag{2.22}$$

Differentiating $\langle Ae_1, e_1\rangle = \langle Be_1, e_1\rangle = \gamma$ in the direction e_1, where $B = \frac{1}{2}(A + {}^tA)$, we have $\langle Be_1, h(e_1, e_1)\rangle = 0$. Thus we have

$$\langle Ce_1, h(e_1, e_1)\rangle = \langle Ae_1, h(e_1, e_1)\rangle = \delta k \quad \text{by (2.4) and (2.11).}$$

From this and (2.22) we have

$$\langle Ce_{n+1}, e_{n+2}\rangle = \frac{\delta^2 k}{\alpha(\beta - \gamma)}. \tag{2.23}$$

Since $\langle Ae_{n+2}, e_{n+1}\rangle = \langle Ae_{n+1}, e_{n+2}\rangle - 2\langle Ce_{n+1}, e_{n+2}\rangle$, from (2.18) and (2.23) we have the desired equality (2.20). Since $\sum_{i=1}^{n+2}\langle Ae_i, e_i\rangle = \gamma + (n-1)\beta - \frac{\gamma^2 + (n-1)\beta^2 + \delta^2}{\alpha^2} + \langle Ae_{n+2}, e_{n+2}\rangle = \operatorname{tr} A$, we have

$$-\frac{\delta^2}{\alpha^2} + \langle Ae_{n+2}, e_{n+2}\rangle = \text{constant}. \tag{2.24}$$

Differentiating this in the direction e_1, we get

$$-2\delta\frac{e_1\delta}{\alpha^2} + \langle A\overline{\nabla}_{e_1}e_{n+2}, e_{n+2}\rangle + \langle Ae_{n+2}, \overline{\nabla}_{e_1}e_{n+2}\rangle = 0.$$

From this and (2.13), we obtain

$$2\delta\frac{e_1\delta}{\alpha^2} + k\langle Ae_1, e_{n+2}\rangle + k\langle Ae_{n+2}, e_1\rangle$$
$$+ \frac{\delta}{\alpha}\{\langle Ae_{n+1}, e_{n+2}\rangle + \langle Ae_{n+2}, e_{n+1}\rangle\} = 0.$$

From (2.4), (2.18), (2.19), (2.20) and this, we have

$$\delta(-2e_1\delta + \beta k) = 0.$$

If $\delta = 0$, then from (2.3), (2.4) and (2.23) we can see that A is symmetric. This contradicts to the assumption. Hence we have

$$e_1\delta = \frac{1}{2}\beta k. \tag{2.25}$$

Then from (2.19), (2.24) and (2.25) we have

$$-\frac{\delta^2}{\alpha^2} + \beta - \frac{\delta^2}{\beta-\gamma} - \frac{(n-1)\beta}{k\alpha^2}\{(\gamma - \frac{1}{2}\beta)k + \frac{\delta^2 k}{\beta-\gamma}\} = \text{constant}.$$

Thus

$$\frac{2\gamma-\beta}{\alpha^2(\beta-\gamma)}\delta^2 = \text{constant}.$$

If $2\gamma \neq \beta$, then we have $\delta = \text{constant}$ and $k = 0$ from (2.25). This is a contradiction to the asssumption dim $(Im\ h) = 2$. Hence we must have

$$2\gamma = \beta. \tag{2.26}$$

From (2.18), (2.19), (2.25) and (2.26) we have

(2.27)
$$Ae_{n+1} = -\frac{2\delta k}{\alpha}e_1 - \frac{(4n-3)\gamma^2 + \delta^2}{\alpha^2} + \frac{\delta^2 k}{\alpha\gamma}e_{n+2},$$

(2.28)
$$Ae_{n+2} = -\frac{2n-3}{2n-1}\delta e_1 + \frac{2(n-1)\gamma(\delta^2-\gamma^2)}{\alpha^3 k}e_{n+1} + (2\gamma + \frac{\delta^2}{\alpha^2})e_{n+2}.$$

From (2.20), (2.25), (2.26) and (2,28) we have

$$\langle Ae_{n+2}, e_{n+1}\rangle = -\frac{\delta^2 k}{\alpha\gamma} = \frac{2(n-1)\gamma(\delta^2-\gamma^2)}{\alpha^3 k}.$$

Hence we have

$$\delta^2 k^2 = -\frac{2(n-1)\gamma^2(\delta^2-\gamma^2)}{\alpha^2}. \tag{2.29}$$

Thus, from (2.28) and (2.29) we have

$$Ae_{n+2} = -\frac{2n-3}{2n-1}\delta e_1 - \frac{\delta^2 k}{\alpha\gamma}e_{n+1} + (2\gamma + \frac{\delta^2}{\alpha^2})e_{n+2}. \tag{2.30}$$

From (2.2), (2.3), (2.27) and (2.30) we get a representation of A with respect to $e_1, \cdots, e_n, e_{n+1}, e_{n+2}$ such that

$$\begin{pmatrix} \gamma & 0 & 0 & \cdots & 0 & -\frac{2\delta k}{\alpha} & -\frac{2n-3}{2n-1}\delta \\ 0 & 2\gamma & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2\gamma & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\gamma & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{(4n-3)\gamma^2+\delta^2}{\alpha^2} & -\frac{\delta^2 k}{\alpha\gamma} \\ \delta & 0 & 0 & \cdots & 0 & \frac{\delta^2 k}{\alpha\gamma} & 2\gamma + \frac{\delta^2}{\alpha^2} \end{pmatrix}.$$

Hence we have the characteristic polynomial of A,

$$\det(A - tI) = (2\gamma - t)^{n-1} \begin{vmatrix} \gamma - t & -\frac{2\delta k}{\alpha} & -\frac{2n-3}{2n-1}\delta \\ 0 & -\frac{(4n-3)\gamma^2+\delta^2}{\alpha^2} - t & -\frac{\delta^2 k}{\alpha\gamma} \\ \delta & \frac{\delta^2 k}{2\gamma} & 2\gamma + \frac{\delta^2}{\alpha^2} - t \end{vmatrix}.$$

By (2.1), $(2\gamma - t)^n$ must divide $\det(A - tI)$. Thus

$$\begin{vmatrix} -\gamma & -\frac{2\delta k}{\alpha} & -\frac{2n-3}{2n-1}\delta \\ 0 & -\frac{(4n-3)\gamma^2+\delta^2}{\alpha^2} - 2\gamma & -\frac{\delta^2 k}{\alpha\gamma} \\ \delta & \frac{\delta^2 k}{\alpha\gamma} & \frac{\delta^2}{\alpha^2} \end{vmatrix} = 0.$$

Thus we have

$$\delta^2 \begin{vmatrix} -\gamma & -\frac{2\delta k}{\alpha} & -\frac{2n-3}{2n-1} \\ 0 & \frac{\gamma^2-\delta^2}{\alpha^2} & -\frac{\delta k}{\alpha\gamma} \\ 1 & \frac{\delta k}{\alpha\gamma} & \frac{1}{\alpha^2} \end{vmatrix} = 0.$$

Hence $\delta^2(3\delta^2k^2 + \frac{2n-4}{2n-1}\gamma(\gamma^2 - \delta^2)) = 0$. Since $\delta \neq 0$, from this and (2.29) we have

$$k^2(3 - \frac{n-2}{n-1}) = 0.$$

Since $k \neq 0$, we have $3(n-1) - (n-2) = 0$. This is impossible. Thus we must have $\dim(Im\ h) = 1$. Going back to (2.1) and differentiating in the direction $e_j (j = 1, \cdots, n)$, we have

$$\overline{\nabla}_{e_j} Ce_1 = Ch(e_j, e_1) + C\nabla_{e_j} e_1, \tag{2.31}$$

$$\overline{\nabla}_{e_j} Ce_2 = Ch(e_j, e_2) + C\nabla_{e_j} e_2 = 0. \tag{2.32}$$

From (2.32) we can deduce that $Ch(e_j, e_2)$ is normal and parallel to Ce_1 for every j. Since $\dim(Im\ h)$=1, $Ch(e_j, e_1)$ is also parpllel to Ce_1. Thus we can conclude that $\overline{\nabla}_{e_j} Ce_1$ is parallel to Ce_1 from (2.32). This means that the unit normal vector $\frac{Ce_1}{|Ce_1|}$ is a constant vector in E^{n+2}. So every component of W is contained in a hyperplane of E^{n+2}. By continuity M^n must be contained in a hyperplane of E^{n+2}. The classification thorem for hypersurfaces satisfying (*) says that M^n is an ordinary sphere [1,5,11]. □

PROPOSITION 2.2. *If M^n satisfies $\Delta x = Ax$ for a nonsymmetric matrix A, then M^n is an ordinary sphere.*

Proof. If the matrix A is singular, then M^n is an ordinary sphere by Lemma 1.2. So we assume that A is nonsingular. Lemma 1.5 implies that the rank of $C = \frac{1}{2}(A - {}^tA)$ is 2. Without loss of generality we may assume that the skew symmetric matrix C is of the following form

$$\begin{pmatrix} C' & 0 \\ 0 & 0 \end{pmatrix},$$

where $C' = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ is a 2×2 matrix. By similar discussion to that in the first part of proof of Lemma 2.1 we can choose local orthonormal frame $e_1, \cdots, e_n$ of M^n such that

$$Ce_1 \neq 0, \quad Ce_i = 0 \quad \text{for } i = 2, \cdots, n. \tag{2.33}$$

Now consider the projection map

$$M^n \longrightarrow E^{n+1} \subseteq E^{n+2}$$

given by $(x_1, x_2, \cdots, x_{n+2}) \longrightarrow (x_1, \cdots, x_{n+1})$, where x_i are the ith coordinate function of M^n. Suppose that the rank of this map is $n-1$ at every point of M^n. Then $E_{n+2} = (0, 0, \cdots, 0, 1)$ is tangential to M^n. Thus we have $\langle Ax, AE_{n+2}\rangle = 0$. Differentiating this in the direction E_{n+2}, we get $\langle AE_{n+2}, AE_{n+2}\rangle = 0$ and hence $AE_{n+2} = 0$. This contradicts to the assumption that A is nonsingular. Thus we may assume that the rank of projecton $(x_1, \cdots, x_{n+2}) \longrightarrow (x_1, \cdots, x_{n+1})$ is n at a point $p \in M^n$. (2.33) implies that the rank of projection $(x_1, x_2, \cdots x_{n+2}) \longrightarrow (x_1, x_2)$ is 1. Using this and the inverse mapping theorem, we may assume that M^n is locally the graph of $(y_1, f(y_1), y_2, \cdots, y_n, g(y_1, \cdots, y_n))$, where $y_1, \cdots, y_n$ are arbitrary, f is a function of y_1 and g is a suitable function of $y_1, \cdots, y_n$. Now we can observe that

$$\frac{\partial}{\partial y_i}\langle Ax, x\rangle = 0, \quad i = 2, \cdots, n.$$

Thus

$$\langle Ax, x\rangle = r(y_1),$$

where r is a function of y_1. Thus, for a constant c, on the submanifold $V = \{(c, f(c), y_2, \cdots, y_n, g(c, y_2, \cdots, y_n)\}$ of M^n, we have

$$\langle Ax, x\rangle = r(c), \quad \langle {}^tAAx, x\rangle = \alpha^2.$$

where $x = (c, f(c), y_2, \cdots, y_n, g)$ and α is a constant. Thus, for $A = (a_{ij})$ and ${}^tAA = (b_{ij})$,

$$a_{11}c^2 + a_{22}\{f(c)\}^2 + (a_{12} + a_{21})c \cdot f(c) + \sum_{i=3}^{n+2} a_{ii}y_{i-1}^2 + 2\sum_{i=3}^{n+2} a_{1i}cy_{i-1}$$
$$+ 2\sum_{i=3}^{n+2} a_{2i}f(c)y_{i-1} + 2\sum_{\substack{i<j \\ 3\le i\le n+1}} a_{ij}y_{i-1}y_{j-1} = r(c).$$

$$b_{11}c^2 + b_{22}\{f(c)\}^2 + 2b_{12}c \cdot f(c) + \sum_{i=3}^{n+2} b_{ii}y_{i-1}^2 + 2\sum_{i=3}^{n+2} b_{1i}cy_{i-1}$$
$$+ 2\sum_{i=3}^{n+2} b_{2i}f(c)y_{i-1} + 2 \sum_{\substack{i<j \\ 3\leq i\leq n+1}} b_{ij}y_{i-1}y_{j-1} = \alpha^2$$

where $y_{n+1} = g$.

Consider the polynomial $P(u_1, u_2, \cdots, u_n)$ and $Q(u_1, \cdots, u_n)$ given by

$$P = \sum_{i=3}^{n+2} a_{ii}u_{i-2}^2 + 2\sum_{i=3}^{n+2} a_{1i}cu_{i-2} + 2\sum_{i=3}^{n+2} a_{2i}f(c)u_{i-2}$$
$$+2 \sum_{\substack{i<j \\ 3\leq i\leq n+1}} a_{ij}u_{i-2}u_{j-2}+a_{11}c^2+a_{22}\{f(c)\}^2+(a_{12}+a_{21})c\cdot f(c)-r(c),$$

$$Q = \sum_{i=3}^{n+2} b_{ii}u_{i-2}^2 + 2\sum_{i=3}^{n+2} b_{1i}cu_{i-2} + 2\sum_{i=3}^{n+2} b_{2i}f(c)u_{i-2}$$
$$+ 2 \sum_{\substack{i<j \\ 3\leq i\leq n+1}} b_{ij}u_{i-2}u_{j-2} + b_{11}c^2 + b_{22}\{f(c)\}^2 + 2b_{12}c \cdot f(c) - \alpha^2.$$

If $P(u_1, \cdots, u_n), Q(u_1, \cdots, u_n)$ have no common factors in the polynomial ring $R[u_1, \cdots, u_{n-1}][u_n]$ over $R[u_1, \cdots, u_{n-1}]$, so they also have no common factors in $R(u_1, \cdots u_{n-1})[u_n]$. Since $R(u_1, \cdots, u_{n-1})[u_n]$ is a PID, $(P, Q) = 1$ in $R(u_1, \cdots, u_{n-1})[u_n]$, so $rQ + sP = 1$ for some $r, s \in R(u_1, \cdots, u_{n-1})[u_n]$. There is a non-zero $d \in R[u_1, \cdots, u_{n-1}]$ such that $dr = a, ds = b \in R[u_1, \cdots, u_n]$. Therefore $aP + bQ = d$. Since $y_2, \cdots, y_n, g$ satisfy P and Q, we have $d(y_2, \cdots, y_n) = 0$. Since $y_2, \cdots, y_n$ are arbitrary, $d = 0$, which is a contradiction. Hence P and Q have a common factor. If P is reducible, i.e., $P = (c_1u_1 + \cdots + c_nu_n + c_{n+1})(e_1u_1 + \cdots + e_nu_n + e_{n+1})$, where c_i and e_i are constants, then we have $c_1y_2 + \cdots + c_{n-1}y_n + c_ng + c_{n+1} = 0$ or $e_1y_2 + \cdots + e_{n-1}y_n + e_ng +$

$e_{n+1} = 0$. Assume $c_1 y_2 + \cdots + c_{n-1} y_n + c_n g + c_{n+1} = 0$. Differentiating $\langle Ax, A\frac{\partial}{\partial y_2}x\rangle = 0$ on V in y_2, where $\frac{\partial}{\partial y_2}x = (0, 0, 1, 0, \cdots, \frac{\partial g}{\partial y_2})$, we have

$$\langle A\frac{\partial}{\partial y_2}x, A\frac{\partial}{\partial y_2}x\rangle = 0.$$

This imply that $A\frac{\partial}{\partial y_2}x = 0$. Since A is nonsingular, this is a contradiction. Thus Q is a constant multiple of P, i.e. $Q = \beta P$ for some constant β. So we have

$$\begin{aligned} &b_{ii} = \beta a_{ii}, \quad i = 3, \cdots, n+2, \\ &b_{ij} = \beta a_{ij}, \quad i < j, 3 \le i \le n+1, \\ &(b_{1i}c + b_{2i}f(c)) = \beta(a_{1i}c + a_{2i}f(c)), \quad i = 3, \cdots, n+2. \end{aligned} \tag{2.34}$$

Since c is arbitrary, from the third equality in (2.34) we have

$$b_{1i} = \beta a_{1i}, \quad b_{2i} = \beta a_{2i}, \quad i = 3, \cdots, n+2 \tag{2.35}$$

or $f(y_1) = a y_1$ for a constant a. If (2.35) holds, then

$$({}^tAA - \beta A)E_i = ({}^tAA - \beta\, {}^tA)E_i = 0, \quad i = 3, \cdots, n+2,$$

where $E_i = (\overbrace{0, \cdots, 0}^{i-1}, 1, 0, \cdots, 0)$. Hence $AE_i = \beta E_i$. Thus A must be of the form (2.1). If $f(y_1) = a y_1$, then $U = \{(y_1, f, y_2, \cdots, y_n, g)\}$ is contained in a hyperplane of E^{n+2}. The classification theorem for hypersurfaces satisfying $(*)$ [1,5,11] and the condition that A is nonsingular imply U is an open part of an ordinary sphere. Thus

$$A\frac{\partial}{\partial y_i}x = \beta'\frac{\partial}{\partial y_i}x, \quad i = 2, \cdots, n$$

for a constant β'. Since g is not linear, we find $AE_i = \beta' E_i$ for $i = 3, \cdots, n+2$. So A is of the form in (2.1). Hence M^n is an ordinary sphere by Lemma 2.1. □

Proof of theorem. Suppose that M^n is a compact submanifold of E^{n+2} with constant mean curvature and M^n satisfies $(*)$. Then we may assume that $b = 0$ in $(*)$ by Lemma 1.1. If A is symmetric, then M^n is an ordinary sphere or a minimal hypersurface of a hypersphere in E^{n+2} or a product of spheres by the results of Th. Hasanis and Th. Vlachos [10]. If A is nonsymmetric, then Proposition 2.2 implies that M^n is an ordinary sphere. The converse is an easy computation. □

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