A CLASS OF COMPACT SUBMANIFOLDS
WITH CONSTANT MEAN CURVATURE

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0. Introduction

Let $M^n$ be a connected submanifold of a Euclidean space $E^m$, equipped with the induced metric. Denote by $\Delta$ the Laplacian operator of $M^n$ and by $x$ the position vector. A well-known T. Takahashi's theorem [13] says that $\Delta x = \lambda x$ for some constant $\lambda$ if and only if $M^n$ is either minimal submanifold of $E^m$ or minimal submanifold in a hypersphere of $E^m$. In [9], O. Garay studied the hypersurfaces $M^n$ in $E^{n+1}$ satisfying $\Delta x = Dx$, where $D$ is a diagonal matrix, and he classified such hypersurfaces. Garay's condition can be seen as a generalization of T. Takahashi's condition. But one can observe that Garay's condition is not coordinate-invariant. F. Dillen, J. Pas and L. Verstraelen generalized T. Takahashi's condition, following Garay's idea, in a way which is independent of the choice of coordinates. They considered submanifolds in $E^m$ satisfying

\[(*) \quad \Delta x = Ax + b,\]

where $A$ is a $m \times m$ constant matrix and $b$ is a constant vector in $E^m$, and proved that the only surfaces in $E^3$ satisfying (*) are minimal surfaces, spheres and circular cylinders [4]. Submanifolds satisfying three authors' condition were studied by several authors [1,2,3,5,6,7,8,11]. Recently, Th. Hasanis and Th. Vlachos classified submanifold of codimension 2 with constant mean curvature satisfying (*) for a diagonal

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matrix $A$ and $b = 0$ [12]. In this paper, we are going to deal with a part of the following problem:

Classify submanifolds of codimension 2 with constant mean curvature satisfying $(\ast)$ for a constant matrix $A$ (not necessarily diagonal or symmetric) and a constant vector $b$.

Our main result is given by the following.

**Theorem.** Let $M^n$ be a compact submanifold of $E^{n+2}$ with constant mean curvature. Then $M^n$ satisfies $(\ast)$ for a constant matrix $A$ and a constant vector $b$ in $E^{n+2}$ if and only if $M^n$ is one of the following:

1. an ordinary sphere in $E^{n+1}$,
2. a minimal hypersurface of a hypersphere $S^{n+1}$,
3. a Riemann product $S^n(r_1) \times S^{n-m}(r_2)$, where $1 \leq m \leq n - 1$ and $\frac{r_1}{r_2} \neq \sqrt{\frac{m}{n-m}}$.

1. Preliminaries

Let $M^n$ be an $n$-dimensional submanifold of $E^m$. We denote by $\nabla, \nabla, h, H$ and $x$, the Euclidean connection of $E^m$, the induced connection, the second fundamental form, the mean curvature vector and the position vector of $M^n$, respectively. Then the Gauss formula is obtained by

$$\nabla_X Y = \nabla_X Y + h(X, Y) \quad (1.1)$$

for vector fields $X, Y$ tangent to $M^n$. Let $e_1, e_2, \cdots, e_m$ be an adapted local orthonormal frame in $E^m$ such that $e_1, \cdots, e_n$ tangent to $M^n$ and $e_{n+1}, e_{n+2}, \cdots, e_m$ normal to $M^n$. Then we have

$$\Delta x = \sum_{i=1}^{n} h(e_i, e_i) = H. \quad (1.2)$$

The connection form $\omega^B_A$ are defined by

$$de_A = \sum \omega^B_A e_B, \ A, B, C, \cdots = 1, \cdots, m. \quad (1.3)$$

Then we have $\omega^A_B = -\omega^B_A$. We will use the same notation $\langle \ , \ \rangle$ for the induced metric of $M^n$ and that of $E^m$. We need the following lemmas.
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**Lemma 1.1.** [4] Let \( M^n \) be a compact submanifold of \( E^m \) and let \( M^n \) satisfy \((*)\). Then we may assume \( b = 0 \).

**Proof.** Now assume that \( M^n \) is compact and satisfies \((*)\). Integrating \((*)\) over \( M^n \) and using divergence theorem, implies that

\[
Ax_0 + b = 0
\]

for a constant vector \( x_0 \) in \( E^{n+2} \). Using this, then \((*)\) further implies that

\[
\Delta(x - x_0) = A(x - x_0).
\]

\( \square \)

**Lemma 1.2.** Let \( M^n \) be a compact submanifold of \( E^{n+2} \) satisfying \((*)\) for a singular matrix \( A \). Then \( M^n \) is an ordinary sphere in a hyperplane of \( E^{n+2} \).

**Proof.** By Lemma 1.1, we may assume \( b = 0 \). Since \( A \) is singular, we may assume that \( A_{n+2} = \alpha_1 A_1 + \cdots + \alpha_{n+1} A_{n+1} \), where \( A_i \) are the \( i \)th row vector of \( A \) and \( \alpha_i \) are suitable constants. Then \((*)\) implies that

\[
\Delta x_{n+2} = \langle A_{n+2}, x \rangle = \langle \sum_{i=1}^{n+1} \alpha_i A_i, x \rangle = \sum_{i=1}^{n+1} \alpha_i \Delta x_i,
\]

where \( x_i \) are the \( i \)th coordinate function of \( M^n \). So we have

\[
\Delta(x_{n+2} - \sum \alpha_i x_i) = 0.
\]

This and Hopf's lemma imply that

\[
x_{n+2} - \sum \alpha_i x_i = \text{constant}.
\]

Thus \( M^n \) is contained in a hyperplane of \( E^{n+2} \). The conclusion follows from the classification theorem of hypersurfaces satisfying \((*)\) [1,5,11].

\( \square \)

**Lemma 1.3.** Let \( M^n \) be a submanifold of \( E^m \) satisfying \((*)\). Then every vector field \( CX \), where \( C \) is the skew symmetric matrix \( \frac{1}{2} (A - \text{tr} A) \) and \( X \) is a local tangent vector of \( M^n \), is normal to \( M^n \).
Proof. Since $Ax + b$ is normal to $M^n$, we have

$$\langle Ax + b, X \rangle = 0, \quad \langle Ax + b, Y \rangle = 0$$

for local tangent vectors $X$, $Y$ of $M^n$. Differentiating these in $Y, X$ respectively, by (1.1) we have

$$\langle AY, X \rangle + \langle Ax + b, h(Y, X) \rangle = 0$$
$$\langle AX, Y \rangle + \langle Ax + b, h(X, Y) \rangle = 0.$$

From these and the symmetry of $h$, we can conclude that $CX$ is normal to $M^n$ for a tangent vector $X$. 

Lemma 1.4. Under the same hypothesis of Lemma 1.3, we have the following formula

$$\sum_{i=1}^{n} \langle Ae_i, e_i \rangle = -\langle Ax + b, Ax + b \rangle,$$

where $e_1, \ldots, e_n$ are local orthonormal tangent frame of $M^n$.

Proof. By similar method to that in the proof of Lemma 1.3, we have

$$\langle Ae_i, e_i \rangle = -\langle Ax + b, h(e_i, e_i) \rangle$$

for $i = 1, \ldots, n$. By summation we obtain

$$\sum_{i=1}^{n} \langle Ae_i, e_i \rangle = -\langle Ax + b, \sum_{i=1}^{n} h(e_i, e_i) \rangle.$$

From this and (1.2), the conclusion follows.

Lemma 1.5. Let $M^n$ be a compact submanifold of $E^{n+2}$ satisfying $(\ast)$. Then the rank of the skew symmetric matrix $C = \frac{1}{2}(A - {}^tA)$ is at most 2.
Proof. Suppose $b = 0$ (Lemma 1.1). Since $M^n$ is compact, the function $\langle Ax, x \rangle$ has minimum and maximum. So we may assume that the function $\langle Ax, x \rangle$ has an extremal value different from zero at a point $p(\in M^n)$. In a local orthonormal tangent frame $e_1, e_2, \cdots, e_n$ on a neighborhood of $p$, differentiating $\langle Ax, x \rangle$, we have

$$e_i \langle Ax, x \rangle = 0 \quad \text{at} \quad p.$$ 

This imply that

$$\langle Ae_i, x \rangle = \langle e_i, ^tAx \rangle = 0 \quad \text{at} \quad p.$$ 

Thus $\langle Ce_i, x \rangle = 0$ at $p$. Decomposing $x$ as $x^N + x^T$, where $x^N$ is normal to $M^n$ and $x^T$ is tangent to $M^n$, we have

$$\langle Ce_i, x^N \rangle = \langle Cx^N, x^N \rangle = 0 \quad \text{at} \quad p.$$ 

From these, considering $Ce_1, Ce_2, \cdots, Ce_n, Cx^N$ are normal to $M^n$ and $x^N \neq 0$ at $p$, we can see that the rank of $C$ is at most 2. 

\[\square\]

2. Proof of theorem

Throughout this section we will assume that $M^n$ is a compact submanifold of $E^{n+2}$ with constant mean curvature.

**Lemma 2.1.** If $M^n$ satisfies $\Delta x = Ax$ for an $(n + 2) \times (n + 2)$ nonsymmetric matrix $A$ of the following form

\[
\begin{pmatrix}
a_{11} & a_{12} & 0 \\
a_{21} & a_{22} & 0 \\
0 & 0 & \beta I
\end{pmatrix},
\]

where $\beta I$ is an $n \times n$ diagonal matrix with all diagonal entries $\beta$ and $a_{ij}(i, j = 1, 2)$ are constants, then $M^n$ is an ordinary sphere.

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Proof. Since $A$ is not symmetric, the rank of the skew symmetric matrix $C = \frac{1}{2}(A - tA)$ is 2. Let $E_1, E_2, E_3, \cdots, E_{n+2}$ be the standard basis of $\mathbb{R}^{n+2}$. Then $\ker C$ is spanned by $E_3, \cdots, E_{n+2}$. If the dimension of the normal space $\{CX_p | X_p \in T_p M \}$ at $p (p \in M^n)$ is 2, then $E_3, \cdots, E_{n+2}$ span the tangent space $T_p M$. This is impossible, since $CE_i = 0$ for $i = 3, \cdots, n+2$. Hence the dimension of $\{CX_p | X_p \in T_p M \}$ is at most 1 at every point $p \in M^n$. Now suppose that the dimension of $\{CX | X \in TM \}$ is 0 at every point of an open subset $U$ of $M^n$. Then $E_3, \cdots, E_{n+2}$ span the tangent space of $U$. This means that $M^n$ is a minimal submanifold of $\mathbb{R}^{n+2}$, since the mean curvature of $M^n$ is constant. This is impossible. Thus we may assume that $\dim \{CX | X \in TM^n \}$ is 1 at every point of a dense open subset $W$ of $M^n$. We will work in $W$. Since $\dim \{CX | X \in TW \}$ is 1 at every point of $W$, we can choose a local orthonormal frame $e_1, \cdots, e_n$ of $W$ such that

$$Ce_1 \neq 0, \quad Ce_i = 0 \quad \text{for } i = 2, \cdots, n.$$  

The second equation in (2.2) and (2.1) imply that

$$(2.3) \quad Ae_i = \beta e_i, \quad i = 2, \cdots, n.$$  

Since the mean curvature of $M^n$ is constant, (1.2) implies

$$\langle Ax, Ax \rangle = \langle H, H \rangle = \alpha^2$$

for a positive constant $\alpha$, where $H$ is the mean curvature vector field of $M^n$. By Lemma 1.4 we have $\sum_{i=1}^{n} \langle Ae_i, e_i \rangle = -\langle Ax, Ax \rangle = -\alpha^2$. By this and (2.3) we obtain $\langle Ae_1, e_1 \rangle + (n-1)\beta = -\alpha^2$. This implies that $\langle Ae_1, e_1 \rangle = -\alpha^2 - (n-1)\beta$. We will let $-\alpha^2 - (n-1)\beta = \gamma$. Let $e_{n+1}, e_{n+2}$ be orthonormal normal frame of $M^n$ such that $e_{n+1} = \frac{1}{\alpha} Ax$. Then we have

$$(2.4) \quad Ae_1 = \gamma e_1 + \delta e_{n+2},$$

where $\delta = \alpha \omega^{n+2}_{n+1}(e_1)$, since $\langle Ae_1, Ax \rangle = 0$. Now we will show that the dimension of $\text{Im} h = \{h(X, Y) | X$ and $Y$ are local tangent vectors$\}$ is
1. Suppose that the dimension of $\text{Im} h$ is 2, locally. Differentiating the second equation in (2.2) in the direction $e_j$ \((j = 1, \cdots, n)\), we have

\[
(2.5) \quad C h(e_i, e_j) = -C \nabla_{e_j} e_i, \quad i = 2, \cdots, n.
\]

(2.5) and Lemma 1.3 imply

\[
\langle Ch(e_i, e_j), e_1 \rangle = -\langle h(e_i, e_j), Ce_1 \rangle = 0.
\]

for \(i = 2, \cdots, n\) and \(j = 1, \cdots, n\). Hence \(h(e_i, e_j)\) are parallel each other for \(i = 2, \cdots, n\) and \(j = 1, \cdots, n\). (2.3) and (2.4) imply that

\[
(2.6) \quad \langle Ax, h(e_i, e_i) \rangle = -\langle Ae_i, e_i \rangle = \beta
\]

for \(i = 2, \cdots, n\) and

\[
(2.7) \quad \langle Ax, h(e_i, e_j) \rangle = -\langle Ae_i, e_j \rangle = 0
\]

for \(i \neq j, \quad i, j = 1, \cdots, n\).

Since \(h(e_i, e_j)(i = 1, \cdots, n, j = 2, \cdots, n)\) are parallel each other, (2.6) and (2.7) imply

\[
(2.8) \quad h(e_2, e_2) = \cdots = h(e_n, e_n),
\]

(2.9) \quad \hbox{for } i \neq j.

(2.5) and (2.9) imply

\[
(2.10) \quad \langle \nabla_{e_j} e_i, e_1 \rangle = 0
\]

for \(i \neq j, \quad j = 1, \cdots, n, \quad i = 2, \cdots, n\). From (2.3),(2.4),(2.8) and the equation \(Ax = \alpha e_{n+1} = \sum h(e_i, e_i)\), we have

\[
(2.11) \quad h(e_1, e_1) = -\frac{\gamma}{\alpha} e_{n+1} + k e_{n+2},
\]

(2.12) \quad \hbox{for some function } k. \quad \text{The assumption that } \dim \text{Im } h = 2 \text{ implies that } k \neq 0. \text{ And we have}

\[
(2.13) \quad \nabla_{e_1} e_{n+2} = -k e_1 - \frac{\delta}{\alpha} e_{n+1}
\]
Differentiating (2.4) in the direction $e_1$, we get
\[ Ah(e_1, e_1) = \gamma h(e_1, e_1) + (e_1 \delta) e_{n+2} + \delta \nabla_{e_1} e_{n+2}, \]

since $\nabla_{e_1} e_1 = 0$ by (2.10). Substituting (2.11) and (2.13) into the above equation, we have
\begin{equation}
-\frac{\gamma}{\alpha} A e_{n+1} + k A e_{n+2} = (-\delta k) e_1 + \left(-\frac{\gamma^2 + \delta^2}{\alpha}\right) e_{n+1} + (\gamma k + e_1 \delta) e_{n+2}.
\end{equation}

Differentiating (2.3) in the direction $e_2$ and using (2.4), we have
\[ Ah(e_2, e_2) = \beta h(e_2, e_2) + (\beta - \gamma) \omega^1_2(e_2) e_1 - \omega^1_2(e_2) \delta e_{n+2}. \]

Substituting (2.12) into the above equation, we get
\begin{equation}
-\frac{\beta}{\alpha} A e_{n+1} - \frac{k}{n-1} A e_{n+2}
= (\gamma - \beta) \omega^1_2(e_2) e_1 - \frac{\beta^2}{\alpha} e_{n+1} + \left(-\frac{\beta k}{n-1} - \omega^1_2(e_2) \delta\right) e_{n+2}.
\end{equation}

Differentiating $\langle A e_1, e_2 \rangle = 0$ in the direction $e_2$, we have
\[ \langle A \nabla_{e_2} e_1, e_2 \rangle + \langle A e_1, h(e_2, e_2) \rangle + \langle A e_1, \nabla_{e_2} e_2 \rangle = 0. \]

From (2.3), (2.4), (2.9) and this it follows that
\[ \langle A e_1, h(e_2, e_2) \rangle = \omega^1_2(e_2)(\beta - \gamma). \]

Thus
\begin{equation}
-\frac{\delta k}{n-1} = \omega^1_2(e_2)(\beta - \gamma).
\end{equation}

If $\beta = \gamma$, then (2.16) implies that $\delta = 0$. Then we can deduce that
\[ A e_{n+1} = \beta e_{n+1} \quad \text{and} \quad A e_{n+2} = \beta e_{n+2} \]
from (2.14) and (2.15). This contradicts to the assumption $A$ is not symmetric. Thus we have
\[ \omega^1_2(e_2) = \frac{1}{\beta - \gamma} \left(-\frac{\delta k}{n-1}\right). \]

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Substituting this into (2.15), we have

\[(2.17) \quad -\frac{\beta}{\alpha} Ae_{n+1} - \frac{k}{n-1} Ae_{n+2} = (\gamma - \beta) \omega_2^1 (e_2) e_1 - \frac{\beta^2}{\alpha} e_{n+1} + \left( - \frac{\beta k}{n-1} + \frac{\delta^2 k}{(\beta - \gamma)(n-1)} \right) e_{n+2}.\]

From (2.14), (2.17) and \(\alpha^2 = -\gamma - (n-1)\beta\), we have

\[(2.18) \quad Ae_{n+1} = -\frac{2\delta k}{\alpha} e_1 - \gamma^2 + (n-1)\beta^2 + \delta^2 e_{n+1} + \frac{1}{\alpha} \left\{ e_1 \delta + (\gamma - \beta) k + \frac{\delta^2 k}{\beta - \gamma} \right\} e_{n+2},\]

\[(2.19) \quad Ae_{n+2} = \frac{2(n-1)\beta \delta}{\alpha^2} + \delta e_1 + \frac{(n-1)\beta (\gamma^2 - \beta \gamma + \delta^2)}{k \alpha^3} e_{n+1} + \left\{ \beta - \frac{\delta^2}{\beta - \gamma} - \frac{(n-1)\beta}{k \alpha^2} (e_1 \delta + (\gamma - \beta) k + \frac{\delta^2 k}{\beta - \gamma}) \right\} e_{n+2}.\]

Now we will derive the following equality

\[(2.20) \quad \langle Ae_{n+2}, e_{n+1} \rangle = \frac{1}{\alpha} \left\{ e_1 \delta + (\gamma - \beta) k - \frac{\delta^2 k}{\beta - \gamma} \right\}.\]

Using \(e_{n+1} = \frac{1}{\alpha} \{ h(e_1, e_1) + (n-1) h(e_2, e_2) \} \) and \(e_{n+2} = \frac{n-1}{k \alpha^3} \{ \gamma h(e_2, e_2) - \beta h(e_1, e_1) \}\), we have

\[(2.21) \quad \langle Ce_{n+1}, e_{n+2} \rangle = \frac{n-1}{k \alpha^3} \langle Ch(e_1, e_1) + (n-1) Ch(e_2, e_2), \gamma h(e_2, e_2) - \beta h(e_1, e_1) \rangle \]

\[= \frac{n-1}{k \alpha^3} \{ \gamma \langle Ch(e_1, e_1), h(e_2, e_2) \rangle - (n-1) \beta \langle Ch(e_2, e_2), h(e_1, e_1) \rangle \} \]

\[= \frac{n-1}{k \alpha^3} \{ -(n-1) \beta - \gamma \} \langle Ch(e_2, e_2), h(e_1, e_1) \rangle \]

\[= \frac{n-1}{k \alpha} \langle Ch(e_2, e_2), h(e_1, e_1) \rangle.\]
Differentiating $Ce_2 = 0$ in the direction $e_2$, we have

$$Ch(e_2, e_2) = -\omega^1_2(e_2)Ce_1 = \frac{\delta k}{(\beta - \gamma)(n-1)} Ce_1.$$ 

From this and (2.21) we obtain

$$\langle Ce_{n+1}, e_{n+2} \rangle = \frac{\delta}{\alpha(\beta - \gamma)} \langle Ce_1, h(e_1, e_1) \rangle. \quad (2.22)$$

Differentiating $\langle Ae_1, e_1 \rangle = \langle Be_1, e_1 \rangle = \gamma$ in the direction $e_1$, where $B = \frac{1}{2}(A + \tau A)$, we have $\langle Be_1, h(e_1, e_1) \rangle = 0$. Thus we have

$$\langle Ce_1, h(e_1, e_1) \rangle = \langle Ae_1, h(e_1, e_1) \rangle = \delta k \quad \text{by (2.4) and (2.11).}$$

From this and (2.22) we have

$$\langle Ce_{n+1}, e_{n+2} \rangle = \frac{\delta^2 k}{\alpha(\beta - \gamma)}. \quad (2.23)$$

Since $\langle Ae_{n+2}, e_{n+1} \rangle = \langle Ae_{n+1}, e_{n+2} \rangle - 2\langle Ce_{n+1}, e_{n+2} \rangle$, from (2.18) and (2.23) we have the desired equality (2.20). Since $\sum_{i=1}^{n+2} \langle Ae_i, e_i \rangle = \gamma + (n-1)\beta - \frac{\gamma^2 + (n-1)\beta^2 + \delta^2}{\alpha^2} + \langle Ae_{n+2}, e_{n+2} \rangle = \text{tr } A$, we have

$$-\frac{\delta^2}{\alpha^2} + \langle Ae_{n+2}, e_{n+2} \rangle = \text{constant.} \quad (2.24)$$

Differentiating this in the direction $e_1$, we get

$$-2\delta \frac{e_1 \delta}{\alpha^2} + \langle A\nabla e_1, e_{n+2}, e_{n+2} \rangle + \langle Ae_{n+2}, \nabla e_1, e_{n+2} \rangle = 0.$$ 

From this and (2.13), we obtain

$$2\delta \frac{e_1 \delta}{\alpha^2} + k\langle Ae_1, e_{n+2} \rangle + k\langle Ae_{n+2}, e_1 \rangle + \frac{\delta}{\alpha} \{ \langle Ae_{n+1}, e_{n+2} \rangle + \langle Ae_{n+2}, e_{n+1} \rangle \} = 0.$$
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From (2.4), (2.18), (2.19), (2.20) and this, we have

\[ \delta(-2e_1\delta + \beta k) = 0. \]

If \( \delta = 0 \), then from (2.3), (2.4) and (2.23) we can see that \( A \) is symmetric. This contradicts to the assumption. Hence we have

\[ (2.25) \quad e_1\delta = -\frac{1}{2} \beta k. \]

Then from (2.19), (2.24) and (2.25) we have

\[-\frac{\delta^2}{\alpha^2} + \beta - \frac{\delta^2}{\beta - \gamma} - \frac{(n-1)\beta}{k\alpha^2}\left\{\left(\gamma - \frac{1}{2} \beta\right)k + \frac{\delta^2 k}{\beta - \gamma}\right\} = \text{constant}.\]

Thus

\[ \frac{2\gamma - \beta}{\alpha^2(\beta - \gamma)}\delta^2 = \text{constant}. \]

If \( 2\gamma \neq \beta \), then we have \( \delta = \text{constant} \) and \( k = 0 \) from (2.25). This is a contradiction to the assumption \( \dim(Im\ h) = 2 \). Hence we must have

\[ (2.26) \quad 2\gamma = \beta. \]

From (2.18), (2.19), (2.25) and (2.26) we have

\[ (2.27) \quad Ae_{n+1} = -\frac{2\delta k}{\alpha} e_1 - \frac{(4n-3)\gamma^2 + \delta^2}{\alpha^2} e_1 + \frac{\delta^2 k}{\alpha \gamma} e_{n+2}, \]

\[ (2.28) \quad Ae_{n+2} = -\frac{2n - 3}{2n - 1} \delta e_1 + \frac{2(n - 1)\gamma(\delta^2 - \gamma^2)}{\alpha^3 k} e_{n+1} + (2\gamma + \frac{\delta^2}{\alpha^2}) e_{n+2}. \]

From (2.20), (2.25), (2.26) and (2.28) we have

\[ \langle Ae_{n+2}, e_{n+1} \rangle = -\frac{\delta^2 k}{\alpha \gamma} = \frac{2(n - 1)\gamma(\delta^2 - \gamma^2)}{\alpha^3 k}. \]
Hence we have
\[(2.29) \quad \delta^2 k^2 = -\frac{2(n-1)\gamma^2(\delta^2 - \gamma^2)}{\alpha^2}.\]

Thus, from (2.28) and (2.29) we have
\[(2.30) \quad Ae_{n+2} = -\frac{2n-3}{2n-1}\delta e_1 - \frac{\delta^2 k}{\alpha\gamma} e_{n+1} + (2\gamma + \frac{\delta^2}{\alpha^2})e_{n+2}.\]

From (2.2), (2.3), (2.27) and (2.30) we get a representation of $A$ with respect to $e_1, \cdots, e_n, e_{n+1}, e_{n+2}$ such that
\[
\begin{pmatrix}
\gamma & 0 & 0 & \cdots & 0 & -\frac{2\delta k}{\alpha} & -\frac{2n-3}{2n-1}\delta \\
0 & 2\gamma & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 2\gamma & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2\gamma & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -\frac{(4n-3)\gamma^2 + \delta^2}{\alpha^2} & -\frac{\delta^2 k}{\alpha\gamma} \\
\delta & 0 & 0 & \cdots & 0 & \frac{\delta^2 k}{\alpha\gamma} & 2\gamma + \frac{\delta^2}{\alpha^2}
\end{pmatrix}.
\]

Hence we have the characteristic polynomial of $A$,
\[
\det(A - tI) = (2\gamma - t)^{n-1} \begin{vmatrix}
\gamma - t & -\frac{2\delta k}{\alpha} & -\frac{2n-3}{2n-1}\delta \\
0 & -(4n-3)\gamma^2 + \delta^2 & -t \\
\delta & \frac{\delta^2 k}{\alpha\gamma} & 2\gamma + \frac{\delta^2}{\alpha^2} - t
\end{vmatrix}.
\]

By (2.1), $(2\gamma - t)^n$ must divide $\det(A - tI)$. Thus
\[
\begin{vmatrix}
-\gamma & -\frac{2\delta k}{\alpha} & -\frac{2n-3}{2n-1}\delta \\
0 & -(4n-3)\gamma^2 + \delta^2 & -2\gamma \\
\delta & \frac{\delta^2 k}{\alpha\gamma} & 2\gamma + \frac{\delta^2}{\alpha^2}
\end{vmatrix} = 0.
\]

Thus we have
\[
\begin{vmatrix}
\delta^2 & -\gamma & -\frac{2\delta k}{\alpha} & -\frac{2n-3}{2n-1} \\
0 & \gamma^2 - \delta^2 & -\frac{\delta k}{\alpha\gamma} & 1 \\
1 & \frac{\delta k}{\alpha\gamma} & -\frac{1}{\alpha^2}
\end{vmatrix} = 0.
\]
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Hence \( \delta^2 (3 \delta^2 k^2 + \frac{2n-4}{2n-1} \gamma (\gamma^2 - \delta^2)) = 0 \). Since \( \delta \neq 0 \), from this and (2.29) we have

\[
k^2 \left( 3 - \frac{n - 2}{n - 1} \right) = 0.
\]

Since \( k \neq 0 \), we have \( 3(n - 1) - (n - 2) = 0 \). This is impossible. Thus we must have \( \dim(\text{Im } h) = 1 \). Going back to (2.1) and differentiating in the direction \( e_j (j = 1, \cdots, n) \), we have

\[
(2.31) \quad \nabla e_j C e_1 = C h(e_j, e_1) + C \nabla e_j e_1,
\]

\[
(2.32) \quad \nabla e_j C e_2 = C h(e_j, e_2) + C \nabla e_j e_2 = 0.
\]

From (2.32) we can deduce that \( C h(e_j, e_2) \) is normal and parallel to \( C e_1 \) for every \( j \). Since \( \dim(\text{Im } h) = 1 \), \( C h(e_j, e_1) \) is also parallel to \( C e_1 \). Thus we can conclude that \( \nabla e_j C e_1 \) is parallel to \( C e_1 \) from (2.32).

This means that the unit normal vector \( \frac{C e_1}{|C e_1|} \) is a constant vector in \( E^{n+2} \). So every component of \( W \) is contained in a hyperplane of \( E^{n+2} \). By continuity \( M^n \) must be contained in a hyperplane of \( E^{n+2} \). The classification theorem for hypersurfaces satisfying (*) says that \( M^n \) is an ordinary sphere \([1,5,11]\).

\( \square \)

**Proposition 2.2.** If \( M^n \) satisfies \( \Delta x = A x \) for a nonsymmetric matrix \( A \), then \( M^n \) is an ordinary sphere.

**Proof.** If the matrix \( A \) is singular, then \( M^n \) is an ordinary sphere by Lemma 1.2. So we assume that \( A \) is nonsingular. Lemma 1.5 implies that the rank of \( C = \frac{1}{2} (A - A^t) \) is 2. Without loss of generality we may assume that the skew symmetric matrix \( C \) is of the following form

\[
\begin{pmatrix}
C' & 0 \\
0 & 0
\end{pmatrix},
\]

where \( C' = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \) is a \( 2 \times 2 \) matrix. By similar discussion to that in the first part of proof of Lemma 2.1 we can choose local orthonormal frame \( e_1, \cdots, e_n \) of \( M^n \) such that

\[
(2.33) \quad C e_1 \neq 0, \quad C e_i = 0 \quad \text{for } i = 2, \cdots, n.
\]
Now consider the projection map

\[ M^n \rightarrow E^{n+1} \subseteq E^{n+2} \]

given by \((x_1, x_2, \ldots, x_{n+2}) \rightarrow (x_1, \ldots, x_{n+1})\), where \(x_i\) are the \(i\)th coordinate function of \(M^n\). Suppose that the rank of this map is \(n-1\) at every point of \(M^n\). Then \(E_{n+2} = (0, 0, \ldots, 0, 1)\) is tangential to \(M^n\). Thus we have \(\langle Ax, AE_{n+2} \rangle = 0\). Differentiating this in the direction \(E_{n+2}\), we get \(\langle AE_{n+2}, AE_{n+2} \rangle = 0\) and hence \(AE_{n+2} = 0\). This contradicts to the assumption that \(A\) is nonsingular. Thus we may assume that the rank of projecton \((x_1, \ldots, x_{n+2}) \rightarrow (x_1, \ldots, x_{n+1})\) is \(n\) at a point \(p \in M^n\). (2.33) implies that the rank of projection \((x_1, x_2, \cdots x_{n+2}) \rightarrow (x_1, x_2)\) is 1. Using this and the inverse mapping theorem, we may assume that \(M^n\) is locally the graph of \((y_1, f(y_1), y_2, \cdots, y_n, g(y_1, \cdots, y_n))\), where \(y_1, \cdots, y_n\) are arbitrary, \(f\) is a function of \(y_1\) and \(g\) is a suitable function of \(y_1, \cdots, y_n\). Now we can observe that

\[
\frac{\partial}{\partial y_i} \langle Ax, x \rangle = 0, \quad i = 2, \ldots, n.
\]

Thus

\[
\langle Ax, x \rangle = r(y_1),
\]

where \(r\) is a function of \(y_1\). Thus, for a constant \(c\), on the submanifold \(V = \{(c, f(c), y_2, \cdots, y_n, g(c, y_2, \cdots, y_n))\}\) of \(M^n\), we have

\[
\langle Ax, x \rangle = r(c), \quad \langle ^tAAx, x \rangle = \alpha^2.
\]

where \(x = (c, f(c), y_2, \cdots, y_n, g)\) and \(\alpha\) is a constant. Thus, for \(A = (a_{ij})\) and \(^tAA = (b_{ij})\),

\[
a_{11}c^2 + a_{22}\{f(c)\}^2 + (a_{12} + a_{21})c \cdot f(c) + \sum_{i=3}^{n+2} a_{ii}y_{i-1}^2 + 2 \sum_{i=3}^{n+2} a_{i1}cy_{i-1} + 2 \sum_{i=3}^{n+2} a_{2i}f(c)yg_{i-1} + 2 \sum_{i<j}^{3 \leq i \leq n+1} a_{ij}yg_{i-1}y_{j-1} = r(c).
\]
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\[ b_{11}c^2 + b_{22}\{f(c)\}^2 + 2b_{12}c \cdot f(c) + \sum_{i=3}^{n+2} b_{ii}y_{i-1}^2 + 2 \sum_{i=3}^{n+2} b_{1i}cy_{i-1} \]
\[ + 2 \sum_{i=3}^{n+2} b_{2i}f(c)y_{i-1} + 2 \sum_{3 \leq i \leq n+1} \sum_{i < j} b_{ij}y_{i-1}y_{j-1} = \alpha^2 \]

where \( y_{n+1} = g \).

Consider the polynomial \( P(u_1, u_2, \cdots, u_n) \) and \( Q(u_1, \cdots, u_n) \) given by

\[ P = \sum_{i=3}^{n+2} a_{ii}u_{i-2}^2 + 2 \sum_{i=3}^{n+2} a_{1i}cu_{i-2} + 2 \sum_{i=3}^{n+2} a_{2i}f(c)u_{i-2} \]
\[ + 2 \sum_{3 \leq i \leq n+1} \sum_{i < j} a_{ij}u_{i-2}u_{j-2} + a_{11}c^2 + a_{22}\{f(c)\}^2 + (a_{12} + a_{21})c \cdot f(c) - r(c), \]

\[ \]

\[ Q = \sum_{i=3}^{n+2} b_{ii}u_{i-2}^2 + 2 \sum_{i=3}^{n+2} b_{1i}cu_{i-2} + 2 \sum_{i=3}^{n+2} b_{2i}f(c)u_{i-2} \]
\[ + 2 \sum_{3 \leq i \leq n+1} \sum_{i < j} b_{ij}u_{i-2}u_{j-2} + b_{11}c^2 + b_{22}\{f(c)\}^2 + 2b_{12}c \cdot f(c) - \alpha^2. \]

If \( P(u_1, \cdots, u_n), Q(u_1, \cdots, u_n) \) have no common factors in the polynomial ring \( R[u_1, \cdots, u_{n-1}][u_n] \) over \( R[u_1, \cdots, u_{n-1}] \), so they also have no common factors in \( R(u_1, \cdots, u_{n-1})[u_n] \). Since \( R(u_1, \cdots, u_{n-1})[u_n] \) is a PID, \((P, Q) = 1\) in \( R(u_1, \cdots, u_{n-1})[u_n] \), so \( rQ + sP = 1 \) for some \( r, s \in R(u_1, \cdots, u_{n-1})[u_n] \). There is a non-zero \( d \in R[u_1, \cdots, u_{n-1}] \) such that \( dr = a, ds = b \in R[u_1, \cdots, u_n] \). Therefore \( aP + bQ = d \).

Since \( y_2, \cdots, y_n, g \) satisfy \( P \) and \( Q \), we have \( d(y_2, \cdots, y_n) = 0 \). Since \( y_2, \cdots, y_n \) are arbitrary, \( d = 0 \), which is a contradiction. Hence \( P \) and \( Q \) have a common factor. If \( P \) is reducible, i.e., \( P = (c_1u_1 + \cdots + c_{n-1}u_{n-1} + c_{n+1})(e_1u_1 + \cdots + e_{n-1}u_{n-1} + e_{n+1}) \), where \( c_i \) and \( e_i \) are constants, then we have \( c_1y_2 + \cdots + c_{n-1}y_{n-1} + c_{n+1} = 0 \) or \( e_1y_2 + \cdots + e_{n-1}y_{n-1} + e_{n+1} = 0 \).
\[ e_{n+1} = 0. \] Assume \( c_1 y_2 + \cdots + c_{n-1} y_n + c_n g + c_{n+1} = 0. \) Differentiating \( \langle Ax, A \frac{\partial}{\partial y_2} x \rangle = 0 \) on \( V \) in \( y_2 \), where \( \frac{\partial}{\partial y_2} x = (0, 0, 1, 0, \cdots, \frac{\partial}{\partial y_2}) \), we have
\[ \langle A \frac{\partial}{\partial y_2} x, A \frac{\partial}{\partial y_2} x \rangle = 0. \]
This imply that \( A \frac{\partial}{\partial y_2} x = 0. \) Since \( A \) is nonsingular, this is a contradiction. Thus \( Q \) is a constant multiple of \( P \), i.e. \( Q = \beta P \) for some constant \( \beta \). So we have
\[ b_{ii} = \beta a_{ii}, \quad i = 3, \cdots, n+2, \]
\[ b_{ij} = \beta a_{ij}, \quad i < j, 3 \leq i \leq n+1, \]
\[ (b_{1i} c + b_{2i} f(c)) = \beta (a_{1i} c + a_{2i} f(c)), \quad i = 3, \cdots, n+2. \]
Since \( c \) is arbitrary, from the third equality in (2.34) we have
\[ \beta a_{1i} = b_{2i}, \quad i = 3, \cdots, n+2 \]
or \( f(y_1) = ay_1 \) for a constant \( a \). If (2.35) holds, then
\[ (t^t AA - \beta A) E_i = (t^t AA - \beta^t A) E_i = 0, \quad i = 3, \cdots, n+2, \]
where \( E_i = (0, \cdots, 0, 1, 0, \cdots, 0) \). Hence \( AE_i = \beta E_i \). Thus \( A \) must be of the form (2.1). If \( f(y_1) = ay_1 \), then \( U = \{(y_1, f, y_2, \cdots, y_n, g)\} \) is contained in a hyperplane of \( E^{n+2} \). The classification theorem for hypersurfaces satisfying (*) \([1,5,11]\) and the condition that \( A \) is nonsingular imply \( U \) is an open part of an ordinary sphere. Thus
\[ A \frac{\partial}{\partial y_i} x = \beta' \frac{\partial}{\partial y_i} x, \quad i = 2, \cdots, n \]
for a constant \( \beta' \). Since \( g \) is not linear, we find \( AE_i = \beta' E_i \) for \( i = 3, \cdots, n+2 \). So \( A \) is of the form in (2.1). Hence \( M^n \) is an ordinary sphere by Lemma 2.1.

**Proof of theorem.** Suppose that \( M^n \) is a compact submanifold of \( E^{n+2} \) with constant mean curvature and \( M^n \) satisfies (*). Then we may assume that \( b = 0 \) in (*) by Lemma 1.1. If \( A \) is symmetric, then \( M^n \) is an ordinary sphere or a minimal hypersurface of a hypersphere in \( E^{n+2} \) or a product of spheres by the results of Th. Hasanis and Th. Vlachos \([10]\). If \( A \) is nonsymmetric, then Proposition 2.2 implies that \( M^n \) is an ordinary sphere. The converse is an easy computation. \( \Box \)

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