Bull. Korean Math. Soc. **34** (1997), No. 2, pp. 185–191

A NOTE ON ORDERED FILTERS OF IMPLICATIVE SEMIGROUPS

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1. Introduction

The notions of implicative semigroup and ordered filter were introduced by M. W. Chan and K. P. Shum [3]. The first is a generalization of implicative semilattice (see W. C. Nemitz [6] and T. S. Blyth [2]) and has a close relation with the implication in mathematical logic and set theoretic difference (see G. Birkhoff [1] and H. B. Curry [4]). For the general development of implicative semilattice theory the ordered filters play an important role, which is shown by W. C. Nemitz [6]. Motivated by this, M. W. Chan and K. P. Shum [3] established some elementary properties, and constructed quotient structure of implicative semigroups via ordered filters. Y. B. Jun, J. Meng and X. L. Xin [5] discussed the ordered filters of commutative implicative semigroups. For the deep study of implicative semigroups it is necessary to establish more complete theory of ordered filters for it.

In this paper, we introduce a special set in an implicative semigroup. By using this set we give an equivalent condition of an ordered filter. Finally, we prove that an ordered filter can be represented by the union of such sets.

We recall some definitions and results.

By a *negatively partially ordered semigroup* (briefly, *n.p.o. semigroup*) we mean a set S with a partial ordering " $\leq$ " and a binary operation " $\cdot$ " such that for all $x, y, z \in S$, we have:

Received September 16, 1996.
1991 Mathematics Subject Classification: 20M12, 06F05, 06A06, 06A12.
Key words and phrases: Implicative semigroup, ordered filter.
*Supported by the LG Yonam Foundation, 1995.

(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(2) $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$,
(3) $x \cdot y \leq x$ and $x \cdot y \leq y$.

An n.p.o. semigroup $(S; \leq, \cdot)$ is said to be *implicative* if there is an additional binary operation $* : S \times S \to S$ such that for any elements x, y, z of S,

(4) $z \leq x * y$ if and only if $z \cdot x \leq y$.

The operation $*$ is called *implication.* From now on, an implicative n.p.o. semigroup is simply called an *implicative semigroup.*

An implicative semigroup $(S; \leq, \cdot, *)$ is said to be *commutative* if it satisfies

(5) $x \cdot y = y \cdot x$ for all $x, y \in S$,

that is, $(S, \cdot)$ is a commutative semigroup.

In any implicative semigroup $(S; \leq, \cdot, *)$, the following hold: for every $x, y \in S$,

(i) $x * x = y * y$
(ii) $x * x$ is the greatest element, written 1, of $(S, \leq)$.

PROPOSITION 1.1 ([3; Theorem 1.4]). *Let S be an implicative semigroup. Then for every $x, y, z \in S$, the following hold:*

(6) $x \leq 1$, $x * x = 1$, $x = 1 * x$,
(7) $x \leq y * (x \cdot y)$,
(8) $x \leq x * x^2$,
(9) $x \leq y * x$,
(10) *if* $x \leq y$ *then* $x * z \geq y * z$ *and* $z * x \leq z * y$,
(11) $x \leq y$ *if and only if* $x * y = 1$,
(12) $x * (y * z) = (x \cdot y) * z$,
(13) *if S is commutative, then* $x * y \leq (s \cdot x) * (s \cdot y)$ *for all s in S.*

DEFINITION 1.2 ([3; DEFINITION 2.1]). Let S be an implicative semigroup and let F be a nonempty subset of S. Then F is called an *ordered filter* of S if

(F1) $x \cdot y \in F$ for every $x, y \in F$, that is, F is a subsemigroup of S.
(F2) if $x \in F$ and $x \leq y$, then $y \in F$.

The following result gives an equivalent condition of an ordered filter.

PROPOSITION 1.3 ([5; Proposition 2]). *Suppose S is an implicative semigroup. Then a non-empty subset F of S is an ordered filter if and only if it satisfies the following conditions:*

(F3) $1 \in F$,
(F4) $x * y \in F$ *and* $x \in F$ *imply* $y \in F$.

2. Main results

DEFINITION 2.1. Let S be an implicative semigroup and let $x, y \in S$. We define

$$S(x, y) := \{z \in S | x * (y * z) = 1\}.$$

Obviously $1, y \in S(x, y)$ for all $x, y \in S$. Note that if S is a commutative implicative semigroup, then $x \in S(x, y)$ for all $x, y \in S$.

REMARK 2.2. Let S be an implicative semigroup. Then the set $S(x, y)$ is, in general, not an ordered filter of S as shown in the following example.

EXAMPLE 2.3. Consider an implicative semigroup $S := \{1, a, b, c, d, 0\}$ with Cayley tables (Tables 1 and 2) and Hasse diagram (Figure 1) as follows:

$\cdot$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

Table 1

$*$	1	a	b	c	d	0
1	0	a	y	x	v	u
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Table 2

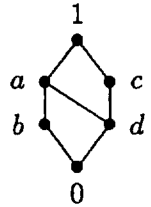

Figure 1

We know that $S(a,b) = \{1,a,b\}$ is an ordered filter of S, but $S(1,a) = \{1,a\}$ is not an ordered filter, since $a * b \in S(1,a)$ and $a \in S(1,a)$, but $b \notin S(1,a)$.

Using (6) and (11) we have the following proposition.

PROPOSITION 2.4. *Let S be an implicative semigroup. If $y \in S$ satisfies $y * z = 1$ for all $z \in S$, then $S(x,y) = S = S(y,x)$ for all $x \in S$.*

EXAMPLE 2.5. Let $S := \{1,a,b,c,d\}$ be an implicative semigroup with Cayley tables (Tables 3 and 4) and Hasse diagram (Figure 2) as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

Table 3

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Table 4

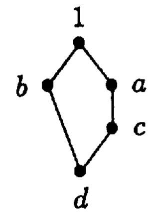

Figure 2

It is easy to check that S satisfies the left self-distributive law under " $*$ ", i.e., $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in S$. By Proposition 2.4 we have $S(x, d) = S(d, x) = S$ for all $x \in S$. Furthermore we know that $S(1,1) = \{1\}$, $S(1, a) = S(a, 1) = S(a, a) = S(a, b) = \{1, a\}$, $S(1, b) = S(b, 1) = S(b, b) = \{1, b\}$, $S(1, c) = S(a, c) = S(c, 1) = S(c, a) = S(c, c) = \{1, a, c\}$, $S(b, a) = \{1, a, b\}$, and $S(c, b) = S$ are ordered filters of S.

The following theorem is a generalization of Example 2.5.

THEOREM 2.6. *Let S be an implicative semigroup satisfying the left self-distributive law under " $*$ ". Then the set $S(x, y)$ is an ordered filter of S for all $x, y \in S$.*

Proof. Let $a * b \in S(x, y)$ and $a \in S(x, y)$. Then $x * (y * (a * b)) = 1$ and $x * (y * a) = 1$. It follows from the left self-distributivity of S that

$$\begin{aligned} 1 &= x * (y * (a * b)) \\ &= x * ((y * a) * (y * b)) \\ &= (x * (y * a)) * (x * (y * b)) \\ &= 1 * (x * (y * b)) \\ &= x * (y * b), && \text{[by (6)]} \end{aligned}$$

whence $b \in S(x, y)$. Therefore $S(x, y)$ is an ordered filter of S. □

By using the set $S(x, y)$ we give an equivalent condition of an ordered filter.

THEOREM 2.7. *Let F be a non-empty subset of a commutative implicative semigroup S. Then F is an ordered filter if and only if*

$S(x,y) \subseteq F$ for all $x, y \in F$.

Proof. Assume that F is an ordered filter and let $x, y \in F$. If $z \in S(x,y)$, then $x*(y*z) = 1 \in F$. Since $x, y \in F$, by (F4) we have $z \in F$. Hence $S(x,y) \subseteq F$.

Conversely, suppose $S(x,y) \subseteq F$ for all $x, y \in F$. Note that $1 \in S(x,y) \subseteq F$. Let $a * b \in F$ and $a \in F$. Since $a * ((a * b) * b) = 1$, we have $b \in S(a, a * b) \subseteq F$ because $a * b \in F$ and $a \in F$. This completes the proof. □

THEOREM 2.8. *If F is an ordered filter of a commutative implicative semigroup S, then $F = \bigcup_{x,y \in F} S(x,y)$.*

Proof. Let F be an ordered filter and let $z \in F$. Clearly, $z \in S(z, 1)$, and hence

$$F \subseteq \bigcup_{z \in F} S(z,1) \subseteq \bigcup_{x,y \in F} S(x,y).$$

Now let $z \in \bigcup_{x,y \in F} S(x,y)$. Then there exist $a, b \in F$ such that $z \in S(a,b)$. It follows from Theorem 2.7 that $z \in F$. This means that $\bigcup_{x,y \in F} S(x,y) \subseteq F$. This completes the proof. □

COROLLARY 2.9. *If F is an ordered filter of a commutative implicative semigroup S, then $F = \bigcup_{x \in F} S(x, 1)$.*

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