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IMPLICATIVE FILTERS OF LATTICE IMPLICATION ALGEBRAS

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1. Introduction

In order to research the logical system whose propositional value is given in a lattice, Y. Xu [4] proposed the concept of lattice implication algebras, and discussed their some properties in [3] and [4]. Y. Xu and K. Qin [5] introduced the notions of filter and implicative filter in a lattice implication algebra, and investigated their properties. In this paper, in the first place, we give an equivalent condition of a filter, and provide some equivalent conditions that a filter is an implicative filter in a lattice implication algebra. By using these results, we construct an extension property for implicative filter.

2. Preliminaries

DEFINITION 1.1 (Xu [4]). By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution " $'$ " and a binary operation " $\rightarrow$ " satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $x \rightarrow x = 1$,
(I3) $x \rightarrow y = y' \rightarrow x'$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,

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for all $x, y, z \in L$

We can define a partial ordering $\leq$ on a lattice implication algebra L by $x \leq y$ if and only if $x \to y = 1$.

EXAMPLE 2.2 (Xu and Qin [5]). Let $L := \{0, a, b, c, 1\}$. Define the partially ordered relation on L as $0 < a < b < c < 1$, and define $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$ for all $x, y \in L$ and " $'$ "and " $\to$ " as follows:

x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	a	c	1	1
1	0	a	b	c	1

Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra.

OBSERVATION (Xu [4]). *In a lattice implication algebra L, the following hold for all $x, y, z \in L$:*

(1) $0 \to x = 1$,
(2) $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$,
(3) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(4) $x \to ((x \to y) \to y) = 1$.

A nonvoid subset J of a lattice L is called a *filter* of L if

(i) $a \in J$, $x \in L$ and $a \leq x$ imply $x \in J$,
(ii) $a \in J$ and $b \in J$ imply $a \wedge b \in J$.

In what follows, L would mean a lattice implication algebra unless otherwise specified.

3. Implicative filters

In [5], Y. Xu and K. Qin defined the notions of filter and implicative filter in a lattice implication algebra.

DEFINITION 3.1 (Xu and Qin [5]). Let $(L, \vee, \wedge, ', \rightarrow)$ be a lattice implication algebra. A subset F of L is called a *filter* of L if it satisfies for all $x, y \in L$:

(F1) $1 \in F$,

(F2) $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$.

A subset F of L is called an *implicative filter* of L, if it satisfies (F1) and

(F3) $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ imply $x \rightarrow z \in F$

for all $x, y, z \in L$.

Denote by $\mathcal{F}(L)$ (resp. $\mathcal{F}_I(L)$) the set of all filters (resp. implicative filters) of L.

The following proposition is clear.

PROPOSITION 3.2. *Every filter F of L has the following property:*

$$x \leq y \text{ and } x \in F \text{ imply } y \in F.$$

PROPOSITION 3.3 (Xu and Qin [5]). *In a lattice implication algebra, every implicative filter is a filter, but not converse.*

In the first place, we give an equivalent condition of a filter.

THEOREM 3.4. *Let F be a non-empty subset of L. Then $F \in \mathcal{F}(L)$ if and only if it satisfies for all $x, y \in F$ and $z \in L$:*

(F4) $x \leq y \rightarrow z$ *implies* $z \in F$.

Proof. Necessity follows from Proposition 3.2 and (F2). Suppose F satisfies (F4). Since $x \leq x \rightarrow I$ for all $x \in F$, we have $I \in F$ by (F4). Let $x \rightarrow y \in F$ and $x \in F$. Using (4) and (F4), we get $y \in F$, whence $F \in \mathcal{F}(L)$. This completes the proof. □

THEOREM 3.5. *Let F be a filter of L such that*

(i) $x \rightarrow (y \rightarrow (y \rightarrow z)) \in F$ *and* $x \in F$ *imply* $y \rightarrow z \in F$

for all $x, y, z \in L$. Then $F \in \mathcal{F}_I(L)$.

Proof. Let $x \to (y \to z) \in F$ and $x \to y \in F$ for all $x, y, z \in L$. Using (I1) and (3) we have

$$x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z)).$$

It follows from Proposition 3.2 that $(x \to y) \to (x \to (x \to z)) \in F$. Since $x \to y \in F$, we get $x \to z \in F$ by (ii). Therefore $F \in \mathcal{F}_I(L)$. This completes the proof. □

We give an equivalent condition that a filter is an implicative filter.

THEOREM 3.6. *Let $F \in \mathcal{F}(L)$. Then the following are equivalent:*
(i) $F \in \mathcal{F}_I(L)$.
(ii) $x \to (x \to y) \in F$ *implies* $x \to y \in F$.
(iii) $x \to (y \to z) \in F$ *implies* $(x \to y) \to (x \to z) \in F$.

Proof. (i) ⇒ (ii) Let $F \in \mathcal{F}_I(L)$ and let $x \to (x \to y) \in F$. Since $x \to x = 1 \in F$, it follows from (F3) that $x \to y \in F$, which proves (ii).

(ii) ⇒ (iii) Suppose (ii) holds and let $x \to (y \to z) \in F$. Using (I1), (2) and (3), we have

$$x \to (y \to z) \leq x \to ((x \to y) \to (x \to z)).$$

Thus, by Proposition 3.2 and (I1), we get

$$x \to (x \to ((x \to y) \to z)) = x \to ((x \to y) \to (x \to z)) \in F.$$

It follows from (ii) and (I1) that

$$x \to ((x \to y) \to z) = (x \to y) \to (x \to z) \in F.$$

(iii) ⇒ (i) Assume that (iii) holds and let $x \to (y \to z) \in F$ and $x \to y \in F$. By (iii), we have $(x \to y) \to (x \to z) \in F$ and $x \to y \in F$. It follows from (F2) that $x \to z \in F$. Thus $F \in \mathcal{F}_I(L)$, ending the proof. □

Let F be a non-empty subset of L and let $a \in L$. Define

$$F_a := \{x \in L | a \to x \in F\}.$$

Note that if $F \in \mathcal{F}(L)$, then $F_1 = F$ and $1 \in F_a$.

REMARK 3.7. Let $F \in \mathcal{F}(L)$. Then there exists $a \in L$ such that $F_a \notin \mathcal{F}(L)$.

EXAMPLE 3.8. Let L be a lattice implication algebra as in Example 2.2. Consider $\{1\} \in \mathcal{F}(L)$. Then $\{1\}_b = \{b, c, 1\} \notin \mathcal{F}(L)$, since $b \to a = c \in \{1\}_b$, but $a \notin \{1\}_b$.

By using F_a, we provide an equivalent condition that a filter is an implicative filter.

THEOREM 3.9. *Let $F \in \mathcal{F}(L)$. Then the following are equivalent:*
(i) $F \in \mathcal{F}_I(L)$.
(ii) $F_a \in \mathcal{F}(L)$ *for all* $a \in L$.

Proof. Let $F \in \mathcal{F}_I(L)$ and let $x, x \to y \in F_a$ for all $a \in L$. Then $a \to (x \to y) \in F$ and $a \to x \in F$. Since $F \in \mathcal{F}_I(L)$, it follows that $a \to y \in F$, i.e., $y \in F_a$. This proves that $F_a \in \mathcal{F}(L)$ for all $a \in L$.

Conversely, suppose $F_a \in \mathcal{F}(L)$ for all $a \in L$. Let $x \to (y \to z) \in F$ and $x \to y \in F$. Then $y \to z \in F_x$ and $y \in F_x$, which imply that $z \in F_x$, i.e., $x \to z \in F$. Hence $F \in \mathcal{F}_I(L)$. This completes the proof. □

COROLLARY 3.10. *Let $F \in \mathcal{F}_I(L)$ and $a \in L$. Then F_a is the least filter of L containing F and a.*

Proof. By Theorem 3.9, $F_a \in \mathcal{F}(L)$. Let G be a filter of L containing F and a. If $x \in F_a$, then $a \to x \in F \subseteq G$. It follows that $x \in G$, whence $F_a \subseteq G$. This completes the proof. □

Finally, we give an extension property for implicative filter.

THEOREM 3.11. (Extension property for implicative filter) *Let $F \in \mathcal{F}_I(L)$. If $G \in \mathcal{F}(L)$ contains F, then $G \in \mathcal{F}_I(L)$.*

Proof. Let $x \to (y \to z) \in G$. Then

$$\begin{aligned} & x \to (y \to ((x \to (y \to z)) \to z)) \\ & = (x \to (y \to z)) \to (x \to (y \to z)) && \text{[by (I1)]} \\ & = 1 \in F. && \text{[by (I2)]} \end{aligned}$$

Since $F \in \mathcal{F}_I(L)$, it follows from Theorem 3.6(iii) that

$$(x \to y) \to (x \to ((x \to (y \to z)) \to z)) \in F,$$

whence

$$\begin{aligned}
&(x \to (y \to z)) \to ((x \to y) \to (x \to z)) \\
&= (x \to y) \to ((x \to (y \to z)) \to (x \to z)) && \text{[by (I1)]} \\
&= (x \to y) \to (x \to ((x \to (y \to z)) \to z)) \in F \subseteq G.
\end{aligned}$$

Since $x \to (y \to z) \in G$ and $G \in \mathcal{F}(L)$, we have $(x \to y) \to (x \to z) \in G$. Thus, by Theorem 3.6, we know that $G \in \mathcal{F}_I(L)$. This completes the proof. □

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