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THE ROLE OF $T(X)$ IN THE IDEAL THEORY OF BCI-ALGEBRAS

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1. Introduction

To develope the theory of BCI-algebrs, the ideal theory plays an important role. The first author [4] introduced the notion of T-ideal in BCI-algebras. In this paper, we first construct a special set, called T-part, in a BCI-algebra X. We show that the T-part of X is a subalgebra of X. We give equivalent conditions that the T-part of X is an ideal. By using T-part, we provide an equivalent condition that every ideal is a T-ideal.

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$ is called a *BCK-algebra*. In any BCI-algebra X one can define a partial order $\leq$ by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X has the following properties for any $x, y, z \in X$:

(1) $x * 0 = x$,
(2) $(x * y) * z = (x * z) * y$,

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(3) $x \leq y$ implies that $x * z \leq y * z$ and $z * y \leq z * x$,
(4) $(x * z) * (y * z) \leq x * y$,
(5) $x * (x * (x * y)) = x * y$,
(6) $0 * (x * y) = (0 * x) * (0 * y)$.

A nonempty subset I of X is called an *ideal* of X if it satisfies

(i) $0 \in I$,
(ii) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Any ideal I has the property: $y \in I$ and $x \leq y$ imply $x \in I$.

In general, an ideal I of X need not be a subalgebra. If I is also a subalgebra of X, we say that I is a *closed ideal*, equivalently, an ideal I is closed if and only if $0 * x \in I$ whenever $x \in I$.

J. Meng and X. L. Xin [3] systematically investigated the theory of atoms and branches of BCI-algebras. An element a of X is called an *atom* if, for all $x \in X$, $x * a = 0$ implies $x = a$, that is, a is a minimal element of $(X, \leq)$. Obviously, 0 is an atom of X. The sets $X_+ := \{x \in X | 0 \leq x\}$ and $L(X) := \{a \in X | \ a$ is an atom of $X\}$ are called *BCK-part* and *p-semisimple part* of X, respectively. We know that $(X_+, *, 0)$ is a BCK-algebra, and $(L(X), *, 0)$ is a p-semisimple BCI-algebra. For any $a \in L(X)$, the set $V(a) := \{x \in X | a \leq x\}$ is called a *branch* of X. It is clear that $V(0) = X_+$.

2. T-parts and T-ideals

We begin with the following definition.

Main Definition. Let X be a BCI-algebra. The set

$$T(X) := \{y \in X | y = (0 * x) * x \text{ for some } x \in X\}$$

is called the *T-part* of X.

Clearly, $0 \in T(X)$.

Theorem 2.1. *Let X be a BCI-algebra. Then $T(X)$ is a subalgebra of X.*

Proof. Let $a, b \in T(X)$. Then $a = (0 * x) * x$ and $b = (0 * y) * y$ for some $x, y \in X$. Thus

$$\begin{aligned}
a * b &= ((0 * x) * x) * ((0 * y) * y) \\
&= ((0 * ((0 * y) * y)) * x) * x && \text{[by (2)]} \\
&= (((0 * (0 * y)) * (0 * y)) * x) * x && \text{[by (6)]} \\
&= (((0 * x) * (0 * y)) * (0 * y)) * x && \text{[by (2)]} \\
&= ((0 * (x * y)) * (0 * y)) * x && \text{[by (6)]} \\
&= ((0 * (0 * y)) * x) * (x * y) && \text{[by (2)]} \\
&= ((0 * x) * (0 * y)) * (x * y) && \text{[by (2)]} \\
&= (0 * (x * y)) * (x * y). && \text{[by (6)]}
\end{aligned}$$

Hence $a * b \in T(X)$, which completes the proof. □

LEMMA 2.2 (MENG AND XIN [3]). *Let X be a BCI-algebra. If $a \in L(X)$, then $a * x \in L(X)$ for all $x \in X$.*

THEOREM 2.3. *If X is a BCI-algebra, then $T(X) \subseteq L(X)$.*

Proof. Let $a \in T(X)$. Then $a = (0*x)*x$ for some $x \in X$. It follows from (2), (5), (6) and Lemma 2.2 that

$$\begin{aligned}
a &= (0 * x) * x \\
&= (0 * (0 * (0 * x))) * x \\
&= (0 * x) * (0 * (0 * x)) \\
&= 0 * (x * (0 * x)) \in L(X).
\end{aligned}$$

Hence $T(X) \subseteq L(X)$. □

Since $L(X)$ is a p-semisimple BCI-algebra, by Theorems 2.1 and 2.3 and [1, Theorem 6] we have

COROLLARY 2.4. *The T-part $T(X)$ of X is an ideal of $L(X)$.*

In general, the T-part $T(X)$ of a BCI-algebra X may not be an ideal of X as shown in the following example.

EXAMPLE 2.5. Consider a BCI-algebra $X := \{0, 1, 2, 3, 4, 5\}$ with Cayley table (Table 1) and Hasse diagram (Figure 1) as follows (see [2]):

*	0	1	2	3	4	5
0	0	0	3	2	3	3
1	1	0	3	2	3	3
2	2	2	0	3	0	0
3	3	3	2	0	2	2
4	4	2	1	3	0	1
5	5	2	1	3	1	0

Table 1

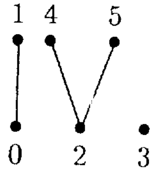

Figure 1

Then $T(X) = \{0, 2, 3\}$ $(= L(X))$ is not an ideal of X, since $4 * 3 = 3 \in T(X)$, but $4 \notin T(X)$.

Now we give equivalent conditions that $T(X)$ is an ideal of a BCI-algebra X.

THEOREM 2.6. *Let X be a BCI-algebra. The following are equivalent:*

(i) *$T(X)$ is an ideal of X.*
(ii) *$x * a = y * a$ implies $x = y$ for all $x, y \in X_+$ and $a \in T(X)$.*
(iii) *$x * a = 0 * a$ implies $x = 0$ for all $x \in X_+$ and $a \in T(X)$.*

Proof. (i) $\Rightarrow$ (ii) Let $T(X)$ be an ideal of X and assume $x * a = y * a$ for all $x, y \in X_+$ and $a \in T(X)$. Then

$$
\begin{aligned}
(x * y) * a &= (x * a) * y && \text{[by (2)]} \\
&= (y * a) * y && \text{[by assumption]} \\
&= (y * y) * a && \text{[by (2)]} \\
&= 0 * a \in T(X). && \text{[by (III) and Theorem 2.1]}
\end{aligned}
$$

Since $T(X)$ is an ideal of X, it follows that $x * y \in T(X)$. On the other hand, note that $x * y \in X_+$ and $X_+ \cap T(X) \subseteq X_+ \cap L(X) = \{0\}$. Thus we have $x * y = 0$ or $x \leq y$. Similarly we get $y \leq x$, and therefore $x = y$.

(ii) $\Rightarrow$ (iii) Since $0 \in X_+$, it is straightforward.

(iii) $\Rightarrow$ (i) Assume that (iii) holds. Let $s * t \in T(X)$ and $t \in T(X)$ for all $s, t \in X$. Denote $u = 0 * (0 * s)$. Then $u \in L(X)$. Since $u = 0 * (0 * s) \leq s$, it follows from (3) that $u * t \leq s * t$, i.e., $s * t \in V(u * t)$, so that $s * t = u * t$, since $s * t \in T(X) \subseteq L(X)$. Hence

$$\begin{aligned}(s * u) * t &= (s * t) * u \\ &= (u * t) * u \\ &= (u * u) * t \\ &= 0 * t,\end{aligned}$$

which implies from (iii) that $s * (0 * (0 * s)) = s * u = 0$, since $s * u \in X_+$. Therefore $s = 0 * (0 * s) \in L(X)$. As $T(X)$ is an ideal of $L(X)$, we get $s \in T(X)$, and $T(X)$ is an ideal of X. This completes the proof. □

X. H. Zhang [4] introduced the notion of T-ideals in BCI-algebras.

DEFINITION 2.7 (ZHANG [4]). A non-empty subset A of a BCI-algebra X is called a T-*ideal* of X if it satisfies:

(i) $0 \in A$,
(ii) $x * (y * z) \in A$ and $y \in A$ imply $x * z \in A$

for all $x, y, z \in X$.

Every T-ideal of a BCI-algebra is an ideal (see [4, Theorem 1]), but not converse. In fact, consider the BCI-algebra $X := \{0, 1, 2, 3, 4, 5\}$ as in Example 2.5. The set $A := \{0, 1\}$ is an ideal of X, but not a T-ideal of X, since $4 * (0 * 3) = 1 \in A$, but $4 * 3 = 3 \notin A$.

LEMMA 2.8 (Zhang [4]). *If A is a T-ideal of a BCI-algebra X, then $(0 * x) * x \in A$ for all $x \in A$.*

By using the T-part of a BCI-algebra, we give an equivalent condition that every ideal is a T-ideal.

THEOREM 2.9. *Let A be an ideal of a BCI-algebra X. Then A is a T-ideal if and only if $T(X) \subseteq A$.*

Proof. Necessity follows from Lemma 2.8. Suppose $T(X) \subseteq A$. Let $x * (y * z) \in A$ and $y \in A$ for all $x, y, z \in X$. Since

$$
\begin{aligned}
&((x * z) * (x * (y * z))) * y \\
&\leq ((y * z) * z) * y && \text{[by (I) and (3)]} \\
&= ((y * y) * z) * z && \text{[by (2)]} \\
&= (0 * z) * z \in T(X) \subseteq A,
\end{aligned}
$$

it follows that $x * z \in A$. Hence A is a T-ideal of X, ending the proof. □

COROLLARY 2.10. (Extension property for T-ideal) *Let A and B be ideals of a BCI-algebra X. If $A \subseteq B$ and A is a T-ideal of X, then B is also a T-ideal of X.*

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