Bull. Korean Math. Soc. **34** (1997), No. 2, pp. 205–209

A NOTE ON k-NIL RADICALS IN BCI-ALGEBRAS

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1. Introduction

Hong et al. [2] and Jun et al. [4] introduced the notion of k-nil radical in a BCI-algebra, and investigated its some properties. In this paper, we discuss the further properties on the k-nil radical. Let A be a subset of a BCI-algebra X. We show that the k-nil radical of A is the union of branches. We prove that if A is an ideal then the k-nil radical $[A; k]$ is a p-ideal of X, and that if A is a subalgebra, then the k-nil radical $[A; k]$ is a closed p-ideal, and hence a strong ideal of X.

We recall some definitions and results.

By a *BCI-algebra* we mean an algebra $(X; *, 0)$ of type (2, 0) satisfying the axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = y * x = 0$ implies $x = y$,

for all x, y and z in X. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. A BCI-algebra X is said to be *p-semisimple* if $X_+ = \{0\}$, where $X_+ := \{x \in X | 0 \leq x\}$, the BCK-part of X.

In any BCI-algebra X, the following hold:

(1) $x * 0 = x$.

Received October 4, 1996.

1991 Mathematics Subject Classification: 03G25, 06F35, 16N99.

Key words and phrases: k-nil radical, atom, branch, (closed) ideal, p-ideal, strong ideal.

*Supported by the Basic Science Research Institute Program, Ministry of Education, 1996, Project No. BSRI-96-1406. **Supported by Shaanxi Provincial Natural Science Foundation, P. R. China.

(2) $(x * y) * z = (x * z) * y$.
(3) $0 * (0 * (0 * x)) = 0 * x$.
(4) $0 * (x * y) = (0 * x) * (0 * y)$.

In what follows, X would mean a BCI-algebra unless otherwise specified.

A non-empty subset A of X is called a *subalgebra* of X if $x * y \in A$ whenever $x, y \in A$.

A non-empty subset A of X is called an *ideal* of X if $0 \in A$ and if $x * y, y \in A$ imply that $x \in A$. An ideal A of X is said to be *closed* if $0 * x \in A$ whenever $x \in A$. We note that every closed ideal of X is a subalgebra of X.

A non-empty subset A of X is called a *p-ideal* of X if $0 \in A$ and if $(x * z) * (y * z) \in A$ and $y \in A$ imply that $x \in A$. An ideal A of X is said to be *strong* if $x * a \in X \setminus A$ whenever $x \in A$ and $a \in X \setminus A$.

For any elements x, y in X, let us write $x * y^k$ for $(...((x * y) * y) * ...) * y$ where y occurs k times.

LEMMA 1.1. (Huang [3]) *For any x, y in X and any positive integer k, we have*

(i) $0 * (x * y)^k = (0 * x^k) * (0 * y^k)$.
(ii) $0 * (0 * x)^k = 0 * (0 * x^k)$.

DEFINITION 1.2. (Meng et al. [7]) An element a of X is called an *atom* if $z * a = 0$ implies $z = a$ for all $z \in X$.

Denote by $L(X)$ the set of all atoms of X. Clearly, $0 \in L(X)$. Moreover $0 * (0 * x) \in L(X)$ for all $x \in X$, which is denoted by a_x. Note that $L(X)$ is a subalgebra of X.

For any non-empty subset A of X, denote by $L(A)$ the set $\{0 * (0 * x) | x \in A\}$. Obviously, $A \cap L(X) \subseteq L(A)$. For any $a \in L(X)$, the set $V(a) := \{x \in X | a \leq x\}$ is called a *branch* of X. Obviously, $V(0) = X_+$.

PROPOSITION 1.3. (Meng et al. [7]) *Let X be a BCI-algebra. Then*

(i) *if $a \in L(X)$, then $a * x \in L(X)$ for all $x \in X$.*
(ii) *if $a, b \in L(X)$ and $x \in V(b)$, then $a * x = a * b$ for all $x \in X$.*
(iii) *$x \in V(a_x)$ for all $x \in X$.*
(iv) *$L(X)$ is a p-semisimple BCI-algebra.*
(v) *if A is a subalgebra of X, then $L(A)$ is a subalgebra of $L(X)$.*

2. k-nil radicals

DEFINITION 2.1. (Hong et al. [2]) Let A be a subset of X. For given positive integer k, the *k-nil radical* of A, denote by $[A;k]$, is the set of all elements of X satisfying $0 * x^k \in A$, i.e.,

$$[A;k] := \{x \in X : 0 * x^k \in A\}.$$

We note from Proposition 1.3(i) that $0 * x^k \in L(X)$ for all $x \in X$, whence $[A;k] = \emptyset$ whenever $A \cap L(X) = \emptyset$.

The following proposition shows that the k-nil radical is the union of branches.

PROPOSITION 2.2. *Let A be a subset of X and let k be a positive integer. Then*

$$[A;k] = \cup\{V(a_x) | 0 * x^k \in A\}.$$

Proof. Let $x \in [A;k]$. Then $0 * x^k \in A$. Since $x \in V(a_x)$ for all $x \in X$, we have $x \in \cup\{V(a_x)|0 * x^k \in A\}$.

Conversely, let $y \in \cup\{V(a_x)|0 * x^k \in A\}$. Then $y \in V(a_x)$ for some $x \in X$ with $0 * x^k \in A$. Since $0 \in L(X)$, it follows from Proposition 1.3(ii) and (3) that

$$0 * y = 0 * a_x = 0 * (0 * (0 * x)) = 0 * x.$$

By induction, we know that $0 * y^k = 0 * x^k \in A$, whence $y \in [A;k]$. This completes the proof. □

COROLLARY 2.3. *Let A be a subset of X. If $0 \in A$, then $X_+ \subseteq [A;k]$ for every positive integer k.*

Proof. Note that $V(0) = X_+$. Since $0 * 0^k = 0 \in A$, it follows from Proposition 2.2 that $X_+ \subseteq [A;k]$. □

LEMMA 2.4. (Hong et al. [2]) *Let A be a subalgebra of X and k a positive integer. Then*

(i) *if $x \in [A;k]$, then $0 * x \in [A;k]$.*
(ii) *if $x * y \in [A;k]$, then $y * x \in [A;k]$.*
(iii) *$[A;k]$ is a subalgebra of X containing A.*

PROPOSITION 2.5. *Let A be a subalgebra of X and let k be a positive integer. Then $x \in [A;k]$ if and only if $0 * x \in [A;k]$.*

Proof. Necessity follows from Lemma 2.4(i). Let $0*x \in [A;k]$. Since $0 \in [A;k]$, it follows from Lemma 2.4(iii) that $a_x = 0*(0*x) \in [A;k]$, whence $0*a_x^k \in A$. But $0*x^k = 0*a_x^k$, and so $0*x^k \in A$ or $x \in [A;k]$. This completes the proof. □

LEMMA 2.6. (Lei et al. [5]) *Any subalgebra of a p-semisimple algebra is an ideal.*

LEMMA 2.7. (Meng et al. [6]) *Let A be a non-empty subset of X. Then A is a p-ideal of X if and only if it satisfies:*

(i) *$L(A)$ is an ideal of $L(X)$,*
(ii) *$A = \cup\{V(a) | a \in L(A)\}$.*

THEOREM 2.8. *Let A be a subalgebra of X and k a positive integer. Then $[A;k]$ is a closed p-ideal of X.*

Proof. Since $L([A;k])$ is a subalgebra of $L(X)$, it follows from Lemma 2.6 that $L([A;k])$ is an ideal of $L(X)$. By means of Propositions 2.2 and 2.5 and Lemma 2.7, we know that $[A;k]$ is a closed p-ideal of X. □

Since any closed ideal is a subalgebra, we have the following corollary.

COROLLARY 2.9. (Hong et al. [2]) *If A is a closed ideal of X, then so is $[A;k]$ for every positive integer k.*

Note that an ideal of X is strong if and only if it is a closed p-ideal (see [1, Theorem 11]). Hence we have

COROLLARY 2.10. *If A is a subalgebra of X, then $[A;k]$ is a strong ideal of X for every positive integer k.*

LEMMA 2.11. (Hong et al. [2]) *If A is an ideal of X, then the k-nil radical of A is an ideal of X for every positive integer k.*

LEMMA 2.12. (Meng et al. [6]) *An ideal A of X is a p-ideal if and only if $X_+ \subseteq A$.*

THEOREM 2.13. *If A is an ideal of X, then the k-nil radical of A is a p-ideal of X for every positive integer k.*

Proof. Since A is an ideal of X, we have $0 \in A$, and so $X_+ \subseteq [A;k]$ by Corollary 2.3. By Lemmas 2.11 and 2.12, we know that $[A;k]$ is a p-ideal of X. □

References

[1] S. M. Hong, Y. B. Jun and J. Meng, *On strong ideals and p-ideals in BCI-algebras*, to appear in Math. Japon.

[2] S. M. Hong, Y. B. Jun and E. H. Roh, *k-nil radicals in BCI-algebras*, Far East J. Math. Sci. (submitted).

[3] W. Huang, *Nil-radical in BCI-algebras*, Math. Japon. **37** (1992), 363-366.

[4] Y. B. Jun, S. M. Hong and E. H. Roh, *k-nil radicals in BCI-algebras II*, Comm. Korean Math. Soc. (submitted).

[5] T. D. Lei and C. C. Xi, *p-radical in BCI-algebras*, Math. Japon. **30** (1985), 511-517.

[6] J. Meng, Y. B. Jun and E. H. Roh, *The role of $B(X)$ and $L(X)$ in the ideal theory of BCI-algebras*, to appear in Indian J. Pure Appl. Math.

[7] J. Meng and X. L. Xin, *Characterizations of atoms in BCI-algebras*, Math. Japon. **37** (1992), 359-361.

[8] X. H. Zhang, H. Jiang and S. A. Bhatti, *On p-ideals of a BCI-algebra*, Punjab University J. Math. **27** (1994), 121-128.

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