A WEAKLY NEGATIVE STRUCTURE 
OF STOCHASTIC ORDERING\(^1\)

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1. Introduction

Lehmann [13] introduced the concept of positive(negative) dependence together with some other dependence concepts. Since then, a great many works have been studied on the subject and its extensions and numerous multivariate inequalities have been obtained. For a references of available results, see Karlin and Rinott [12], Ebrahimi and Ghosh [8] and Sampson [14]. Whereas a number of dependence notions exist for multivariate processes (see Friday [10]), recently, Ebrahimi [7] introduced some new dependence concepts of the hitting times of stochastic processes.

Most of the dependence concepts introduced in the literature are stronger than the positive(negative) dependence. For this reason, Baek [3] introduced some new weakly quadrant dependence concepts in terms of the finite-dimensional distributions of the hitting times of the components of a vector process. These concepts not only help us to understand the structure of functionals such as hitting times of the given vector process but also have the potential for new and useful inequalities for stochastic processes. Moreover, the concept of dependence is a form of qualitative bivariate dependence which has led to many applications in applied probability, reliability, and statistical inference such as analysis of variance, multivariate tests of hypothesis and sequential testing.

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Since \( WNQD \) is a qualitative form of dependence, it would seem difficult, or impossible to compare different pairs of stochastic processes as to their "degree of \( WNQD \)-ness". For these reasons, in this paper we introduce a new notion of a more weakly negative quadrant dependence of two stochastic processes.

The importance of this paper lies in the fact that this new notion is weaker than the more negative quadrant dependence. In particular, we give a partial ordering which permits us to compare pairs of \( WNQD \) bivariate vector processes of interest as to their "degree of \( WNQD \)-ness".

In Section 2, we develop some definitions and notations of \( WNQD \) ordering processes. In Section 3, we derive useful closure properties of \( WNQD \) ordering. We show that \( WNQD \) ordering is closed under convolution, limit in distribution, compound distribution, a mixture of certain types, transformations of stochastic processes by univariate increasing convex functions and convex combination. Finally, in section 4, we present several examples of hitting times possessing various \( WNQD \) ordering processes.

2. Notation and definitions

First, in this section, we present notations and basic facts used throughout the paper. In what follows increasing(decreasing) means non-decreasing(non-increasing) and positive(negative) means non-negative (non-positive). Suppose that we are given a bivariate stochastic processes \( \{(X_{11}(t), X_{21}(t))|t \geq 0\}, \{(X_{12}(t), X_{22}(t))|t \geq 0\} \). The state space of \( (X_{11}(t), X_{21}(t)) \) and \( (X_{12}(t), X_{22}(t)) \) will be taken to be any subset, \( E = E_1 \times E_2 \), of the plane \( \mathbb{R}^2 \).

For any states \( a_i \) in \( E_i, i = 1, 2 \), we define the random times as follows

\[
T_{ij}(a_i) = \inf\{t|X_{ij}(t) \leq a_i, \ 0 \leq t \leq \infty\}, \ j = 1, 2.
\]

In other words, \( T_{ij}(a_i) \) is the hitting time that the \( ij \)th component process \( X_{ij}(t) \) reaches or goes below \( a_i \) (see [7]). The stochastic processes \( \{T_{ij}(a_i)|a_i \in E_i\} \) will be referred to as the hitting time processes of the processes \( X_{ij}(t), \ i, j = 1, 2 \). If we base the dependence between
processes on the dependence of their hitting times, we then have the following definitions.

**Definition 2.1.** [6]. The bivariate stochastic process \( \{(X_{12}(t), X_{22}(t)) | t \geq 0\} \) is said to be more negatively quadrant dependent than \( \{(X_{11}(t), X_{21}(t)) | t \geq 0\} \) if

\[
P(T_{12}(a_1) > t_1, T_{22}(a_2) > t_2) \\
\leq P(T_{11}(a_1) > t_1, T_{21}(a_2) > t_2) \quad \text{for all } t_i \geq 0, \ a_i \in E_i, \ i = 1, 2.
\]

**Definition 2.2.** [1]. The bivariate stochastic process \( \{(X_{12}(t), X_{22}(t)) | t \geq 0\} \) is said to be weakly negative quadrant dependent of the first type \( (WNQD1) \) if

\[
\int_{x_1}^{\infty} \int_{x_2}^{\infty} P(\cap_{i=1}^{2} T_{i2}(a_i) > t_1) dt_2 dt_1 \\
\leq \int_{x_1}^{\infty} \int_{x_2}^{\infty} \Pi_{i=1}^{2} P(T_{i2}(a_i) > t_i) dt_2 dt_1 \quad \text{for all } t_i \geq 0, \ a_i \in E_i, \ i = 1, 2.
\]

**Definition 2.3.** [1]. The bivariate stochastic process \( \{(X_{12}(t), X_{22}(t)) | t \geq 0\} \) is said to be weakly negative quadrant dependent of the second type \( (WNQD2) \) if

\[
\int_{0}^{x_1} \int_{0}^{x_2} P(\cap_{i=1}^{2} T_{i2}(a_i) > t_1) dt_2 dt_1 \\
\leq \int_{0}^{x_1} \int_{0}^{x_2} \Pi_{i=1}^{2} P(T_{i2}(a_i) > t_i) dt_2 dt_1 \quad \text{for all } t_i \geq 0, \ a_i \in E_i, \ i = 1, 2.
\]

Moreover, \( \{(X_{12}(t), X_{22}(t)) | t \geq 0\} \) (or the distribution \( H \)) is said to be weakly negative quadrant dependent \( (WNQD) \) if they satisfy both \( WNQD1 \) and \( WNQD2 \).

Before we state more definitions, we let \( \beta = \beta(F, G) \) denote the class of bivariate distribution function \( H \) having specified marginal distribution functions \( F \) and \( G \), where \( F \) and \( G \) are nondegenerate, and we then consider \( \beta^+ \), a subclass of \( \beta \), defined by
$$\beta^+ = \{ H(t_1, t_2) | H \text{ is } WNQD, H(t_1, \infty) = F(t_1), H(\infty, t_2) = G(t_2) \}.$$ 

When $H_1$ and $H_2$ belong to $\beta^+$, we may now define the following definitions.

**Definition 2.4.** The bivariate distribution $H_2$ is said to be more weakly negative quadrant dependent of the first type than $H_1$ if

$$\int_0^\infty \int_0^\infty P(T_{12}(a_1) > t_1, T_{22}(a_2) > t_2) dt_1 dt_2$$

$$\leq \int_0^\infty \int_0^\infty P(T_{11}(a_1) > t_1, T_{21}(a_2) > t_2) dt_1 dt_2$$

(2.1)

for all $t_i \geq 0$, $i = 1, 2$. We write $H_2 > (WNQD1)H_1$.

**Definition 2.5.** The bivariate distribution $H_2$ is said to be more weakly negative quadrant dependent of the second type than $H_1$ if

$$\int_0^{x_1} \int_0^{x_2} P(T_{12}(a_1) > t_1, T_{22}(a_2) > t_2) dt_1 dt_2$$

$$\leq \int_0^{x_1} \int_0^{x_2} P(T_{11}(a_1) > t_1, T_{21}(a_2) > t_2) dt_1 dt_2$$

(2.2)

for all $t_i \geq 0$, $i = 1, 2$. We write $H_2 > (WNQD2)H_1$.

Moreover, the bivariate distribution $H_2$ is said to be more weakly negative quadrant dependent than $H_1$ if they satisfy both $H_2 > (WNQD1)H_1$ and $H_2 > (WNQD2)H_1$. We write $H_2 > (WNQD)H_1$.

From the Definition 2.1, 2.4 and 2.5 we then have the following Theorem 2.6.

**Theorem 2.6.** Let $H_1$ and $H_2$ be bivariate distribution with specified marginals $F$ and $G$. Assume that the bivariate distribution $H_2$ is more negatively quadrant dependent than $H_1$. Then $H_2$ is more weakly negative quadrant dependent than $H_1$. 

214
3. Closure properties of \((\beta^+, > (WNQD))\)

In this section, we establish preservation of the \(WNQD\) ordering under convolution, limit in distribution, compound distribution, mixture of a certain type, transformations of univariate increasing convex functions, and convex combination. First note that by theorem of Alzaid [1], (2.1) and (2.2) are equivalent to 
\[ E(f(T_{12}(a_1))g(T_{22}(a_2))) \leq E(f(T_{11}(a_1))g(T_{21}(a_2))) \]
for all increasing positive convex functions \(f\) and \(g\).

Below, we show that the ordering is preserved under convolution. We need the following Lemma 3.1 which is of independent interest.

**Lemma 3.1.** Let (a) \(\{(X_{11}(t), X_{21}(t))| t \geq 0\}\) and \(\{(X_{12}(t), X_{22}(t))| t \geq 0\}\) have distributions \(H_1\) and \(H_2\), where \(H_1, H_2\) belong to \(\beta^+, (b) \{(X_{12}(t), X_{22}(t))| t \geq 0\} > (WNQD)\{(X_{11}(t), (X_{21}(t))| t \geq 0\}, and (c) \(Z_1, Z_2\) with an arbitrary \(WNQD\) distribution function \(H\) independent of both of \(\{(X_{11}(t), X_{21}(t))| t \geq 0\}\) and \(\{(X_{12}(t), X_{22}(t))| t \geq 0\}\).

Then \((X_{12}(t) + Z_1, X_{22}(t) + Z_2) > (WNQD)(X_{11}(t) + Z_1, X_{21}(t) + Z_2)\).

**Proof.** First, we will show that \((X_{12}(t) + Z_1, X_{22}(t) + Z_2)\) is \(WNQD1\) (WNQD2). Consider any hitting times \(W_{ij}(a_i)\) given by \(W_{ij}(a_i) = \inf\{t|X_{ij}(t) + Z_i \leq a_i, t \geq 0\}, i, j = 1, 2\).

Then,

\[
\text{Cov}(f(W_{12}(a_1)), g(W_{22}(a_2)))
= \text{Cov}(f(T_{12}(a_1 - Z_1)), g(T_{22}(a_2 - Z_2)))
= \text{Cov}(E(f(T_{12}(a_1 - Z_1))|Z_1, Z_2), E(g(T_{22}(a_2 - Z_2))|Z_1, Z_2))
+ E(\text{Cov}(f(T_{12}(a_1 - Z_1)), g(T_{22}(a_2 - Z_2))|Z_1, Z_2)) \geq 0.
\]

Note that the first and second terms are greater than or equal to zero for all functions \(f\) and \(g\) such that \(f\) is increasing positive convex(negative concave) and \(g\) is decreasing negative concave(positive convex) function. Thus by Theorem 3 of Alzaid(1990), \((X_{12}(t) + Z_1, X_{22}(t) + Z_2)\) is \(WNQD\). Similarly we can show that \((X_{11}(t) + Z_1, X_{21}(t) + Z_2)\) is also \(WNQD\).

Next, we will show that \((X_{12}(t) + Z_1, X_{22}(t) + Z_2) > (WNQD)(X_{11}(t) + Z_1, X_{21}(t) + Z_2)\) i.e.,
\[ E(f(T_{12}(a_1 - Z_1))g(T_{22}(a_2 - Z_2))) \leq E(f(T_{11}(a_1 - Z_1))g(T_{21}(a_2 - Z_2))) \]
for any increasing positive convex functions \( f \) and \( g \).

Now,

\[
E(f(T_{12}(a_1 - Z_1))g(T_{22}(a_2 - Z_2))) \\
= E(E(f(T_{12}(a_1 - Z_1))g(T_{22}(a_2 - Z_2))|Z_1, Z_2)) \\
= E(E(f(T_{12}(a_1 - Z_1))g(T_{22}(a_2 - Z_2)))|Z_1, Z_2)) \\
\leq E(E(f(T_{11}(a_1 - Z_1))g(T_{21}(a_2 - Z_2)))|Z_1, Z_2)) \\
= E(f(T_{11}(a_1 - Z_1))g(T_{21}(a_2 - Z_2))).
\]

The inequality follows from the assumption that \((X_{12}(t), X_{22}(t)) > (W_{NQD})(X_{11}(t), X_{21}(t))\). \( \square \)

**Theorem 3.2.** Suppose that the stochastic process (a) \( \{(X_{12}(t), X_{22}(t))| t \geq 0\} \) is more weakly negative quadrant dependent than \( \{(X_{11}(t), X_{21}(t))| t \geq 0\} \), (b) \( \{(Y_{12}(t), Y_{22}(t))| t \geq 0\} \) is more weakly negative quadrant dependent than \( \{(Y_{11}(t), Y_{21}(t))| t \geq 0\} \), and (c) let \( \{(X_{12}(t), X_{22}(t))| t \geq 0\} \) and \( \{(Y_{12}(t), Y_{22}(t))| t \geq 0\} \) be independent processes, \( \{(X_{11}(t), X_{21}(t))| t \geq 0\} \) and \( \{(Y_{11}(t), Y_{21}(t))| t \geq 0\} \) be independent processes. Then \( \{(X_{12}(t) + Y_{12}(t), X_{22}(t) + Y_{22}(t))| t \geq 0\} > (W_{NQD})\{(X_{11}(t) + Y_{11}(t), X_{21}(t) + Y_{21}(t))| t \geq 0\}\).

**Proof.** By assumption, \((X_{12}(t), X_{22}(t)) > (W_{NQD})(X_{11}(t), X_{21}(t))\). Specifying \((Z_1(t), Z_2(t))\) to be \((Y_{12}(t), Y_{22}(t))\), we apply Lemma 3.1 to obtain

\[
(X_{12}(t) + Y_{12}(t), X_{22}(t) + Y_{22}(t)) \\
> (W_{NQD})(X_{11}(t) + Y_{11}(t), X_{21}(t) + Y_{21}(t)).
\] (3.1)

Next, we use the assumption \((Y_{12}(t), Y_{22}(t)) > (W_{NQD})(Y_{11}(t), Y_{21}(t))\), specifying \((Z_1(t), Z_2(t))\) to be \((X_{11}(t), X_{21}(t))\), and again use Lemma 3.1 yielding

\[
(X_{11}(t) + Y_{12}(t), X_{21}(t) + Y_{22}(t)) \\
> (W_{NQD})(X_{11}(t) + Y_{11}(t), X_{21}(t) + Y_{21}(t)).
\] (3.2)
By combining (3.1) and (3.2),

\[(X_{12}(t) + Y_{12}(t), X_{22}(t) + Y_{22}(t)) > (WNQD)(X_{11}(t) + Y_{12}(t), X_{21}(t) + Y_{22}(t)) > (WNQD)(X_{11}(t) + Y_{11}(t), X_{21}(t) + Y_{21}(t)).\]

Thus \( (X_{12}(t) + Y_{12}(t), X_{22}(t) + Y_{22}(t)) > (WNQD)(X_{11}(t) + Y_{11}(t), X_{21}(t) + Y_{21}(t)) \). This completes the proof. \(\square\)

The next theorem demonstrates that, under suitable conditions, limits of the \(WNQD\) ordering processes inherit the \(WNQD\) ordering structure.

**Theorem 3.3.** Let (a) \( \{(X_{n1}(t), X_{n2}(t))|t \geq 0\}, \{(Y_{n1}(t), Y_{n2}(t))|t \geq 0\} \) have distributions \( H_n, H'_n \) for every \( n \) and \( H_n > (WNQD)H'_n \), (b) \( \{(X_1(t), X_2(t))|t \geq 0\}, \{(Y_1(t), Y_2(t))|t \geq 0\} \) have distributions \( H, H' \), (c) \( \{(X_{n1}(t), X_{n2}(t))|t \geq 0\}, \{(Y_{n1}(t), Y_{n2}(t))|t \geq 0\} \) have all sample paths and they are right continuous on \([0, \infty)\) with finite left limits at all \( t \), and (d) \( H_n \rightarrow H \) and \( H'_n \rightarrow H' \) be weakly as \( n \rightarrow \infty \), respectively. Then \( H > (WNQD)H' \).

**Proof.** Denote by \( C(H) \) and \( C(H') \) the sets of continuity points of \( H \) and \( H' \), respectively. Let \( D = C(H) \cap C(H') \). It follows from our assumptions that \( H(t_1, t_2) \geq H'(t_1, t_2) \) for all \( (t_1, t_2) \in D \). Since \( D \) is a dense set in \( \mathbb{R}^2, H > (WNQD)H' \).

The following theorem is another application of Theorem 3.2 which is very important in recognizing \(WNQD\) ordering in compound distributions which arise naturally in stochastic processes. \(\square\)

**Theorem 3.4.** Let (a) \( (Y_1, S_1), (Y_2, S_2), \ldots \) be independent random processes, (b) \( (X_1, K_1), (X_2, K_2), \ldots \) be independent random processes, (c) \( (Y_i, S_i) \) and \( (X_i, K_i), i = 1, 2, \ldots n \) are \( WNQD \) random process (d) \( (Y_i, S_i) > (WNQD)(X_i, K_i), i = 1, 2, \ldots \), and (e) \( N(t) \) be a Poisson
process which is independent of \((Y_i, S_i)\) and \((X_i, K_i), i = 1, 2, \cdots\). Then

\[
(Z_{12}(t)) = \sum_{i=1}^{N(t)} Y_i, \quad Z_{22}(t) = \sum_{i=1}^{N(t)} S_i
\]

\[
(WNQD)(Z_{11}(t)) = \sum_{i=1}^{N(t)} X_i, \quad Z_{21}(t) = \sum_{i=1}^{N(t)} K_i
\]

Our next result deals with the preservation of the WNQD ordering under mixture. In order to motivate our definition of a subclass of \(\beta^+\) in which the WNQD ordering is preserved under mixture we need a definition and a similar result of Ebrahimi and Ghosh [8].

**Definition 3.5.** [8] A stochastic process \(\{X_{22}(t)\}_{t \geq 0}\) is stochastically increasing(decreasing) in \(\{X_{12}(t)\}_{t \geq 0}\) if \(E(f(T_{22}(a_1))|T_{12}(a_1) = t_1)\) is increasing(decreasing) in \(t_1\) for all \(a_1 \in E_i, i = 1, 2, \) and positive increasing convex function \(f\).

We shall use the abbreviation \(SI\) and \(SD\) for stochastically increasing and decreasing respectively.

**Remark 1.** A stochastic process \(\{(X_{12}(t), X_{22}(t))\}_{t \geq 0}\) given a scalar \(\lambda\), is WNQD1(WNQD2) if and only if \(Cov[f(T_{22}(a_2))|T_{12}(a_1) = t_1] \geq 0\) for all functions \(f\) and \(g\) such that \(f\) is non-decreasing, non-negative(non-positive concave), and \(g\) is non-increasing, non-positive concave(non-negative convex).

**Theorem 3.6.** Let (a) \(\{(X_{12}(t), X_{22}(t))\}_{t \geq 0}\) given a scalar \(\lambda\), a stochastic process be conditionally WNQD, and (b) \(\{X_{12}(t)\}_{t \geq 0}\) be \(SI\) and \(\{X_{22}(t)\}_{t \geq 0}\) be \(SD\) in \(\lambda\), or (b') \(\{X_{12}(t)\}_{t \geq 0}\) be \(SD\) and \(\{X_{22}(t)\}_{t \geq 0}\) be \(SI\) in \(\lambda\). Then \(\{(X_{12}(t), X_{22}(t))\}_{t \geq 0}\) is WNQD.

The next theorem deals with the preservation of the WNQD ordering under mixture. We may now define the class \(\beta^+_{\lambda}\) by

\[
\beta^+_{\lambda} = \{H_{\lambda}|H(t_1, \infty|\lambda) = F(t_1|\lambda), H(\infty, t_2|\lambda) = G(t_2|\lambda), H_{\lambda}|\lambda\ is\ WNQD,\ F\ is\ SD\ and\ G\ is\ SI\ in\ \lambda,\ or\ F\ is\ SI\ and\ G\ is\ SD\ in\ \lambda\}.
\]

Now consider \((\beta^+_{\lambda}, > (WNQD))\). The following theorem shows that if two elements of \(\beta^+_{\lambda}\) are ordered according to \(> (WNQD)\), then after mixing \(\lambda\), the resulting element in \(\beta^+\) preserves the same order.
A weakly negative structure of stochastic ordering

**Proposition 3.7.** Let the stochastic processes \((X_{12}(t), X_{22}(t))\) and \((X_{11}(t), X_{21}(t))\), given a scalar \(\lambda\), belong to \(\beta^+_\lambda\), respectively and \(||(X_{12}(t), X_{22}(t))\|\lambda > (WNQD)||X_{11}(t), X_{21}(t))||\lambda\) for all \(\lambda\). Then, unconditionally, \((X_{12}(t), X_{22}(t)), (X_{11}(t), X_{21}(t))\) belong to \(\beta^+\) and \((X_{12}(t), X_{22}(t)) > (WNQD)(X_{11}(t), X_{21}(t))\).

**Proof.** From Theorem 3.6, \((X_{12}(t), X_{22}(t))\) and \((X_{11}(t), X_{21}(t))\) are \(WNQD\).

Now,

\[
E(f(T_{12}(a_1))g(T_{22}(a_2))) = E_\lambda(E(f(T_{12}(a_1))g(T_{22}(a_2))|\lambda)) \\
\leq E_\lambda(E(f(T_{11}(a_1))g(T_{21}(a_2))|\lambda)) \\
= E(f(T_{11}(a_1))g(T_{21}(a_2))).
\]

The inequality comes from the fact that \((X_{12}(t), X_{22}(t))|\lambda > (WNQD)(X_{11}(t), X_{21}(t))|\lambda\) for all \(\lambda\).

Next, we show that the \(WNQD\) ordering is invariant under transformations of stochastic processes by increasing convex functions. ∎

**Theorem 3.8.** Let \((a)\ \{(X_{ij}(t), X_{ij}(t))^{H_j}|t \geq 0\}, i = 1, 2, 3, \cdots, n\) be \(n\)-independent pairs from a bivariate distribution \(H_j\) with continuous increasing sample paths, \(j = 1, 2, (b)\ \ H_1\) and \(H_2\) belong to \(\beta^+\) such that \(H_2 > (WNQD)H_1\), and \((c)\ g_1\) and \(g_2\) are positive convex functions and they are increasing in each of their arguments when all other arguments are fixed. Then the processes \((Y_{12}(t), Y_{22}(t)) > (WNQD) (Y_{11}(t), Y_{21}(t))\), given by \(Y_{1i}(t) = g_1(X_{1i}(t), \cdots, X_{ni}(t))\), \(Y_{2i}(t) = g_2(X'_{1i}(t), \cdots, X'_{ni}(t)), i = 1, 2.\)

**Proof.** First, we will show that the \(WNQD1\) ordering holds. The proof will be given for the case \(n = 2\). For the general \(n\), the proof is similar. Fix \(t_i \geq 0, i = 1, 2\) and introduce the variables \(V_i = X_{2i}(t_i), V_i' = X_{2i}'(t_i), U_i = sup_{0 \leq s < t_i}(g_1(X_{1i}(s), X_{2i}(s))), \) and \(U_i' = sup_{0 \leq s < t_i}(g_2(X_{1i}'(s), X_{2i}'(s))), i = 1, 2\), where for simplicity, \(t_1, t_2\) have been suppressed in \(V_i, V_i', U_i\) and \(U_i'\). Consider any hitting times of \(Y_{1i}(s) = g_1(X_{1i}(s), X_{2i}(s))\), \(Y_{2i}(s) = g_2(X_{1i}'(s), X_{2i}'(s)), i = 1, 2\) given by

\[
W_{ij}(a_i) = inf\{s|Y_{ij}(s) \leq a_i, s \geq 0\}, i, j = 1, 2.
\]

219
It suffices to show that

\[ \int_{x_1}^{\infty} \int_{x_2}^{\infty} P(W_{12}(a_1) > t_1, W_{22}(a_2) > t_2) dt_1 dt_2 \]

\[ \leq \int_{x_1}^{\infty} \int_{x_2}^{\infty} P(W_{11}(a_1) > t_1, W_{21}(a_2) > t_2) dt_1 dt_2, \quad i = 1, 2. \]

Note the facts that \( U_i = \sup_{0 \leq s < t_i} (g_1(X_{1i}(s), V_i)), U_i' = \sup_{0 \leq s < t_i} (g_2(X_{1i}')(s), V_i')), \) \( i = 1, 2, \) and that, by the hypothesis, random variables \((V_2, V_2')\) and \((V_1, V_1')\) are satisfied the following

\[ (V_2, V_2') > (WNQD1)(V_1, V_1'). \]

Now, we obtain

\[ \int_{x_1}^{\infty} \int_{x_2}^{\infty} P(W_{12}(a_1) > t_1, W_{22}(a_2) > t_2) dt_1 dt_2 \]

\[ = \int_{x_1}^{\infty} \int_{x_2}^{\infty} P(U_1 < a_1, U_2 < a_2) dt_1 dt_2 \]

\[ = \int_{x_1}^{\infty} \int_{x_2}^{\infty} E[P(U_1 < a_1, U_2 < a_2|V_2, V_2')] dt_1 dt_2 \]

\[ \leq \int_{x_1}^{\infty} E[P(U_1 < a_1|V_2)] dt_1 \int_{x_2}^{\infty} E[P(U_2 < a_2|V_2')] dt_2 \]

\[ \leq \int_{x_1}^{\infty} \int_{x_2}^{\infty} E[P(U_1 < a_1, U_2 < a_2|V_1, V_1')] dt_1 dt_2 \]

\[ = \int_{x_1}^{\infty} \int_{x_2}^{\infty} P(W_{11}(a_1) > t_1, W_{21}(a_2) > t_2) dt_1 dt_2 \]

The proof of the \(WNQD2\) ordering is similar. \( \square \)

Next, we now turn our attention to a simple but important property of the class \( \beta^+ \).

**Theorem 3.9.** The class \( \beta^+ = \{H|H(t_1, t_2) \text{ is } WNQD, H(t_1, \infty) = F(t_1), H(\infty, t_2) = G'(t_2)\} \) is convex.
A weakly negative structure of stochastic ordering

Proof. Let $H_1, H_2$ in $\beta^+$ and for $\alpha$ in $(0, 1), H = \alpha H_1 + (1 - \alpha) H_2$. Then we will show that $H$ is convex combination of $H_1$ and $H_2$. Since each of the $H_1$ and $H_2 \in \beta^+$,

$$
\int_{x_1}^{\infty} \int_{x_2}^{\infty} P_H(T_{12}(a_1) > t_1, T_{22}(a_2) > t_2) dt_1 dt_2
= \alpha \int_{x_1}^{\infty} \int_{x_2}^{\infty} P_{H_1}(T_{12}(a_1) > t_1, T_{22}(a_2) > t_2) dt_1 dt_2
+ (1 - \alpha) \int_{x_1}^{\infty} \int_{x_2}^{\infty} P_{H_2}(T_{12}(a_1) > t_1, T_{22}(a_2) > t_2) dt_1 dt_2 \\
\leq \alpha \int_{x_1}^{\infty} \int_{x_2}^{\infty} P_H(T_{12}(a_1) > t_1) P_H(T_{22}(a_2) > t_2) dt_1 dt_2
+ (1 - \alpha) \int_{x_1}^{\infty} \int_{x_2}^{\infty} P_H(T_{12}(a_1) > t_1) P_H(T_{22}(a_2) > t_2) dt_1 dt_2 \\
= \int_{x_1}^{\infty} \int_{x_2}^{\infty} P_H(T_{12}(a_1) > t_1) P_H(T_{22}(a_2) > t_2) dt_1 dt_2.
$$

(3.3)

Hence $H$ is WNQD1. The proof of the WNQD2 ordering is similar to the proof of the WNQD1. \hfill \Box

Moreover,

$$
\lim_{t_1 \to \infty} H(t_1,t_2) = \alpha G(t_2) + (1 - \alpha) G(t_2) = G(t_2),
$$

and

$$
\lim_{t_2 \to \infty} H(t_1,t_2) = \alpha F(t_1) + (1 - \alpha) F(t_1) = F(t_1)
$$

It follows from (3.3), (3.4), (3.5) that $H \in \beta^+$. Thus $\beta^+$ is convex.

4. Examples

Example 4.1. Consider bivariate processes $\{(X_{n1},Y_{n1})|n \geq 1\}, \{(X_{n2},Y_{n2})|n \geq 1\}$ such that $(X_{11},Y_{11}), (X_{21},Y_{21}), \cdots$ are independent and $(X_{12},Y_{12}), (X_{22},Y_{22}), \cdots$ are independent processes. Then it is easy to check that $(X_{n2},Y_{n2}) > (WNQD)(X_{n1},Y_{n1}), n \geq 1$ whenever $(X_{i2},Y_{i2}) > (WNQD)(X_{i1},Y_{i1}),$ for each $i = 1, 2, \cdots$
Example 4.2. Consider a system with four components which is subjected to shocks. Let \( N(t) \) be the number of shocks received by time \( t \) and \( \{(X_k, S_k) | k = 1, 2, \ldots\} \) and \( \{(Y_k, L_k) | k = 1, 2, \ldots\} \) are sequences of damages to components 1, 2, 3 and 4 by shock \( k \), respectively. Define the compound Poisson processes by

\[
Z_{11}(t) = \sum_{k=1}^{N(t)} X_k, \quad Z_{12}(t) = \sum_{k=1}^{N(t)} Y_k, \quad Z_{21}(t) = \sum_{k=1}^{N(t)} L_k, \quad Z_{22}(t) = \sum_{k=1}^{N(t)} S_k.
\]

This follows by application of Theorem 3.4 implies \( (Z_{12}(t), Z_{22}(t)) > (\text{WNQD})(Z_{11}(t), Z_{21}(t)) \) for every \( t \geq 0 \) whenever \( (X_i, S_i) > (\text{WNQD})(Y_i, L_i) \), for each \( i = 1, 2, 3, \ldots \).

Example 4.3. Let \( Z_j = (Z_{1j}, Z_{2j}) \) and \( W_j = (W_{1j}, W_{2j}), j \geq 0 \) be a sequence of i.i.d. bivariate vectors such that \( (Z_{10}, Z_{20}) > (\text{WNQD})(W_{10}, W_{20}) \) random variables with marginal uniform distribution on the interval \([0, 1]\), respectively. Consider the sequences \( (X_{1n}(t), X_{2n}(t)) \) and \( (Y_{1n}(t), Y_{2n}(t)) \)(in \( n \geq 1 \)) of bivariate processes defined by

\[
X_n(t) = (X_{1n}(t), X_{2n}(t)) = (\sqrt{n}(F_{1n}(t) - t), \sqrt{n}(F_{2n}(t) - t)),
\]

\[
Y_n(t) = (Y_{1n}(t), Y_{2n}(t)) = (\sqrt{n}(G_{1n}(t) - t), \sqrt{n}(G_{2n}(t) - t)), t \in [0, 1],
\]

where for \( i = 1, 2, F_{in}(t) = n^{-1} \sum_{j=1}^{n} I(Z_{ij} \leq t), G_{in}(t) = n^{-1} \sum_{j=1}^{n} I(W_{ij} \geq t) \) are usual empirical c.d.f. of the i.i.d random variables \( Z_{i1}, Z_{i2}, \ldots, Z_{in} \) and \( W_{i1}, W_{i2}, \ldots, W_{in} \), respectively. Note that \( X_n(t) \) and \( Y_n(t) \) are simply the combination of the two(dependent)one-dimensional empirical processes, respectively. Such processes have been used by Goel and Ramalllingam(1987) to study matching problems. Fix \( i = 1, 2 \), then for all real \( a_i \), it is easy to verify that the hitting times \( T_i(a_i) = \inf\{t | X_{in}(t) \leq a_i \} \) and \( S_i(a_i) = \inf\{t | Y_{in}(t) \leq a_i \} \) are increasing functions of \( Z_{i1}, \ldots, Z_{in}, W_{i1}, \ldots, W_{in} \), respectively. In view of this fact, if we fixed \( n \geq 1 \), then we can argue (see Tong(1980), p. 84) that for all \( a_i, i = 1, 2, (T_{1}(a_1), T_{2}(a_2)) > (\text{WNQD})(S_{1}(a_1), S_{2}(a_2)) \) random variables. We conclude that \( (X_{1n}(t), X_{2n}(t)) > (\text{WNQD})(Y_{1n}(t), Y_{2n}(t)) \), for each \( n \geq 1 \). It is easy to check that \( (X_{1n}(t), X_{2n}(t)) \) converges
A weakly negative structure of stochastic ordering

weakly to \((X_1(t), X_2(t))\) and \((Y_{1n}(t), Y_{2n}(t))\) converges weakly to \((Y_1(t), Y_2(t))\) as \(n \to \infty\) on the time interval \([0,1]\). Hence, using the Theorem 3.3, we can obtain that \((X_1(t), X_2(t)) > (WNQD)(Y_1(t), Y_2(t))\).

References


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