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A WEAKLY NEGATIVE STRUCTURE OF STOCHASTIC ORDERING[1)]

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1. Introduction

Lehmann [13] introduced the concept of positive(negative) dependence together with some other dependence concepts. Since then, a great many works have been studied on the subject and its extensions and numerous multivariate inequalities have been obtained. For a references of available results, see Karlin and Rinott [12], Ebrahimi and Ghosh [8] and Sampson [14]. Whereas a number of dependence notions exist for multivariate processes (see Friday [10]), recently, Ebrahimi [7] introduced some new dependence concepts of the hitting times of stochastic processes.

Most of the dependence concepts introduced in the literature are stronger than the positive(negative) dependence. For this reason, Baek [3] introduced some new weakly quadrant dependence concepts in terms of the finite-dimensional distributions of the hitting times of the components of a vector process. These concepts not only help us to understand the structure of functionals such as hitting times of the given vector process but also have the potential for new and useful inequalities for stochastic processes. Moreover, the concept of dependence is a form of qualitative bivariate dependence which has led to many applications in applied probability, reliability, and statistical inference such as analysis of variance, multivariate tests of hypothesis and sequential testing.

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Since $WNQD$ is a qualitative form of dependence, it would seem difficult, or impossible to compare different pairs of stochastic processes as to their "degree of $WNQD$-ness". For these reasons, in this paper we introduce a new notion of a more weakly negative quadrant dependence of two stochastic processes.

The importance of this paper lies in the fact that this new notion is weaker than the more negative quadrant dependence. In particular, we give a partial ordering which permits us to compare pairs of $WNQD$ bivariate vector processes of interest as to their "degree of $WNQD$-ness".

In Section 2, we develop some definitions and notations of $WNQD$ ordering processes. In Section 3, we derive useful closure properties of $WNQD$ ordering. We show that $WNQD$ ordering is closed under convolution, limit in distribution, compound distribution, a mixture of certain types, transformations of stochastic processes by univariate increasing convex functions and convex combination. Finally, in section 4, we present several examples of hitting times possessing various $WNQD$ ordering processes.

2. Notation and definitions

First, in this section, we present notations and basic facts used throughout the paper. In what follows increasing(decreasing) means non-decreasing(non-increasing) and positive(negative) means non-negative (non-positive). Suppose that we are given a bivariate stochastic processes $\{(X_{11}(t), X_{21}(t))|t \geq 0\}$, $\{(X_{12}(t), X_{22}(t))|t \geq 0\}$. The state space of $(X_{11}(t), X_{21}(t))$ and $(X_{12}(t), X_{22}(t))$ will be taken to be any subset, $E = E_1 \times E_2$, of the plane $\mathbb{R}^2$.

For any states a_i in $E_i, i = 1, 2$, we define the random times as follows

$$T_{ij}(a_i) = \inf\{t|X_{ij}(t) \leq a_i,\ 0 \leq t \leq \infty\},\ j = 1, 2.$$

In other words, $T_{ij}(a_i)$ is the hitting time that the ij <u>th</u> component process $X_{ij}(t)$ reaches or goes below a_i (see [7]). The stochastic processes $\{T_{ij}(a_i)|a_i \in E_i\}$ will be referred to as the hitting time processes of the processes $X_{ij}(t),\ i, j = 1, 2$. If we base the dependence between processes on the dependence of their hitting times, we then have the following definitions.

DEFINITION 2.1.[6]. The bivariate stochastic process $\{(X_{12}(t), X_{22}(t))|t \geq 0\}$ is said to be more negatively quadrant dependent than $\{(X_{11}(t), X_{21}(t))|t \geq 0\}$ if

$$\begin{aligned}&P(T_{12}(a_1) > t_1,\ T_{22}(a_2) > t_2)\\&\leq P(T_{11}(a_1) > t_1,\ T_{21}(a_2) > t_2) \text{ for all } t_i \geq 0,\ a_i \in E_i,\ i = 1, 2.\end{aligned}$$

DEFINITION 2.2.[1]. The bivariate stochastic process $\{(X_{12}(t), X_{22}(t))|t \geq 0\}$ is said to be weakly negative quadrant dependent of the first type $(WNQD1)$ if

$$\begin{aligned}&\int_{x_1}^{\infty}\int_{x_2}^{\infty} P(\cap_{i=1}^{2} T_{i2}(a_i) > t_1)dt_2dt_1\\&\leq \int_{x_1}^{\infty}\int_{x_2}^{\infty} \Pi_{i=1}^{2} P(T_{i2}(a_i) > t_i)dt_2dt_1 \text{ for all } t_i \geq 0,\ a_i \in E_i,\ i = 1, 2.\end{aligned}$$

DEFINITION 2.3.[1]. The bivariate stochastic process $\{(X_{12}(t), X_{22}(t))|t \geq 0\}$ is said to be weakly negative quadrant dependent of the second type $(WNQD2)$ if

$$\begin{aligned}&\int_{0}^{x_1}\int_{0}^{x_2} P(\cap_{i=1}^{2} T_{i2}(a_i) > t_1)dt_2dt_1\\&\leq \int_{0}^{x_1}\int_{0}^{x_2} \Pi_{i=1}^{2} P(T_{i2}(a_i) > t_i)dt_2dt_1 \text{ for all } t_i \geq 0,\ a_i \in E_i,\ i = 1, 2.\end{aligned}$$

Moreover, $\{(X_{12}(t), X_{22}(t))|t \geq 0\}$ (or the distribution H) is said to be weakly negative quadrant dependent $(WNQD)$ if they satisfy both $WNQD1$ and $WNQD2$.

Before we state more definitions, we let $\beta = \beta(F, G)$ denote the class of bivariate distribution function H having specified marginal distribution functions F and G, where F and G are nondegenerate, and we then consider β^+, a subclass of β, defined by

$$\beta^+ = \{H(t_1, t_2) | H \text{ is } WNQD, H(t_1, \infty) = F(t_1), H(\infty, t_2) = G(t_2)\}.$$

When H_1 and H_2 belong to β^+, we may now define the following definitions.

DEFINITION 2.4. The bivariate distribution H_2 is said to be more weakly negative quadrant dependent of the first type than H_1 if

$$\begin{aligned} &\int_{x_1}^{\infty}\int_{x_2}^{\infty} P(T_{12}(a_1) > t_1, T_{22}(a_2) > t_2) dt_1 dt_2 \\ (2.1) \qquad &\leq \int_{x_1}^{\infty}\int_{x_2}^{\infty} P(T_{11}(a_1) > t_1, T_{21}(a_2) > t_2) dt_1 dt_2 \end{aligned}$$

for all $t_i \geq 0,\ i = 1, 2$. We write $H_2 > (WNQD1)H_1$.

DEFINITION 2.5. The bivariate distribution H_2 is said to be more weakly negative quadrant dependent of the second type than H_1 if

$$\begin{aligned} &\int_{0}^{x_1}\int_{0}^{x_2} P(T_{12}(a_1) > t_1, T_{22}(a_2) > t_2) dt_1 dt_2 \\ (2.2) \qquad &\leq \int_{0}^{x_1}\int_{0}^{x_2} P(T_{11}(a_1) > t_1, T_{21}(a_2) > t_2) dt_1 dt_2 \end{aligned}$$

for all $t_i \geq 0,\ i = 1, 2$. We write $H_2 > (WNQD2)H_1$.

Moreover, the bivariate distribution H_2 is said to be more weakly negative quadrant dependent than H_1 if they satisfy both $H_2 > (WNQD1)H_1$ and $H_2 > (WNQD2)H_1$. We write $H_2 > (WNQD)H_1$.

From the Definition 2.1, 2.4 and 2.5 we then have the following Theorem 2.6.

THEOREM 2.6. *Let H_1 and H_2 be bivariate distribution with specified marginals F and G. Assume that the bivariate distribution H_2 is more negatively quadrant dependent than H_1. Then H_2 is more weakly negative quadrant dependent than H_1.*

3. Closure properties of $(\beta^+, > (WNQD))$

In this section, we establish preservation of the $WNQD$ ordering under convolution, limit in distribution, compound distribution, mixture of a certain type, transformations of univariate increasing convex functions, and convex combination. First note that by theorem of Alzaid [1], (2.1) and (2.2) are equivalent to $E(f(T_{12}(a_1))g(T_{22}(a_2))) \leq E(f(T_{11}(a_1))g(T_{21}(a_2)))$ for all increasing positive convex functions f and g.

Below, we show that the ordering is preserved under convolution. We need the following Lemma 3,1 which is of independent interest.

LEMMA 3.1. *Let (a)* $\{(X_{11}(t), X_{21}(t))|t \geq 0\}$ *and* $\{(X_{12}(t), X_{22}(t))|t \geq 0\}$ *have distributions* H_1 *and* H_2, *where* H_1, H_2 *belong to* β^+, *(b)* $\{(X_{12}(t), X_{22}(t))|t \geq 0\} > (WNQD)\{(X_{11}(t), (X_{21}(t))|t \geq 0\}$, *and (c)* (Z_1, Z_2) *with an arbitrary* $WNQD$ *distribution function* H *independent of both of* $\{(X_{11}(t), X_{21}(t))|t \geq 0\}$ *and* $\{(X_{12}(t), X_{22}(t))|t \geq 0\}$.
Then $(X_{12}(t)+Z_1, X_{22}(t)+Z_2) > (WNQD)(X_{11}(t)+Z_1, X_{21}(t)+Z_2)$.

Proof. First, we will show that $(X_{12}(t)+Z_1, X_{22}(t)+Z_2)$ is $WNQD1$ $(WNQD2)$. Consider any hitting times $W_{ij}(a_i)$ given by $W_{ij}(a_i) = \inf\{t|X_{ij}(t) + Z_i \leq a_i, t \geq 0\}, i, j = 1, 2$.
Then,

$$\begin{aligned}
&Cov(f(W_{12}(a_1)),\ g(W_{22}(a_2))) \\
&= Cov(f(T_{12}(a_1 - Z_1)),\ g(T_{22}(a_2 - Z_2))) \\
&= Cov(E(f(T_{12}(a_1 - Z_1))|Z_1, Z_2), E(g(T_{22}(a_2 - Z_2))|Z_1, Z_2)) \\
&\quad + E(Cov(f(T_{12}(a_1 - Z_1)),\ g(T_{22}(a_2 - Z_2))|Z_1, Z_2)) \geq 0.
\end{aligned}$$

Note that the first and second terms are greater than or equal to zero for all functions f and g such that f is increasing positive convex(negative concave) and g is decreasing negative concave(positive convex)function. Thus by Theorem 3 of Alzaid(1990), $(X_{12}(t) + Z_1, X_{22}(t) + Z_2)$ is $WNQD$. Similarly we can show that $(X_{11}(t) + Z_1, X_{21}(t) + Z_2)$ is also $WNQD$.

Next, we will show that $(X_{12}(t)+Z_1, X_{22}(t)+Z_2) > (WNQD)(X_{11}(t) +Z_1, X_{21}(t) + Z_2)$ i.e.,

$E(f(T_{12}(a_1 - Z_1))g(T_{22}(a_2 - Z_2))) \leq E(f(T_{11}(a_1 - Z_1))g(T_{21}(a_2 - Z_2)))$ for any increasing positive convex functions f and g.
Now,

$$\begin{aligned}
&E(f(T_{12}(a_1 - Z_1))g(T_{22}(a_2 - Z_2))) \\
&= E(E(f(T_{12}(a_1 - Z_1))g(T_{22}(a_2 - Z_2))|Z_1, Z_2)) \\
&= E(E(f(T_{12}(a_1 - Z_1)g(T_{22}(a_2 - Z_2)))) \\
&\leqq E(E(f(T_{11}(a_1 - Z_1)g(T_{21}(a_2 - Z_2)))) \\
&= E(f(T_{11}(a_1 - Z_1))g(T_{21}(a_2 - Z_2))).
\end{aligned}$$

The inequality follows from the assumption that $(X_{12}(t), X_{22}(t)) > (WNQD)(X_{11}(t), X_{21}(t))$. □

THEOREM 3.2. *Suppose that the stochastic process (a)* $\{(X_{12}(t), X_{22}(t))|t \geq 0\}$ *is more weakly negative quadrant dependent than* $\{(X_{11}(t), X_{21}(t))|t \geq 0\}$, *(b)* $\{(Y_{12}(t), Y_{22}(t))|t \geq 0\}$ *is more weakly negative quadrant dependent than* $\{(Y_{11}(t), Y_{21}(t))|t \geq 0\}$, *and (c) let* $\{(X_{12}(t), X_{22}(t))|t \geq 0\}$ *and* $\{(Y_{12}(t), Y_{22}(t))|t \geq 0\}$ *be independent processes,* $\{(X_{11}(t), (X_{21}(t))|t \geq 0\}$ *and* $\{(Y_{11}(t), Y_{21}(t))|t \geq 0\}$ *be independent processes. Then* $\{(X_{12}(t) + Y_{12}(t), X_{22}(t) + Y_{22}(t))|t \geq 0\} > (WNQD)\{(X_{11}(t) + Y_{11}(t), X_{21}(t) + Y_{21}(t))|t \geq 0\}$.

Proof. By assumption, $(X_{12}(t), X_{22}(t)) > (WNQD)(X_{11}(t), X_{21}(t))$. Specifying $(Z_1(t), Z_2(t))$ to be $(Y_{12}(t), Y_{22}(t))$, we apply Lemma 3.1 to obtain

$$\begin{aligned}
&(X_{12}(t) + Y_{12}(t), X_{22}(t) + Y_{22}(t)) \\
&> (WNQD)(X_{11}(t) + Y_{12}(t), X_{21}(t) + Y_{22}(t))
\end{aligned} \tag{3.1}$$

Next, we use the assumption $(Y_{12}(t), Y_{22}(t)) > (WNQD)(Y_{11}(t), Y_{21}(t))$, specifying $(Z_1(t), Z_2(t))$ to be $(X_{11}(t), X_{21}(t))$, and again use Lemma 3.1 yielding

$$\begin{aligned}
&(X_{11}(t) + Y_{12}(t), X_{21}(t) + Y_{22}(t)) \\
&> (WNQD)(X_{11}(t) + Y_{11}(t), X_{21}(t) + Y_{21}(t)).
\end{aligned} \tag{3.2}$$

By combining (3.1) and (3.2),

$$\begin{aligned}&(X_{12}(t)+Y_{12}(t),X_{22}(t)+Y_{22}(t))\\ &>(WNQD)(X_{11}(t)+Y_{12}(t),X_{21}(t)+Y_{22}(t))\\ &>(WNQD)(X_{11}(t)+Y_{11}(t),X_{21}(t)+Y_{21}(t)).\end{aligned}$$

Thus $(X_{12}(t)+Y_{12}(t),X_{22}(t)+Y_{22}(t))>(WNQD)(X_{11}(t)+Y_{11}(t),$ $X_{21}(t)+Y_{21}(t))$.
This completes the proof. □

The next theorem demonstrates that, under suitable conditions, limits of the $WNQD$ ordering processes inherit the $WNQD$ ordering structure.

THEOREM 3.3. *Let (a) $\{(X_{n1}(t),X_{n2}(t))|t\geq 0\},\{(Y_{n1}(t),Y_{n2}(t))|t\geq 0\}$ have distributions $H_n,H_n^{'}$ for every n and $H_n>(WNQD)H_n^{'}$, (b) $\{(X_1(t),X_2(t))|t\geq 0\},\{(Y_1(t),Y_2(t))|t\geq 0\}$ have distributions $H,H^{'}$, (c) $\{(X_{n1}(t),X_{n2}(t))|t\geq 0\},\{(Y_{n1}(t),Y_{n2}(t))|t\geq 0\},\{(X_1(t),X_2(t))|t\geq 0\}$ and $\{(Y_1(t),Y_2(t))|t\geq 0\}$ have all sample paths and they are right continuous on $[0,\infty)$ with finite left limits at all t, and (d) $H_n\to H$ and $H_n^{'}\to H^{'}$ be weakly as $n\to\infty$, respectively. Then $H>(WNQD)H^{'}$.*

Proof. Denote by $C(H)$ and $C(H^{'})$ the sets of continuity points of H and $H^{'}$, respectively. Let $D=C(H)\cap C(H^{'})$. It follows from our assumptions that $H(t_1,t_2)\geq H^{'}(t_1,t_2)$ for all $(t_1,t_2)\in D$. Since D is a dense set in $\mathbb{R}^2, H>(WNQD)H^{'}$.

The following theorem is another application of Theorem 3.2 which is very important in recognizing $WNQD$ ordering in compound distributions which arise naturally in stochastic processes. □

THEOREM 3.4. *Let (a) $(Y_1,S_1),(Y_2,S_2),\cdots$ be independent random processes, (b) $(X_1,K_1),(X_2,K_2),\cdots$ be independent random processes, (c) (Y_i,S_i) and $(X_i,K_i), i=1,2,\cdots n$ are $WNQD$ random process (d) $(Y_i,S_i)>(WNQD)(X_i,K_i), i=1,2,\cdots,$ and (e) $N(t)$ be a Poisson* *process which is independent of* (Y_i, S_i) *and* $(X_i, K_i), i = 1, 2, \cdots$. *Then*

$$(Z_{12}(t) = \sum_{i=1}^{N(t)} Y_i, Z_{22}(t) = \sum_{i=1}^{N(t)} S_i)$$

$$> (WNQD)(Z_{11}(t) = \sum_{i=1}^{N(t)} X_i, Z_{21}(t) = \sum_{i=1}^{N(t)} K_i)$$

Our next result deals with the preservation of the $WNQD$ *ordering under mixture. In order to motivate our definition of a subclass of* β^+ *in which the* $WNQD$ *ordering is preserved under mixture we need a definition and a similar result of Ebrahimi and Ghosh* [8].

DEFINITION 3.5.[8]. A stochastic process $\{X_{22}(t)|t \geq 0\}$ is stochastically increasing(decreasing) in $\{X_{12}(t)|t \geq 0\}$ if $E(f(T_{22}(a_2))|T_{12}(a_1) = t_1)$ is increasing(decreasing) in t_1 for all $a_i \in E_i, i = 1, 2$, and positive increasing convex function f.

We shall use the abbreviation SI and SD for stochastically increasing and decreasing respectively.

REMARK 1. A stochastic process $\{(X_{12}(t), X_{22}(t))|t \geq 0\}$, given a scalar λ, is $WNQD1(WNQD2)$ if and only if $Cov[f(T_{12}(a_1)), g(T_{22}(a_2))|\lambda] \geq 0$ for all functions f and g such that f is non-decreasing, non-negative(non-positive concave), and g is non-increasing, non-positive concave(non-negative convex).

THEOREM 3.6. *Let (a)* $\{(X_{12}(t), X_{22}(t))|t \geq 0\}$ *given a scalar* λ, *a stochastic process be conditionally* $WNQD$, *and (b)* $\{X_{12}(t)|t \geq 0\}$ *be* SI *and* $\{X_{22}(t)|t \geq 0\}$ *be* SD *in* λ, *or* $(b')\{X_{12}(t)|t \geq 0\}$ *be* SD *and* $\{X_{22}(t)|t \geq 0\}$ *be* SI *in* λ. *Then*$\{(X_{12}(t), X_{22}(t))|t \geq 0\}$ *is* $WNQD$.

The next theorem deals with the preservation of the $WNQD$ *ordering under mixture. We may now define the class* β_λ^+ *by* $\beta_\lambda^+ = \{H_\lambda | H(t_1, \infty|\lambda) = F(t_1|\lambda), H(\infty, t_2|\lambda) = G(t_2|\lambda), H_\lambda|\lambda$ *is* $WN-QD, F$ *is* SD *and* G *is* SI *in* λ, *or* F *is* SI *and* G *is* SD *in* $\lambda\}$.

Now consider $(\beta_\lambda^+, > (WNQD))$. *The following theorem shows that if two elements of* β_λ^+ *are ordered according to* $> (WNQD)$, *then after mixing* λ, *the resulting element in* β^+ *preserves the same order.*

PROPOSITION 3.7. *Let the stochastic processes* $(X_{12}(t), X_{22}(t))$ *and* $(X_{11}(t), X_{21}(t))$, *given a scalar* λ, *belong to* β_λ^+, *respectively and* $((X_{12}(t), X_{22}(t))|\lambda) > (WNQD)((X_{11}(t), X_{21}(t))|\lambda)$ *for all* λ. *Then, unconditionally,* $(X_{12}(t), X_{22}(t)), (X_{11}(t), X_{21}(t))$ *belong to* β^+ *and* $(X_{12}(t), X_{22}(t)) > (WNQD)(X_{11}(t), X_{21}(t))$.

Proof. From Theorem 3.6, $(X_{12}(t), X_{22}(t))$ and $(X_{11}(t), X_{21}(t))$ are $WNQD$.

Now,

$$\begin{aligned} E(f(T_{12}(a_1))g(T_{22}(a_2))) &= E_\lambda(E(f(T_{12}(a_1))g(T_{22}(a_2))|\lambda)) \\ &\le E_\lambda(E(f(T_{11}(a_1))g(T_{21}(a_2))|\lambda)) \\ &= E(f(T_{11}(a_1))g(T_{21}(a_2))). \end{aligned}$$

The inequality comes from the fact that $(X_{12}(t), X_{22}(t))|\lambda > (WNQD)(X_{11}(t), X_{21}(t))|\lambda$ for all λ

Next, we show that the $WNQD$ ordering is invariant under transformations of stochastic processes by increasing convex functions. □

THEOREM 3.8. *Let (a)* $\{(X_{ij}(t), X_{ij}'(t))^{H_j}|t \ge 0\}, i = 1, 2, 3, \cdots, n$ *be* n*-independent pairs from a bivariate distribution* H_j *with continuous incrasing sample paths,* $j = 1, 2$, *(b)* H_1 *and* H_2 *belong to* β^+ *such that* $H_2 > (WNQD)H_1$, *and (c)* g_1 *and* g_2 *are positive convex functions and they are increasing in each of their arguments when all other arguments are fixed. Then the processes* $(Y_{12}(t), Y_{22}(t)) > (WNQD)(Y_{11}(t), Y_{21}(t))$, *given by* $Y_{1i}(t) = g_1(X_{1i}(t), \cdots, X_{ni}(t))$, $Y_{2i}(t) = g_2(X'_{1i}(t), \cdots, X'_{ni}(t)), i = 1, 2$.

Proof. First, we will show that the $WNQD1$ ordering holds. The proof will be given for the case $n = 2$. For the general n, the proof is similar. Fix $t_i \ge 0, i = 1, 2$ and introduce the variables $V_i = X_{2i}(t_i), V_i' = X_{2i}'(t_i), U_i = \sup_{0 \le s < t_i}(g_1(X_{1i}(s), X_{2i}(s)))$, and $U_i' = \sup_{0 \le s < t_i}(g_2(X_{1i}'(s), X_{2i}'(s))), i = 1, 2$, where for simplicity, t_1, t_2 have been suppressed in V_i, V_i', U_i and U_i'. Consider any hitting times of $Y_{1i}(s) = g_1(X_{1i}(s), X_{2i}(s)), Y_{2i}(s) = g_2(X_{1i}'(s), X_{2i}'(s)), i = 1, 2$ given by

$$W_{ij}(a_i) = \inf\{s|Y_{ij}(s) \le a_i, s \ge 0\}, i, j = 1, 2.$$

It suffices to show that

$$\int_{x_1}^{\infty}\int_{x_2}^{\infty} P(W_{12}(a_1) > t_1, W_{22}(a_2) > t_2)dt_1dt_2$$
$$\leq \int_{x_1}^{\infty}\int_{x_2}^{\infty} P(W_{11}(a_1) > t_1, W_{21}(a_2) > t_2)dt_1dt_2, i = 1,2.$$

Note the facts that $U_i = \sup_{0\leq s<t_i}(g_1(X_{1i}(s), V_i)), U_i^{'} = \sup_{0\leq s<t_i}$ $(g_2(X_{1i}^{'}(s), V_i^{'})), i = 1,2$, and that, by the hypothesis, random variables $(V_2, V_2^{'})$ and $(V_1, V_1^{'})$ are satisfied the following

$$(V_2, V_2^{'}) > (WNQD1)(V_1, V_2^{'}).$$

Now, we obtain

$$\begin{aligned}
&\int_{x_1}^{\infty}\int_{x_2}^{\infty} P(W_{12}(a_1) > t_1, W_{22}(a_2) > t_2)dt_1dt_2\\
&= \int_{x_1}^{\infty}\int_{x_2}^{\infty} P(U_1 < a_1, U_2 < a_2)dt_1dt_2\\
&= \int_{x_1}^{\infty}\int_{x_2}^{\infty} E[P(U_1 < a_1, U_2 < a_2|V_2, V_2^{'})]dt_1dt_2\\
&\leq \int_{x_1}^{\infty} E[P(U_1 < a_1|V_2)]dt_1 \int_{x_2}^{\infty} E[P(U_2 < a_2|V_2^{'})]dt_2\\
&\leq \int_{x_1}^{\infty}\int_{x_2}^{\infty} E[P(U_1 < a_1, U_2 < a_2|V_1, V_1^{'})]dt_1dt_2\\
&= \int_{x_1}^{\infty}\int_{x_2}^{\infty} P(W_{11}(a_1) > t_1, W_{21}(a_2) > t_2)dt_1dt_2
\end{aligned}$$

The proof of the $WNQD2$ ordering is similar. □

Next, we now turn our attention to a simple but important property of the class β^+.

THEOREM 3.9. *The class* $\beta^+ = \{H|H(t_1, t_2)$ *is* $WNQD, H(t_1, \infty) = F(t_1), H(\infty, t_2) = G^{'}(t_2)\}$ *is convex.*

Proof. Let H_1, H_2 in $\beta^+$and for α in $(0,1), H = \alpha H_1 + (1-\alpha)H_2$. Then we will show that H is convex combination of H_1 and H_2. Since each of the H_1 and $H_2 \in \beta^+$,

$$(3.3)\quad \begin{aligned} &\int_{x_1}^{\infty}\int_{x_2}^{\infty} P_H(T_{12}(a_1) > t_1,\ T_{22}(a_2) > t_2)dt_1dt_2 \\ =&\alpha\int_{x_1}^{\infty}\int_{x_2}^{\infty} P_{H_1}(T_{12}(a_1) > t_1,\ T_{22}(a_2) > t_2)dt_1dt_2 \\ &+ (1-\alpha)\int_{x_1}^{\infty}\int_{x_2}^{\infty} P_{H_2}(T_{12}(a_1) > t_1,\ T_{22}(a_2) > t_2)dt_1dt_2 \\ \leq& \alpha\int_{x_1}^{\infty}\int_{x_2}^{\infty} P_H(T_{12}(a_1) > t_1)P_H(T_{22}(a_2) > t_2)dt_1dt_2 \\ &+ (1-\alpha)\int_{x_1}^{\infty}\int_{x_2}^{\infty} P_H(T_{12}(a_1) > t_1)P_H(T_{22}(a_2) > t_2)dt_1dt_2 \\ =&\int_{x_1}^{\infty}\int_{x_2}^{\infty} P_H(T_{12}(a_1) > t_1)P_H(T_{22}(a_2) > t_2)dt_1dt_2. \end{aligned}$$

Hence H is $WNQD1$. The proof of the $WNQD2$ ordering is similar to the proof of the $WNQD1$. □

Moreover,

$$(3.4)\quad \lim_{t_1\to\infty} H(t_1,t_2) = \alpha G(t_2) + (1-\alpha)G(t_2) = G(t_2),$$

and

$$(3.5)\quad \lim_{t_2\to\infty} H(t_1,t_2) = \alpha F(t_1) + (1-\alpha)F(t_1) = F(t_1)$$

It follows from (3.3), (3.4), (3.5) that $H \in \beta^+$. Thus β^+ is convex.

4. Examples

EXAMPLE 4.1. Consider bivariate processes $\{(X_{n1}, Y_{n1})|n \geq 1\}$, $\{(X_{n2}, Y_{n2})|n \geq 1\}$ such that $(X_{11}, Y_{11}), (X_{21}, Y_{21}), \cdots$ are independent and $(X_{12}, Y_{12}), (X_{22}, Y_{22}), \cdots$ are independent processes. Then it is easy to check that $(X_{n2}, Y_{n2}) > (WNQD)(X_{n1}, Y_{n1}), n \geq 1$ whenever $(X_{i2}, Y_{i2}) > (WNQD)(X_{i1}, Y_{i1})$, for each $i = 1, 2, \cdots$

EXAMPLE 4.2. Consider a system with four components which is subjected to shocks. Let $N(t)$ be the number of shocks received by time t and $\{(X_k, S_k)|k = 1, 2, \cdots\}$ and $\{(Y_k, L_k)|k = 1, 2, \cdots\}$ are sequences of damages to components 1, 2, 3 and 4 by shock k, respectively. Define the compound Poisson processes by

$$Z_{11}(t) = \sum_{k=1}^{N(t)} Y_k, Z_{12}(t) = \sum_{k=1}^{N(t)} X_k, Z_{21}(t) = \sum_{k=1}^{N(t)} L_k, Z_{22}(t) = \sum_{k=1}^{N(t)} S_k.$$

This follows by application of Theorem 3.4 implies $(Z_{12}(t), Z_{22}(t)) > (WNQD)(Z_{11}(t), Z_{21}(t))$ for every $t \geq 0$ whenever $(X_i, S_i) > (WNQD)$ (Y_i, L_i), for each $i = 1, 2, 3, \cdots$

EXAMPLE 4.3. Let $Z_j = (Z_{1j}, Z_{2j})$ and $W_j = (W_{1j}, W_{2j}), j \geq 0$ be a sequence of i.i.d. bivariate vectors such that $(Z_{10}, Z_{20}) > (WNQD)(W_{10}, W_{20})$ random variables with marginal uniform distribution on the interval [0, 1], respectively. Consider the sequences $(X_{1n}(t), X_{2n}(t))$ and $(Y_{1n}(t), Y_{2n}(t))$(in $n \geq 1$) of bivariate processes defined by

$$\begin{aligned} X_n(t) &= (X_{1n}(t), X_{2n}(t)) = (\sqrt{n}(F_{1n}(t) - t), \sqrt{n}(F_{2n}(t) - t)), \\ Y_n(t) &= (Y_{1n}(t), Y_{2n}(t)) = (\sqrt{n}(G_{1n}(t) - t), \sqrt{n}(G_{2n}(t) - t)), t \in [0, 1], \end{aligned}$$

where for $i = 1, 2, F_{in}(t) = n^{-1} \sum_{j=1}^{n} I(Z_{ij} \leq t), G_{in}(t) = n^{-1} \sum_{j=1}^{n}$ $I(W_{ij} \geq t)$ are usual empirical c.d.f. of the i.i.d random variables $Z_{i1}, Z_{i2}, \cdots, Z_{in}$ and $W_{i1}, W_{i2}, \cdots, W_{in}$, respectively. Note that $X_n(t)$ and $Y_n(t)$ are simply the combination of the two(dependent)one-dimensional empirical processes, respectively. Such processes have been used by Goel and Ramallingam(1987) to study matching problems. Fix $i = 1, 2$, then for all real a_i, it is easy to verify that the hitting times $T_i(a_i) = \inf\{t|X_{in}(t) \leq a_i\}$ and $S_i(a_i) = \inf\{t|Y_{in}(t) \leq a_i\}$ are increasing functions of $Z_{i1}, \cdots, Z_{in}, W_{i1}, \cdots, W_{in}$, respectively. In view of this fact, if we fixed $n \geq 1$, then we can argue (see Tong(1980)), p. 84) that for all $a_i, i = 1, 2, (T_1(a_1), T_2(a_2)) > (WNQD)(S_1(a_1), S_2(a_2))$ random variables. We conclude that $(X_{1n}(t), X_{2n}(t)) > (WNQD)(Y_{1n}(t), Y_{2n}(t))$, for each $n \geq 1$. It is easy to check that $(X_{1n}(t), X_{2n}(t))$ converges weakly to $(X_1(t), X_2(t))$ and $(Y_{1n}(t), Y_{2n}(t))$ converges weakly to $(Y_1(t), Y_2(t))$ as $n \to \infty$ on the time interval [0,1]. Hence, using the Theorem 3.3, we can obtain that $(X_1(t), X_2(t)) > (WNQD)(Y_1(t), Y_2\ (t))$.

References

[1] Alzaid, A. A., *A weak quadrant dependence concept with Applications*, Comm. Stat. Stoch. Models **6** (1990), 353-363.

[2] Baek, J. I., *The ordering of hitting times of multivariate processes*, J. Kor. Stat. **25** (1996), 545-556.

[3] ———, *A weakly dependence structure of multivariate processes*, (Stat. & Proba. Letters)(to appear), 1997.

[4] Barlow, R. and Proschan, F., *Statistical Theory of Reliability and Life Testing: Probability Models. Holt, Rinehart and Winston, Inc.*, New York, 1975.

[5] Choi, J. E., Baek, J. I. and Youn E. H., *A weakly dependence concept of bivariate stochastic processes*, Comm. Kor. Math Soc. **11** (1996), 831-839.

[6] Ebrahimi, N., *The ordering of negative quadrant dependence*, Comm. Stat. Theor Meth. **11** (1982), 2389-2399.

[7] ———, *Bivariate Processes with Positive or Negative Dependent Structures*, J. Appl. Pro. **24** (1987), 115-122.

[8] Ebrahimi, N. and Ghosh, M., *Multivariate negative dependence*, Commun. Statist A10 (1981), 307-337.

[9] Ebrahimi, N. and Ramallingam. T., *On the dependence structure of hitting times of multivariate processes*, J. Appl. Pro. **26** (1989), 287-295.

[10] Friday, D. S., *Dependence concepts for stochastic processes*, Proc. Nato Advanced Study Instiutes Series **5** (1981), 349-361.

[11] Goel and Ramalingam, *Some properties of the maximum likelihood strategy for repairing a broken random sample*, J. Stat. Planning Inf. **16** (1987), 237-248.

[12] Karlin, S. and Rinott, Y., *Classes of orderings of measures and related correlation inequalities*, J. Multivariate Anal. **10** (1980), 467-498.

[13] Lehmann, E., *Some concepts of dependence*, Ann. Math. Statist **37** (1966), 1137-1153.

[14] Sampson, A. R., *Positive dependenceproperties of elliptically symmetric distribution*, J. of Multivariate Analysis **13** (1983), 375-381

[15] Tong, Y. L., *Probability Inequalities in Multivariate Distributions*, Academic press, New York, 1980.

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