ON ASYMPTOTIC BEHAVIOR OF A RANDOM EVOLUTION

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1. Introduction

In this paper, we study the asymptotic behavior of a random evolution. Some examples of random evolution can be found in Chapter 12 of [2].

In [4][5], Kurtz and Protter worked also on an approximation of solutions of SDE applying their Theorem 5.4 in the same paper. Motivated by theorems by Kurtz and Protter, we now consider a sequence of stochastic differential equations. This study dates back at least to Khasminskii [3], who studies the behavior of trajectory of stochastic process defined by the differential equation with a rapidly varying components,

\[ \frac{dx}{dt} = \epsilon F(x, t, \omega), \quad x(0) = x_0, \]

over a length of time of order \( O(\frac{1}{\epsilon}) \) as \( \epsilon \to 0 \).

Let \( E \) be a separable metric space, \( Z \) be an \( E \)-valued ergodic Markov process with stationary distribution \( \mu \). We assume that \( F : R \times E \to R \) is bounded and has bounded and continuous first order partial derivatives such that \( \int F(x, y)\mu(dy) = 0 \).

Let \( X_n, \quad n = 1, 2, \ldots \), satisfy;

\[ dX_n(t) = nF(X_n(t), Z(n^2 t))dt \]
We shall consider the limit behavior of solution processes, \( X_t \), in an extension of the results of Wong and Zakai [6]: that certain naive approximations of semimartingale differentials lead to a lack of continuity of the corresponding solutions of stochastic differential equations. You may refer this kind of results to [4], [5].

We assume the following hypotheses:

There exists an operator \( A \) which is the generator of \( Z \) such that (letting \( \mathcal{R}(A) \) be the range of \( A \) and \( \mathcal{D}(A) \) be the domain of \( A \)) \( L^2_E(\mu) \) is generated by 1 and \( \mathcal{R}(A) \), and \( \mathcal{D}(A) \) is an algebra.

\( A \) has the eigenvectors \( \{f_k\} \) with eigenvalues \( \{\lambda_k\} \) which satisfy;

**Condition 1.1.**

1) For each \( T > 0 \) there exists a \( M_0 > 0 \) such that \( \sup_{0 \leq s \leq T} E[f_k(Z(s))] < M_0 \) for every \( k \).

2) \( f_0 = 1, \int f_i \cdot f_j d\mu = 0 \) if \( i \neq j \), and \( \int f_i^2(z) d\mu(z) = 1 \) if \( i = 1, 2, \ldots \)

3) \( \sum_{k=0}^{\infty} \frac{1}{\lambda_k} < \infty \)

4) \( R(A) = \langle 1, f_1, f_2, \ldots, f_k, \ldots \rangle = L_E^2(\mu) \),

where \( \langle 1, f_1, f_2, \ldots, \rangle \) is the smallest space generated by \( 1, f_1, f_2, \ldots \).

Now we expand \( F(x, \cdot) \) in \( L_E^2(\mu) \) with \( f_k \). Let

\[
g_k(x) = \int F(x, y) f_k(y) \mu(dy) = \langle F(x, y), f_k(y) \rangle_\mu \quad k = 1, 2, \ldots
\]

(1.2)

\[
g_0(x) = \int F(x, y) 1 \mu(dy) = 0
\]

Then

\[
F(x, y) = \sum_{k=0}^{\infty} g_k(x) f_k(y)
\]

By Bessel's inequality,

\[
\sum_{k=0}^{\infty} |g_k(x)|^2 \leq \int F(x, y)^2 \mu(dy) < \infty.
\]

Then, (1.1) can be rewritten as

\[
X_n(t) = X_n(0) + n \cdot \int_0^t \left( \sum_{k=0}^{\infty} g_k(X_n(s)) \cdot f_k(Z(n^2 s)) \right) ds,
\]

(1.3)
On asymptotic behavior of a random evolution

where the stochastic integral is just a Stieltjes integral and consequently needs no special definition. Finally, we need to assume that there exist \( \eta_k, \quad k = 1, 2, \cdots \) such that

\[
\sup_x |g_k(x)| \leq \frac{1}{\eta_k}, \quad \sum_{k=0}^\infty \frac{1}{|\eta_k|^2} < \infty.
\]

(1.4)

**Example 1.1.** Let \( Z(s) \) be Brownian Motion with state space \([0, \pi]\), which reflects at both end points. Then

\[
A = \{(f, \frac{1}{2} f^{''}) | f \in C^2[0, \pi], f'(0) = f'(\pi) = 0\}
\]

is the generator of \( Z(s) \). The eigenvectors of \( A \) are \( f_k(x) = \sqrt{\frac{2}{\pi}} \cos kx, k = 1, 2, \cdots \) and the eigenvalues \( \lambda_k = -k^2 \). Then our \( \{f_k\} \) and \( \{\lambda_k\} \) satisfies the assumptions.

1) \( \{f_k(x)\} \) is uniformly bounded.
2) \( \sum_{k=1}^{\infty} |\frac{1}{\lambda_k}| = \sum \frac{1}{k^2} < \infty \)
3) \( A(f_k^2) = \frac{2k^2}{\pi} \cos 2kx, \frac{1}{\lambda_k} A(f_k^2) = -\frac{2}{\pi} \cos 2kx \)

Furthermore, let \( F : R \times [0, \pi] \rightarrow R \) be a bounded and even function. Then \( F(x, z) \) can be expanded

\[
F(x, z) = \sum_{k=0}^{\infty} g_k(x) \cos kzd, \quad g_k(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\pi} F(x, z) \cos kzdz
\]

and \( \|g_k\|_\infty \leq \|F\|_\infty \cdot \frac{1}{k} \)

Choosing \( \eta_k = k \) we can see \( g_k \) satisfies the assumption (1.4). \( \square \)

2. Main theorem

Define \( W_n^k(t), Y_n^k(t) \) and \( Z_n^k(t) \) such that

\[
W_n^k(t) = \int_0^t n f_k Z(n^2 s) \, ds = \frac{1}{n} \int_0^{n^2 t} f_k(Z(s)) \, ds
\]

(2.1)

\[
Y_n^k(t) = -\frac{1}{n \lambda_k} f_k(Z(n^2 t)) + \frac{1}{n} \int_0^{n^2 t} A(\frac{1}{\lambda_k} f_k)(Z(s)) \, ds
\]

\[
Z_n^k(t) = \frac{1}{n \lambda_k} f_k(Z(n^2 t))
\]

235
Then \( W_n^k(t) = Y_n^k(t) + Z_n^k(t) \), and (1.1) can be expressed;

\[
X_n(t) = X_n(0) + n \int_0^t F(X_n(s), Z(n^2s)) ds
\]

\[
= X_n(0) + \sum_{k=0}^{\infty} \int_0^t g_k(X_n(s)) dW_n^k(s)
\]

\[
= X_n(0) + \sum_{k=0}^{\infty} \int_0^t g_k(X_n(s)) dY_n^k(s) + \sum_{k=0}^{\infty} \int_0^t g_k(X_n(s)) dZ_n^k(s)
\]

Before we state our main theorem, we first see the limit behavior of \( Y_n^k, k = 1, 2, \ldots \).

**Lemma 2.1.** Let

\[
A_n^{kj}(t) = \frac{1}{n^2} \int_0^{n^2t} A(\frac{f_k}{\lambda_k}, \frac{f_j}{\lambda_j})(Z(s)) - \frac{f_i}{\lambda_j} A(\frac{f_k}{\lambda_k})(Z(s)) - \frac{f_k}{\lambda_k} A(\frac{f_j}{\lambda_j})(Z(s)) ds
\]

for any \( k, j = 1, 2, \ldots \) and let

\[
C_{kj} = \int \frac{2}{\lambda_k} f_k^2(z) d\mu(z) = \frac{2}{|\lambda_k|} \quad \text{if } k = j
\]

\[
= 0 \quad \text{if } k \neq j
\]

Then

\[
A_n^{kj}(t) \to t \cdot C_{kj}, \quad \text{a.s.}
\]

and

\[
\lim_{n \to \infty} E[Y_n^k]_t = \frac{2t}{|\lambda_k|}
\]

**Proof.** Since \( Z(s) \) is ergodic with stationary distribution \( \mu \)

\[
A_n^{kj}(t) \to t \cdot C_{kj} \quad \text{a.s.}
\]
Note that $Y^k_n(t)Y^j_n(t) - A^{kj}_n(t)$ is a martingale and hence,

$$E[Y^k_n, Y^j_n]_t = E[A^{kj}_n(t)]$$

$$E[Y^k_n]_t = \frac{1}{n^2} \int_0^{n^2t} \left[ \frac{1}{\lambda_k^2} A(f_k^2)(Z(s)) - \frac{2f_k^2}{\lambda_k}(Z(s)) \right] ds$$

$$\to \frac{2t}{|\lambda_k|},$$

as $n \to \infty$. □

**Lemma 2.2.** For every $d$, $d = 1, 2, \cdots$ there exists a process $Y = (Y^1, \cdots, Y^d)$ with sample paths in $C_{R^d}[0, \infty)$ such that $(Y^1_n, \cdots, Y^d_n) \Rightarrow (Y^1, \cdots, Y^d)$ and $Y^i_n Y^j_n - C_{ij}$, $i, j = 1, 2, \cdots, d$ are martingales with respect to $\{\mathcal{F}_t^Y\}$. The process $Y$ has independent Gaussian increments.

**Proof.** For each $i, j = 1, 2 \cdots$ $Y^i_nY^j_n - A^{ij}_n(t)$ is an $\mathcal{F}_t^n$-martingale and $A^{ij}_n(t) \rightarrow C_{ij}(t)$. So, by the martingale central limit theorem (Th.7.1.4 [2]) we get the conclusion. □

We shall show that the sequence of solution to equation (1.5), $\{X_n\}$ is relatively compact and get a possible limit. In fact, in (2.2) we show that (*) and (**) are relatively compact in $D_R[0, \infty)$. Then $\{X_n\}$ is also relatively compact, since the limits are continuous. If we apply Theorem 2.2 [4] we can see the limit of (*) and using the ergodic theorem, we will see the limit of (**).

**Theorem 2.1.** Let $Z(s)$ be an ergodic process with generator $A$ and stationary distribution $\mu$ satisfying the above hypotheses. If $X_n(0) \Rightarrow X(0)$, then $\{X_n\}$ is relatively compact and any limit point $X$ satisfies

$$X(t) = X(0) + \sum_{k=0}^{\infty} \int_0^t g_k(X(s))dY^k(s)$$

$$+ \sum_{k=0}^{\infty} \int_0^t \int_E \frac{1}{\lambda_k} g_k(X(s))f_k(z)F(X(s), z)\mu(dz)ds$$

where $Y^k, k = 1, 2 \cdots$ are martingale processes with Gaussian independent increments.

237
Proof. First, for convenience, let's denote

\[
\bar{X}_n(t) = \sum_{k=0}^{\infty} \int_0^t g_k(X_n(s))dY_n^k(s)
\]

(2.4)

\[
\tilde{X}_n(t) = \sum_{k=0}^{\infty} \int_0^t g_k(X_n(s))dZ_n^k(s)
\]

\[
X_n(t) = X_n(0) + \bar{X}_n(t) + \tilde{X}_n(t),
\]

where \(Y_n^k(t) = -\frac{1}{\lambda k} f_k(Z(n^2(t))) + \frac{1}{n} \int_0^{n^2t} A(\frac{f_k}{\lambda k})(Z(s))ds\). We shall show the relative compactness for \(X_n(t)\) in (2.4). \textbf{Step1} To show the relative compactness of \(\{\bar{X}_n\}\) according to (1.4), choose \(\eta_k > 0\) such that

\[
\sup_{0 \leq s \leq t} |g_k(X_n(s))| \leq \frac{1}{\eta_k} \quad \text{for every} \quad n \quad \text{and} \quad \sum \frac{1}{\eta_k^2} < \infty
\]

Then for all \(n\)

\[
E[|\bar{X}_n(t)|^2]
\]

\[
\leq E[\sum_{k=0}^{\infty} \int_0^t g_k^2(X_n(s))d|Y_n^k|] + E[\sum_{k \neq j} \int_0^t g_k(X_n(s))g_j(X_n(s))d|Y_n^k, Y_n^j|] \]

\[
\leq \sum_{k=0}^{\infty} \frac{1}{\eta_k^2} E[|Y_n^k|^t] + \sum_{k \neq j} \left( \int_0^t g_k^2(X_n(s))d|Y_n^k| \right)^{\frac{1}{2}} \left( \int_0^t g_j^2(X_n(s))d|Y_n^j| \right)^{\frac{1}{2}} \]

(2.5)

by Kunita-Watanabe inequality

\[
\rightarrow \sum_{k=0}^{\infty} \frac{1}{\eta_k^2} \frac{2t}{|\lambda_k|} + \sum_{k \neq j} \frac{1}{\eta_k} \left( \frac{2t}{|\lambda_k|} \right)^{\frac{1}{2}} \frac{1}{\eta_j} \left( \frac{2t}{|\lambda_j|} \right)^{\frac{1}{2}} \quad \text{by (2.3)}
\]

\[
= C_0 \cdot t, \quad C_0 = 2\left( \sum_{k=0}^{\infty} \frac{1}{\eta_k^2 |\lambda_k|} + \sum_{k \neq j} \frac{1}{\eta_k |\lambda_k|^{\frac{1}{2}}} \frac{1}{\eta_j |\lambda_j|^{\frac{1}{2}}} \right)
\]

Hence, for each \(\eta > 0\),

\[
\lim_{n \to \infty} P\{|\bar{X}_n(t)| > \left( \frac{C_0 t}{\eta} \right)^{\frac{1}{2}} \} \leq \lim_{n \to \infty} \frac{E[|\bar{X}_n(t)|^2]}{C_0 t} \eta \leq \eta
\]

238
On asymptotic behavior of a random evolution

for every \( n \). Choose \( \Gamma_{\eta,t} = \mathbb{B}(0, (\frac{C_\eta}{\eta})^{\frac{1}{2}}) \), then

\[
\lim_{n \to \infty} \mathbb{P}\{\bar{X}_n(t) \in \Gamma_{\eta,t}\} \geq 1 - \eta.
\]

To see the other criteria for relative compactness for \( \{\bar{X}_n(t)\} \),

\[
\begin{align*}
E[|\bar{X}_n(t + u) - \bar{X}_n(t)|^2 |\mathcal{F}_t] & \leq \sum_{k=0}^{\infty} E[\int_t^{t+u} (g_k(X_n(s))dY_n^k(s))^2] \\
& + \sum_{k \neq j} (\int_t^{t+u} g_k^2(X_n(s))d[Y_n^k]_s)^{\frac{1}{2}} (\int_t^{t+u} g_j^2(X_n(s))d[Y_n^j]_s)^{\frac{1}{2}} |\mathcal{F}_t] \\
& \leq E[\sum_{k=0}^{\infty} \frac{1}{\eta_k^2} ([Y_n^k]_{t+u} - [Y_n^k]_t) |\mathcal{F}_t] \\
& + E[\sum_{k \neq j} \frac{1}{\eta_k} ([Y_n^k]_{t+u} - [Y_n^k]_t)^{\frac{1}{2}} \frac{1}{\eta_j} ([Y_n^j]_{t+u} - [Y_n^j]_t)^{\frac{1}{2}} |\mathcal{F}_t],
\end{align*}
\]

Let

\[
\gamma_n(\delta) = \sum_{k=0}^{\infty} \frac{1}{\eta_k} ([Y_n^k]_{t+\delta} - [Y_n^k]_\delta) \\
+ \sum_{k \neq j} \frac{1}{\eta_k \eta_j} ([Y_n^k]_{t+\delta} - [Y_n^k]_t)^{\frac{1}{2}} ([Y_n^j]_{t+\delta} - [Y_n^j]_t)^{\frac{1}{2}}
\]

Then, we have for \( 0 \leq t \leq T, \ 0 \leq u \leq \delta, \)

\[
E[|\bar{X}_n(t + u) - \bar{X}_n(t)|^2 |\mathcal{F}_t] \leq E[\gamma_n(\delta) |\mathcal{F}_t]
\]

and since \( Y_n^k(t)Y_n^j(t) - A_{n}^{kj}(t) \) is a martingale and \( Y_n^k(t)Y_n^j(t) - [Y_n^k, Y_n^j]_t \) is also a martingale.

\[
\lim_{\delta \to 0} \sup_n E[\gamma_n(\delta)] = \lim_{\delta \to 0} \left( \sum_{k=0}^{\infty} \frac{1}{\eta_k} \frac{2\delta}{|\lambda_k|} + \sum_{k \neq j} \frac{1}{\eta_k \eta_j} \frac{2\delta}{|\lambda_k \lambda_j|^2} \right) = 0
\]

239
Nhansook Cho

Step 2 To show \( \{ \sum_{k=0}^{\infty} \int_0^t g_k(X_n(s))dZ_n^k(s) \} \) is relatively compact, fix \( T > 0 \). Since \([X_n, X_n]_t = 0\) and \([g_k(X_n), Z_n^k]_t = 0\), by integration parts

\[
\int_0^t g_k(X_n(s))dZ_n^k(s) = g_k(X_n(t))Z_n^k(t) - g_k(X_n(0))Z_n^k(0)
\]

\[
- \int_0^t g_k'(X_n(s))Z_n^k(s)dX_n(s)
\]

\[
- \int_0^t g_k(X_n(s))dZ_n^k(s)
\]

Since

\[
\|g_k\|_{\infty} \leq \|F\|_{\infty},\ |g_k'|_{\infty} \leq \|\frac{\partial F}{\partial x}\|_{\infty},
\]

we have for \( 0 \leq t \leq T \),

(2.7)

\[
E[\int_0^t g_k(X_n(s))dZ_n^k(s)]
\]

\[
= E[\|g_k(X_n(t))Z_n^k(t) - g_k(X_n(0))Z_n^k(0) - \int_0^t g_k'(X_n(s))Z_n^k(s)dX_n(s)]
\]

\[
\leq \|g_k\|_{\infty} E[\left| \frac{f_k(Z(n^2t))}{n\lambda_k} \right| + \left| \frac{f_k(Z(0))}{n\lambda_k} \right|] + \|g_k'\|_{\infty} E[\int_0^t \frac{f_k(Z(n^2s))}{\lambda_k} ds]
\]

\[
\leq \frac{1}{\eta_k} \frac{2M_0}{n\lambda_k} + \frac{1}{\lambda_k} \|g_k'\|_{\infty} M_0 t \left( \frac{2M_0}{n\eta_k\lambda_k} + \|\frac{\partial F}{\partial x}\|_{\infty} M_0 t \right),
\]

where \( \sup_{0 \leq s \leq T} E[f_k(Z(s))] < M_0 \) for all \( k \) by Condition 1.1.

Hence for every \( \eta > 0 \), let \( \Gamma_{\eta, t} = B(0, \frac{1}{\eta} \sum_{k=0}^{\infty} \frac{2M_0}{\lambda_k\eta_k} + \frac{1}{\lambda_k} \|g_k'\|_{\infty} M_0 t) \). Then,

\[
\lim_{n \to \infty} P\left\{ \sum_{k=0}^{\infty} \int_0^t g_k(X_n(s))dZ_n^k(s) \in \Gamma_{\eta, t} \right\} \geq 1 - \eta
\]

240
On asymptotic behavior of a random evolution

Also,

\[ |\tilde{X}_n(t + u) - \tilde{X}_n(t)| \]
\[ \leq \sum_{k=0}^{\infty} |g_k(X_n(t + u))Z_n^k(t + u) - g_k(X_n(t))Z_n^k(t)| \]
\[ + \left| \int_t^{t+u} g'_k(X_n(s))Z_n^k(s)dX_n(s) \right| \]
\[ \leq \sum_{k=0}^{\infty} \frac{2M_0}{n\lambda_k} + \|g'_kF\|_{\infty} \frac{M_0u}{\lambda_k} \]

Let

\[ \gamma_n(\delta) = \sum_{k=0}^{\infty} \frac{2M_0}{n\lambda_k} + \|g'_kF\|_{\infty} \frac{M_0\delta}{\lambda_k} \]

Then, for \( 0 \leq t \leq T, 0 \leq u \leq \delta \)

\[ E[|\tilde{X}_n(t + u) - \tilde{X}_n(t)| |\mathcal{F}_t] \leq E[\gamma_n(\delta) |\mathcal{F}_t] \]

and

\[ \lim_{\delta \to 0} \lim_{n \to \infty} E[\gamma_n(\delta)] = \lim_{\delta \to 0} \sum_{k=0}^{\infty} \|g'_kF\|_{\infty} \frac{M_0\delta}{\lambda_k} = 0 \]

Hence \( \{\tilde{X}_n = \sum_{k=0}^{\infty} \int_0^t g_k(X_n(s))dZ_n^k(s)\} \) is relatively compact. So far, we have seen that \( \{X_n(t)\} \) is relatively compact.

Since for all \( n \)

\[ \sum_{k=0}^{\infty} \int_0^t |g_k(X_n(s))dZ_n^k(s)| < \infty \]

according to (2.7), the limit of the series,

\[ \sum_{k=0}^{\infty} \int_0^t g_k(X_n(s))dZ_n^k(s) \]
is the same as the sum of limits of each term. From (2.7)

$$
\sum_{k=0}^{\infty} \int_{0}^{t} g_k(X_n(s))dZ_n^k(s)
$$

$$
= \sum_{k=0}^{\infty} (g_k(X_n(t))Z_n^k(t) - g_k(X_n(0))Z_n^k(0))
- \sum_{k=0}^{\infty} \int_{0}^{t} g_k'(X_n(s))Z_n^k(s)dX_n(s)
$$

It is obvious as $n \to \infty$,

$$
g_k(X_n(t))Z_n^k(t) - g_k(X(0))Z_n^k(s) \to 0.
$$

□

The following lemma is to get a limit of the second series of (2.8).

**Lemma 2.3.** Let $X$ be a limit point of $X_n$. Then along the appropriate subsequence

$$
\int_{0}^{t} g_k'(X_n(s))Z_n^k(s)dX_n(s)
$$

$$
\Rightarrow \int_{0}^{t} \int_{E} \frac{1}{\lambda_k} g_k'(X(s))f_k(z)F(X(s), z)\mu(dz)ds
$$

**Proof.** For $B \subset E$, let

$$
\Gamma_n([0,t] \times B) \equiv \int_{0}^{t} I_B(Z(n^2s))ds
$$

$$
\Gamma([0,t] \times B) \equiv \int_{0}^{t} I_B(Z(s))d\mu(z) \cdot t
$$

By the ergodicity of $Z(s)$, $\Gamma_n \to \Gamma$ a.s. as $n \to \infty$. Let

$$
U_n(t) = \int_{0}^{t} g_k'(X_n(s)) \frac{f_k(Z(n^2s))}{n\lambda_k} nF(X_n(s), Z(n^2s))ds
$$

$$
= \int_{0}^{t} \int_{E} \frac{1}{\lambda_k} g_k'(X_n(s))f_k(z)F(X_n(s), z)\Gamma_n(ds \times dz),
$$
and let $X(t)$ be a weak limit of $X_n(t)$. Since $X$ is continuous on $[0, t]$ and $(X_n, \Gamma_n) \Rightarrow (X, \Gamma)$, we get $U_n(t) \Rightarrow U(t)$, where

$$U(t) = \int_0^t \int_E \frac{1}{\lambda_k} g_k(X(s)) f_k(z) F(X(s), z) d\mu(z) ds.$$  

Finally, we shall show the limit of (*) in (2.2)

**Lemma 2.4.** Let $X$ be a limit point of $X_n$. Then along the appropriate subsequence

$$\sum_{k=0}^{\infty} \int_0^t g_k(X_n(s)) dY_n^k(s) \Rightarrow \sum_{k=0}^{\infty} \int_0^t g_k(X(s)) dY^k(s)$$

**Proof.** Let $X$ be a limit of $X_n$ and we have in Lemma 1.2

$$Y_n^k \Rightarrow Y^k \quad \text{for } k = 1, 2, \ldots$$

Applying the Skorohod representation theorem again, we can assume that $(X_n, Y_n) \rightarrow (X, Y)$ a.s. Note that we have

$$E[\sum_{k=0}^{\infty} |\int_0^t g_k(X_n(s)) dY_n^k(s)|] \leq \sum_{k=0}^{\infty} \frac{1}{\eta_k} \left( \frac{1}{\lambda_k} M_0 t \right)^{1/2} < \infty,$$

uniformly in $n$, so $\sum_{k=0}^{\infty} \int_0^t g_k(X_n(s)) dY_n^k(s)$ converges with probability 1 uniformly in $n$, by the generalized Borel-Cantelli lemma. Since for any $\epsilon$, we can choose $N$ s.t.

$$E[|\sum_{k=N}^{\infty} \int_0^t g_k(X_n(s)) dY_n^k(s) - \sum_{k=N}^{\infty} \int_0^t g_k(X(s)) dY^k(s)|] \leq \epsilon^2,$$

we have

$$P(\sum_{k=N}^{\infty} |\int_0^t g_k(X_n(s)) dY_n^k(s) - \sum_{k=N}^{\infty} \int_0^t g_k(X(s)) dY^k(s)| \geq \epsilon) \leq \epsilon$$

243
Now, for each \( k \), (*) implies that \( Y^k \) satisfies the Condition 2.2(1) [4] and hence, \((X_n, Y^k) \Rightarrow (X, Y^k)\) implies that

\[
\sum_{k=0}^{N} \int_{0}^{t} g_k(X_n(s))dY^k_n(s) \Rightarrow \sum_{k=0}^{N} \int_{0}^{t} g_k(X(s))dY^k(s)
\]

by Theorem 2.2 [4]. It implies that

\[
\sum_{k=0}^{\infty} \int_{0}^{t} g_k(X_n(s))dY^k_n(s) \Rightarrow \sum_{k=0}^{\infty} \int_{0}^{t} g_k(X(s))dY^k(s).
\]

\[
\square
\]

**Example (continued).** Let \( Z(s) \) be Brownian motion with state space \([0, \pi]\), which reflects at both end points. Then

\[
A = \{(f, \frac{1}{2}f'') | f \in C^2[0, \pi], f'(0) = f'(\pi) = 0 \}
\]

is the generator of \( Z(s) \). The eigenfunctions of \( A \) are \( f_k(x) = 2 \cos k(x), k = 1, 2, \cdots \) and the eigenvalues \( \lambda_k = -k^2 \). Then our \( \{f_k(x)\} \) and \( \{\lambda_k\} \) satisfy the assumptions. Let \( F: R \times [-\pi, \pi] \to R \) be a bounded function. Assume for each fixed \( x \in R \), \( F(x, \cdot) \in C^1[0, \pi] \) and is even function. Since \( \int_{-\pi}^{\pi} F(x, z)dz = 0 \), \( F(x, z) \) can be expanded

\[
F(x, z) = \sum_{k=0}^{\infty} \sqrt{\frac{2}{\pi}} g_k(x) \cos k z, \quad g_k(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x, z) \sqrt{\frac{2}{\pi}} \cos kz dz
\]

And the Feller semigroup \( \{S(t)\} \) on \( C^1(R) \) generated by \( A \) has a unique stationary distribution \( \mu \), which is \( \frac{1}{\pi}dx \), \( dx \) is the Lebesgue measure.

Consider an equation,

\[
dX_n(t) = nF(X_n(t), Z(n^2t))dt
\]

Then \( \{X_n(t)\} \) is relatively compact and any limit point \( X(t) \) satisfies

\[
X(t) = \sum_{k=0}^{\infty} \int_{0}^{t} g_k(X(s))dY^k(s)
\]

\[
- \frac{1}{\pi} \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \int_{0}^{\pi} \int_{0}^{\pi} \frac{1}{k^2} g'_k(X(s))g_j(X(s))f_k(x)f_j(x)dxdy
\]

\[
= \sum_{k=0}^{\infty} \int_{0}^{t} g_k(X(s))dY_k(s) - \sum_{k=0}^{\infty} \frac{1}{k^2} \int_{0}^{t} g'_k(X(s))g_k(X(s))ds
\]

244
On asymptotic behavior of a random evolution

since \( \frac{1}{\pi} \int_0^\pi \cos kx \cos jx \, dx = \delta_{k,j} \) Here, \( Y_k, k = 1, 2, \ldots \) are Brownian motions with covariance \( C_{kj} \),

\[
C_{kj} = \begin{cases} 
0 & \text{if } k \neq j \\
\frac{2}{k^2} & \text{if } k = j
\end{cases}
\]

References


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245