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REMARKS ON R-WEAKLY COMMUTING MAPPINGS AND COMMON FIXED POINT THEOREMS

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1. Introduction and Preliminaries

It was the turning point in the "fixed point arena" when the notion of weak commutativity was introduced by Sessa [9] as a sharper tool to obtain common fixed points of mappings. As a result, all the results on fixed point theorems for commuting mappings were easily transformed in the setting of the new notion of weak commutativity of mappings. It gives a new impetus to the studying of common fixed points of mappings satisfying some contractive type conditions and a number of interesting results have been found by various authors. A bulk of results were produced and it was the centre of vigorous research activity in "Fixed Point Theory and its Application in various other Branches of Mathematical Sciences" in last two decades. A major break through was done by Jungck [3] when he proclaimed the new notion what he called "compatibility" of mapping and its usefulness for obtaining common fixed points of mappings was shown by him. Thereafter a flood of common fixed point theorems was produced by various researchers by using the improved notion of compatibility of mappings. In fact, every weak commutive pair of mappings is compatible but the converse is not true ([3]). However, Singh has remarked in his review of [4] that weak commutivity does not imply the existence of a sequence of points satisfying the condition of compatibility.

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Pant [6] introduced the following definitions:

DEFINITION 1.1. Let (X, d) be a metric space and let f, g be self-mappings of X. The mappings f and g are is said to be *R-weakly commuting* if there exists a positive real number R such that

$$d(fgx, gfx) \leq Rd(fx, gx) \tag{1.1}$$

for each $x \in X$. f and g are said to be *R-weakly commuting* if (1.1) holds for some real number $R > 0$.

Under the assumption of R-weak commutivity, Pant [6] proved two common fixed point theorems for a pair of mappings.

THEOREM 1.1. [6] *Let (X, d) be a complete metric space and let f, g be R-weakly commuting self-mappings of X satisfying the condition:*

$$d(fx, fy) \leq \gamma(d(gx, gy)) \tag{1.2}$$

for all $x, y \in X$, where $\gamma : R_+ \to R_+$ is a continuous function such that $\gamma(t) < t$ for each $t > 0$. If $f(X) \subseteq g(X)$ and if either f or g is continuous, then f and g have a unique common fixed point in X.

THEOREM 1.2. [6] *Let (X, d) be a complete metric space and let f, g be R-weakly commuting self-mappings of X satisfying the condition: Given $\epsilon > 0$, there exists $h(\epsilon) > 0$ such that*

$$\epsilon \leq d(gx, gy) < \epsilon + h \Rightarrow d(fx, fy) < \epsilon, \tag{1.3}$$

$$fx = fy \text{ whenever } gx = gy \tag{1.4}$$

If $f(X) \subseteq g(X)$ and if either f or g is continuous, then f and g have a unique common fixed point in X.

Simple statements and elegant proofs of Theorems 1.1 and 1.2 reveal the fact that these two theorems do not hold if we allow both the mappings f and g to be discontinuous on X or the space X is not complete. To this end, we have the following example:

EXAMPLE 1.1. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{2^2}, \cdots\}$ be a metric space with the usual metric $d(x,y) = |x-y|$ for all $x, y \in X$. Define the mappings $f, g : X \to X$ by

$$f(0) = \frac{1}{2^2}, \quad f\Big(\frac{1}{2^n}\Big) = \frac{1}{2^{n+2}}, \qquad g(0) = \frac{1}{2}, \quad g\Big(\frac{1}{2^n}\Big) = \frac{1}{2^{n+1}}$$

for $n = 0, 1, 2, \cdots$, respectively. (X, d) is clearly complete and

$$g(X) = \Big\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots\Big\} \supset \Big\{\frac{1}{2^2}, \frac{1}{2^3}, \cdots\Big\} = f(X).$$

Since f and g commute on X, they are R-weakly commuting for $R > 0$. Define $\gamma(t) = \frac{1}{2}t$ for all $t > 0$. f and g both are not continuous at 0. Furthermore, we have

$$\begin{aligned} d(f(0), f(1)) &= \Big|\frac{1}{4} - \frac{1}{4}\Big| = 0, \\ d(f(0), f(\frac{1}{2})) &= \Big|\frac{1}{4} - \frac{1}{8}\Big| = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4} = \gamma\Big(d\Big(g(0), g\Big(\frac{1}{2}\Big)\Big)\Big), \\ d(f(0), f(\frac{1}{2^2})) &= \Big|\frac{1}{4} - \frac{1}{16}\Big| = \frac{3}{16} = \frac{1}{2} \cdot \frac{3}{8} = \gamma\Big(d\Big(g(0), g\Big(\frac{1}{2^2}\Big)\Big)\Big) \end{aligned}$$

and so on.

Also, for $x = \frac{1}{2^n}$ and $y = \frac{1}{2^m}$ $(n, m = 0, 1, 2, \cdots)$, we have

$$\begin{aligned} d(fx, fy) &= d\Big(f\Big(\frac{1}{2^n}\Big), f\Big(\frac{1}{2^m}\Big)\Big) = \Big|\frac{1}{2^{n+2}} - \frac{1}{2^{m+2}}\Big| \\ &= \frac{1}{2}\Big|\frac{1}{2^{n+1}} - \frac{1}{2^{m+1}}\Big| = \gamma\Big(d\Big(g\Big(\frac{1}{2^n}\Big), g\Big(\frac{1}{2^m}\Big)\Big)\Big) \\ &= \gamma(d(gx, gy)). \end{aligned}$$

Hence all the conditions of Theorem 1.1 are satisfied except continuity of either f or g, but neither f nor g have a fixed point in X.

EXAMPLE 1.2. Let $X = \{1, \frac{1}{2}, \frac{1}{2^2}, \cdots\}$ be a metric space with the usual metric $d(x,y) = |x-y|$ for all $x, y \in X$. Define the mappings $f, g : X \to X$ by

$$f\Big(\frac{1}{2^n}\Big) = \frac{1}{2^{n+2}}, \quad g\Big(\frac{1}{2^n}\Big) = \frac{1}{2^{n+1}}$$

for $n = 0, 1, 2, \cdots$, respectively. We have

$$g(X) = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots\right\} \supset \left\{\frac{1}{2^2}, \frac{1}{2^3}, \cdots\right\} = f(X).$$

(X, d) is clearly not complete. Define $\gamma(t) = \frac{1}{2}t$ for all $t > 0$. Then by the similar arguments as in Example 1.1, all the conditions of Theorem 1.1 are satisfied except the completeness of X, but neither f nor g have a fixed point in X.

Now, there arises a natural question: "How Theorems 1.1 and 1.2 can be improved to the setting of non-complete metric spaces and without continuity of f and g over the whole space X?" We give the partial answer. It seems that Theorem 1.1 can be improved in two ways: either imposing certain restrictions on the space X or by replacing the notion of R-weakly commutativity of mappings with certain improved notion. We choose the second option. In this perspective, we introduce the following definitions:

Let (X, d) be a metric space and let f, g be self-mappings of X.

DEFINITION 1.2. The mappings f and g are said to be *R-weakly commuting of type* (A_f) if there exists a positive real number R such that

$$d(fgx, ggx) \leq Rd(fx, gx) \tag{1.5}$$

for all $x \in X$. f and g are said to be *R-weakly commuting of type* (A_f) if (1.5) holds for some real number $R > 0$.

REMARK 1.1. We have suitable examples which show that R-weakly commuting mappings are not necessarily R-weakly commuting of type (A_f) (see an example of Pant [6]).

DEFINITION 1.3. The mappings f and g are said to be *R-weakly commuting of type* (A_g) if there exists a positive real number R such that

$$d(gfx, ffx) \leq Rd(fx, gx) \tag{1.5}$$

for all $x \in X$. f and g are said to be *R-weakly commuting of type* (A_g) if (1.6) holds for some real number $R > 0$.

2. Main Results

We now state and prove the main theorems:

THEOREM 2.1. *Let (X, d) be a metric space and let f, g be R-weakly commuting self-mappings of type (A_g) or type (A_f) of X satisfying the condition (1.2) for x, y in C, where $\gamma : R_+ \to R_+$ is a continuous function such that $\gamma(t) < t$ for each $t > 0$, and C is a subset of X. If $g(C) \supseteq f(C)$, $f(C)$ is complete and if either f or g is continuous, then f and g have a unique common fixed point.*

Proof. Pick x_0 in C and choose a point x_1 in C such that $fx_0 = gx_1$. This can be done since $g(C) \supseteq f(C)$. In general, having chosen x_n, we can and do choose x_{n+1} such that $fx_n = gx_{n+1}$ for $n = 0, 1, 2, \cdots$. Then we have

$$\begin{aligned} xd(fx_n, fx_{n+1}) &\leq \gamma(d(gx_n, gx_{n+1})) \\ &= \gamma(d(fx_{n-1}, fx_n)) < d(fx_{n-1}, fx_n) \end{aligned}$$

for $n = 1, 2, \cdots$. Thus $\{d(fx_n, fx_{n+1})\}$ is a decreasing sequence of positive real numbers and so it tends to the limit $L \geq 0$. We claim that $L = 0$. For, if $L > 0$, then the inequality

$$d(fx_n, fx_{n+1}) \leq \gamma(d(fx_{n-1}, fx_n))$$

on passing the limit as $n \to \infty$ and in view of continuity of the function γ yields $L \leq \gamma(L) < L$, which is a contradiction. We, therefore, have $L = 0$. and so $d(fx_n, fx_{n+1}) \to 0$ as $n \to \infty$. Thus, to prove that $\{fx_n\}$ is a Cauchy sequence in $f(C)$, it is sufficient to show that a subsequence $\{fx_{2n}\}$ of $\{fx_n\}$ is a Cauchy sequence in $f(C)$. For convenience, let $y_n = fx_n$ for $n = 1, 2, \cdots$. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence in $f(C)$. Then there is an $\epsilon > 0$ such that for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) \geq 2k$ such that

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon. \tag{2.1}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying

$$d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon, \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \tag{2.2}$$

Then, for each even integer $2k$, we have

$$\begin{aligned}\epsilon <&d(y_{2n(k)}, y_{2m(k)})\\ \le&d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1})\\ &+ d(y_{2m(k)-1}, y_{2m(k)}).\end{aligned} \tag{2.3}$$

Since $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$, from (2.2) and (2.3), it follows that

$$d(y_{2n(k)}, y_{2m(k)}) \to \epsilon \quad \text{as } k \to \infty. \tag{2.4}$$

By the triangle inequality,

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d(y_{2m(k)-1}, y_{2m(k)})$$

and so, from (2.4), as $k \to \infty$,

$$d(y_{2n(k)}, y_{2m(k)-1}) \to \epsilon. \tag{2.5}$$

Therefore, by (1.2), we have

$$\begin{aligned}&d(y_{2n(k)}, y_{2m(k)})\\ \le&d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)})\\ =&d(y_{2n(k)}, y_{2n(k)+1}) + d(fx_{2n(k)+1}, fx_{2m(k)})\\ \le&d(y_{2n(k)}, y_{2n(k)+1}) + \gamma(d(gx_{2n(k)+1}, gx_{2m(k)}))\\ =&d(y_{2n(k)}, y_{2n(k)+1}) + \gamma(d(fx_{2n(k)}, fx_{2m(k)-1}))\\ =&d(y_{2n(k)}, y_{2n(k)+1}) + \gamma(d(y_{2n(k)}, y_{2m(k)-1}))\end{aligned} \tag{2.6}$$

and so, as $k \to \infty$ in (2.6),

$$\epsilon \le \gamma(\epsilon) < \epsilon,$$

which is a contradiction. Thus $\{y_{2n}\}$ is a Cauchy sequence and so $\{fx_n\}$ is a Cauchy sequence in $f(C)$. Since $f(C)$ is complete, $\{fx_n\}$ has a limit point z in $f(C)$. By the definition of g, $gx_n \to z$ as $n \to \infty$.

Suppose that the pair $\{f, g\}$ is R-weakly commuting of type (A_g). Assume that the mapping f is continuous. Then $f^2x_n \to fz$ as $n \to \infty$. Again, since f and g are R-weakly commuting of type (A_g), we have

$$d(gfx_n, f^2x_n) \leq Rd(fx_n, gx_n).$$

Letting $n \to \infty$, we obtain $gfx_n \to fz$. We now prove that $z = fz$. If not, then we have

$$\begin{aligned} d(z, fz) &= \lim_{n\to\infty} d(fx_n, ffx_n) \leq \lim_{n\to\infty} \gamma(d(gx_n, gfx_n)) \\ &= \gamma(d(z, fz)) < d(z, fz), \end{aligned}$$

which gives a contradiction, and so $z = fz$. Since $g(C) \supseteq f(C)$, there exists z_1 in C such that $z = fz = gz_1$. Now, we have

$$d(ffx_n, fz_1) \leq \gamma(d(gfx_n, gz_1)) < d(gfx_n, gz_1),$$

which, on letting $n \to \infty$, yields $fz = fz_1$, i.e., $z = fz = fz_1 = gz_1$. It follows that

$$d(fz, gz) = d(fz_1, gfz_1) \leq Rd(fz_1, gz_1) = 0.$$

Thus, we obtain $z = fz = gz$, i.e., z is a common fixed point of f and g. Since the continuity of g implies continuity of f, we obtain the same conclusion of this theorem if g is continuous instead of f and similarly we draw the same conclusion for $\{f, g\}$ pair to be R-weakly commuting of type (A_f)

The uniqueness of the common fixed point z is easy to establish from (1.2). This completes the proof. □

We now give an example which shows the validity of Theorem 2.1 and its seperiority over Theorem 1.1.

EXAMPLE 2.1. Let $X = [-1, 2]$ be a metric space with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and let $C = [-1, 1]$. Define the mappings $f, g : X \to X$ by

$$f(x) = \begin{cases} 1 & \text{if } -1 \le x \le 1 \\ \frac{3}{4} & \text{if } 1 < x < \frac{5}{4} \\ 1 + \frac{1}{32}x^2 & \text{if } \frac{5}{4} \le x \le 2, \end{cases} \quad g(x) = \begin{cases} 1 + \frac{1}{4}x^2 & \text{if } -1 \le x < 1 \\ 1 & \text{if } x = 1 \\ 2 & \text{if } 1 < x < \frac{5}{4} \\ 1 - \frac{1}{8}x^2 & \text{if } \frac{5}{4} \le x \le 2, \end{cases}$$

respectively.

For $-1 \le x < 1$, we have

$$f(x) = 1, \quad g(x) = 1 + \frac{1}{4}x^2, \quad gf(x) = 1, \quad ff(x) = 1,$$

so that $d(gfx, ffx) = 0 \le R\frac{1}{4}x^2 = Rd(fx, gx)$ for all $R > 0$.

For $x = 1$, we have

$$f(1) = 1, \quad g(1) = 1, \quad gf(1) = 1, \quad ff(1) = 1,$$

so that $d(gf(1), ff(1)) = 0 \le R \cdot 0 = Rd(f(1), g(1))$ for all $R > 0$.

For $1 < x < \frac{5}{4}$, we have

$$f(x) = \frac{3}{4}, \quad g(x) = 2, \quad gf(x) = \frac{73}{64}, \quad ff(x) = 1,$$

and so $d(gfx, ffx) = \frac{9}{64} \le R \cdot \frac{5}{4} = Rd(fx, gx)$ for all $R \ge \frac{9}{80}$.

For $\frac{5}{4} \le x \le 2$, we have

$$f(x) = 1 + \frac{1}{32}x^2, \quad g(x) = 1 - \frac{1}{8}x^2, \quad gf(x) = 2, \quad ff(x) = \frac{3}{4},$$

and so $d(gfx, ffx) = \frac{5}{4} \le R\frac{5}{32}x^2 = Rd(fx, gx)$ for all $R \ge \frac{128}{25}$.

Thus, f and g are R-weakly commuting of type (A_g) with $R = 6$. On the other hand, there exists no real number $R > 0$ such that $d(fgx, gfx) \le Rd(fx, gx)$ for all x in X (see, for instance when $-1 \le$

$x < 1$). Hence f and g are not R-weakly commuting on X. Also, for all x, y in C, we have

$$g(C) = \left[1, \frac{5}{4}\right) \supset \{1\} = f(C), \quad d(fx, fy) = 0 \le \gamma(d(gx, gy)).$$

Clearly, $f(C)$ is complete. Thus all the conditions of Theorem 2.1 are satisfied and 1 is the unique common fixed point of f and g.

We now establish a common fixed point theorem for R-weakly commuting pairs of mappings of type (A_g) or type (A_f) satisfying the Meir-Keeler type contractive condition.

THEOREM 2.2. *Let (X, d) be a metric space and let f, g be R-weakly commuting self-mappings of type (A_g) or type (A_f) of X satisfying the condition such that, given $\epsilon > 0$, there exists $h(\epsilon) > 0$ such that* (1.3) *and* (1.4) *holds for x, y in C, where C is a subset of X. If $g(C) \supseteq f(C)$, $f(C)$ is complete and if either f or g is continuous, then f and g have a unique common fixed point.*

Proof. Suppose that $\{x_n\}$ is a sequence in X, as defined in Theorem 2.1, given by the rule $fx_n = gx_{n+1}$ for $n = 1, 2, \cdots$. By (1.3), for all x, y in X with $gx \ne gy$, we have

$$d(fx, fy) < d(gx, gy).$$

Then we have

$$d(fx_n, fx_{n+1}) < d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n). \tag{2.7}$$

Thus $\{d(fx_n, fx_{n+1})\}$ is a monotone decreasing sequence of positive real numbers and so it tends to the limit $L \ge 0$. We asserts that $L = 0$. If it is not true, then suppose that $L > 0$. Now, for a given $h > 0$, there exists a positive integer N such that for all $m \ge N$,

$$L \le d(fx_m, fx_{m+1}) = d(gx_{m+1}, gx_{m+2}) < L + h. \tag{2.8}$$

Select h in (2.8) in accordence with (1.3), for each $m \ge N$, we then obtain $d(fx_{m+1}, fx_{m+2}) < L$, which contradicts (2.8). Therefore, we have

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = \lim_{n\to\infty} d(gx_{n+1}, gx_{n+2}) = 0.$$

Thus, by the same argument, as in the proof of Theorem 2.1, $\{fx_n\}$ is a Cauchy sequence in $f(C)$. Since $f(C)$ is complete, there exists z in $f(C)$ such that $fx_n \to z$ and $gx_n \to z$ as $n \to \infty$.

Suppose that the pair $\{f, g\}$ is R-weakly commuting of type (A_g) and f is continuous. Then $f^2x_n \to fz$ as $n \to \infty$. Further, R-weak commutativity of f and g implies that $gfx_n \to fz$ as $n \to \infty$. We now claim that $fz = z$. If not, it is clear that no subsequence of $\{ffx_n\}$ or of $\{gfx_n\}$ can converge to the point z. It follows that there exist a real number $L > 0$ and positive integers M, N such that for all $m \geq M$ and $n \geq N$,

$$\inf\{d(fx_m, ffx_n)\} = L. \tag{2.9}$$

In view of (1.3), the equation (2.9) yields

$$\inf\{d(fx_m, ffx_n\} < L,$$

which is a contradiction. Therefore, $z = fz$. The rest of the proof follows by using an argument similar to that used in the corresponding part of Theorem 2.1. This completes the proof. □

EXAMPLE 2.2. To illustrate the validity of Theorem 2.2, we consider Example 2.1 again. If $d(gx, gy) = \epsilon > 0$, then $d(fx, fy) = 0 < \epsilon$ for all x, y in C. It follows, therefore, that all the conditions of Theorem 2.2 are satisfied and so f and g have a unique common fixed point $z = 1$.

REMARK. Our results improve and generalize several known results on fixed point theorems which are due to Boyd and Wong [1], Jungck [2], Meir-Keeler [5], Pant [6], Park and Bae [7], Rhoades, Park and Moon [8].

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